A Closed, Expanding Universe with a Radial Curvature Gradient

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Abstract

We propose a novel cosmological model in which the universe is spatially closed and expanding, yet features a radial gradient in curvature that intensifies near its outermost regions. Unlike the standard closed FLRW universe with constant positive spatial curvature, our model introduces a smooth radial deformation of the angular part of the metric. This curvature profile bends geodesics inward as they approach the outer regions, yielding a finite yet unbounded spatial geometry. Notably, unlike many inhomogeneous cosmologies or big bang-type models, the proposed geometry permits geodesic completeness without encountering singularities, due to the regulating effect of the radial curvature gradient. We derive the spacetime metric, compute the Ricci scalar and Einstein tensor, and analyze the physical implications of the associated energy-momentum distribution.

1. Introduction

Modern cosmology is typically framed within the Friedmann–Lemaître–Robertson–Walker (FLRW) class of solutions, which describe homogeneous and isotropic universes. In the closed FLRW case, spatial slices have the topology of a 3-sphere and possess constant positive curvature. However, general relativity admits more general solutions when homogeneity is relaxed.

In this work, we explore a class of spatially closed, expanding universes that feature radially varying curvature. The geometry redirects geodesics inward near its outermost regions, enabling natural geodesic return and topological closure without requiring physical boundaries. This represents an inhomogeneous, anisotropic extension of the standard closed universe that maintains topological finiteness while introducing curvature variation as a physically meaningful structure.

2. The Metric

We consider a time-dependent, spherically symmetric spacetime of the form:

$$ds^2 = -dt^2 + a(t)^2 dr^2 + S_k(r)^2 d\theta^2 + \sin^2\theta\, d\phi^2$$

where $a(t)$ is the cosmological scale factor, and $r \in [0, R]$ is a radial coordinate labeling concentric spatial shells.

The function $S_k(r)$ replaces the usual $\sin r$ term in the standard closed FLRW metric. We define it as:

$$S_k(r) = \sin r + \epsilon \sin^3 r$$

where $\epsilon > 0$ is a deformation parameter that enhances curvature near $r \to \pi$. When $\epsilon = 0$, the model reduces to the closed FLRW universe.

3. Geometry and Curvature

The 3D spatial part of the metric is:

$$d\sigma^2 = dr^2 + S_k(r)^2 d\theta^2 + \sin^2\theta\, d\phi^2$$

To study curvature, we compute the Ricci scalar of the full spacetime:

$$R = 6\frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\alpha}}{\alpha}^2 + \frac{K(r)}{\alpha^2}$$

where the spatial curvature scalar is:

$$K(r) = \frac{2}{S_k(r)^2}\left[1 - \frac{dS_k}{dr}^2 + S_k(r)\frac{d^2 S_k}{dr^2}\right]$$

Using:

$$\frac{dS_k}{dr} = \cos + 3\epsilon \sin^2 r \cos r \quad \frac{d^2 S_k}{dr^2} = -\sin r + 3\epsilon(2\sin r \cos^2 r + \sin^3 r)$$

we find that $K(r)$ increases with r, especially as $r \to \pi$. This establishes the radial curvature gradient central to our model.

4. Einstein Field Equations and Matter Content

Einstein's field equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

connect geometry to matter. For the proposed metric, the nonzero components of the Einstein tensor in an orthonormal frame are:

- Temporal: $G^t_t = -3\left(\frac{\dot{\alpha}}{\alpha}\right)^2 - \frac{3K(r)}{\alpha^2}$

- Radial: $\quad G^r_r = -2\frac{\ddot{\alpha}}{\alpha} - \left(\frac{\dot{\alpha}}{\alpha}\right)^2 - \frac{K(r)}{\alpha^2}$

- Angular: $\quad G^\theta_\theta = G^\phi_\phi = -2\frac{\ddot{\alpha}}{\alpha} - \left(\frac{\dot{\alpha}}{\alpha}\right)^2 - \frac{K(r)}{\alpha^2}$

This implies a diagonal energy-momentum tensor:

$$T^\mu_\nu = diag\ (-\rho(r,t), p_r(r,t), P\perp\ (r,t), P\perp\ (r,t))$$

indicating inhomogeneity due to the radial dependence of $K(r).$ In central regions, a standard fluid may suffice. Near the outer regions, exotic matter or anisotropic pressure may be required to support the geometry.

5. Geodesic Structure

Geodesics satisfy:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

We consider radial null and timelike geodesics. The effective potential for angular motion:

$$V_{eff}(r) \propto \frac{L^2}{S_k(r)^2}$$

where L is angular momentum, diverges as $S_k(r) \to 0$. This produces a repulsive potential barrier near the poles, redirecting geodesics inward.

Unlike many inhomogeneous cosmologies or big bang-type models, the proposed geometry permits geodesic completeness without encountering singularities, due to the regulating effect of the radial curvature gradient. All geodesics are naturally confined, ensuring that the universe is both spatially finite and dynamically bounded without physical edges.

6. Discussion and Implications

Key features of this model include:

- No Artificial Boundaries: The curvature gradient enforces geodesic return without boundary conditions.
- Curvature-Driven Confinement: The geometry ensures geodesic closure via spacetime structure.

- Inhomogeneous Matter Content: The geometry can be supported by standard matter in the central region, and potentially by fields with exotic properties near the edges.
- Observational Consequences:

  - CMB anisotropies may reflect curvature gradients.
  - Structure formation and gravitational lensing could be influenced by the inhomogeneous geodesic deviation.
- Theoretical Opportunities:

- Possible applications in quantum cosmology or higher-dimensional analogues.
- Relevant to inflationary exit scenarios or mechanisms of cosmic acceleration.

7. Conclusion

We have constructed a class of closed, expanding cosmologies with radially increasing spatial curvature. The curvature profile modifies geodesic behavior, ensuring natural geodesic return without singularities or physical boundaries. The metric satisfies Einstein's equations with physically plausible energy-momentum distributions over much of the domain. This model enriches the catalog of viable solutions in general relativity and offers fertile ground for further exploration in both classical and quantum gravitational contexts.

References

1. A. Friedmann, *On the Curvature of Space*, *General Relativity and Gravitation* 31, 1991-2000 (1999).
2. G. Lemaître, *Mon. Not. R. Astron. Soc.* 91, 483 (1931).
3. S. Weinberg, *Gravitation and Cosmology*, Wiley (1972).
4. R. M. Wald, *General Relativity*, University of Chicago Press (1984).
5. C. Clarkson et al., *Cosmology in a lumpy universe*, MNRAS 426, 1121 (2012).
6. K. Bolejko et al., *Structures in the Universe by Exact Methods*, Cambridge Univ. Press (2009).
7. M. Bojowald, *Loop Quantum Cosmology*, Living Rev. Relativ. 11, 4 (2008).
8. V. Faraoni, *Cosmology in Scalar-Tensor Gravity* (2004).