A Flaw in the Proof of Riemann Rearrangement Theorem - An Incomplete Mathe-magic Trick

Marko V. Jankovic

Institute of Electrical Engineering "Nikola Tesla", Belgrade, Serbia
Swiss Rockets Serbia, Belgrade, Serbia

To
CH-ANA

Prologue:

"Every great magic trick consists of three parts or acts. ***The first part*** *is called "****The Pledge".*** *The magician shows you something ordinary: a deck of cards, a bird or a man. He shows you this object. Perhaps he asks you to inspect it to see if it is indeed real, unaltered, normal. But of course … it probably isn't.* ***The second act*** *is called "****The Turn".*** *The magician takes an ordinary something and makes it do something extraordinary. Now you're looking for the secret … but you won't find it, because of course you're not really looking. You don't really want to know.* ***You want to be fooled.*** *But you wouldn't clap yet. Because making something disappear isn't enough; you have to bring it back. That's why every magic trick has* ***a third act, the hardest part,*** *the part we call "****The Prestige".***

Christopher Priest, *The Prestige*, Tor Book, 1997

Abstract In this paper Riemann rearrangement theorem is going to be analyzed on a single example and it is going to be explained that the proof of the theorem is incomplete and wrong, That means that it does not matter how you rearrange the elements of the series, the sum would always stay the same. The reason that “rearranged” series does not have the same sum as the original series, is in the hidden omission of infinite number of elements that are contained in the original series. The content is presented in the form of explanation of a magic trick (since the claim of the theorem sounds as a real magic).

1 Introduction

Here, Riemann series (rearrangement) theorem [1] is going to be very briefly introduced.

Riemann series theorem: Suppose that we have a sequence of real numbers m_k, and that

$$R=\lim_{n\to\infty}\sum_{k=1}^{n} m_k ,$$

is conditionally convergent. Let M be the real number (it also can take the values of ∞ or $-\infty$) .

Then, there exists a permutation σ such that the following holds

$$R_\sigma=\sum_{k=1}^{\infty} m_{\sigma(k)}=M.$$

In the following sections, it is going to be shown that this is not true for a specific series and its “rearrangement” into alternating harmonic series (it can be easily extended for rearrangements of alternating harmonic series that is presented in the literature, and many other series). Performed analysis clearly shows that theorem contains a major flaw in the part of the proof that explains that all elements of the initial series are present in the rearranged series. Different value of sum for “rearranged” series is direct consequence of the omission of infinite number of elements of the initial series – although you cannot specify a single element that is missing. In order to understand that part, reader should have in mind that an integral is actually a sum of infinite number of elements where you cannot specify a single element, but still you can calculate the sum.

2 The first Act – The PLEDGE

Here, one example of the series is going to be presented. Let us start with the following series b_k (we are going to call it zero sum harmonic series – or ZSH series):

$$b_1=1,\ b_2=-\ 1,\ b_3=\frac{1}{2},\ b_4=-\ \frac{1}{2},\ b_5=\frac{1}{3,}\ b_6=-\ \frac{1}{3},\ldots$$

This object is quite frequently analyzed mathematical object and most of the readers are probably quite familiar with it. It is easy to understand that the following holds:

$$B=\sum_{k=1}^{\infty}\frac{1}{k}-\frac{1}{k}=0.$$

It is trivial to understand that ZSH series is conditionally convergent.

3 The second Act – The TURN

In this act mathe-magicians usually change the order of the elements of the series and claim (based on Riemann rearrangement theorem) that it is possible to obtain a different sum. The most usual example is rearrangement of ZSH series, is one that results in alternating harmonic series h_k. It will appear if we rearrange initial series b_k in such a way that every 2 positive elements are followed with a negative element and then we sum second positive term and negative term

$$h_1=1,\ h_2=\frac{1}{2}-\ 1,\ h_3=\frac{1}{3,}\ h_4=\frac{1}{4}-\frac{1}{2},\ldots$$

that results in

$$h_1=1,\ h_2=-\ \frac{1}{2},\ h_3=\frac{1}{3,}\ h_4=-\ \frac{1}{4}\ ,\ h_5=\frac{1}{5,}\ h_6=-\ \frac{1}{6}\ ,\ldots$$

It is well known result that sum of this infinite series is equal to ln(2). More generally speaking, if the initial series is rearranged in such a way that p positive terms are followed by q negative terms, sum of such series is ln(p/q) (see e.g. the Mathologer's video: The best A-A≠0 paradox, or some of the Michael Penn's videos). Usually at this moment, magic mathematical trick is finished.

However, from the Prologue we know that every magic trick should contain the third act – The Prestige, where magician shows back to the public what was hidden.

4 The third Act – The PRESTIGE

In this “act” hidden part is going to be brought back.

Consider now the partial sum of the ZSH series that is obtained after summing first 4*n* elements. It can be written as

$$B(n)=\sum_{k=1}^{2n}\frac{1}{k}-\frac{1}{k}=0.$$

This can be further rewritten as

$$B(n)=\sum_{k=1}^{n}\frac{1}{2k-1}+\frac{1}{2k}-\frac{1}{k}-\sum_{k=n+1}^{2n}\frac{1}{k}=0,$$

or

$$B(n)=\sum_{k=1}^{n}\frac{1}{2k-1}-\frac{1}{2k}-\sum_{k=n+1}^{2n}\frac{1}{k}=0,$$

or

$$B(n)=\sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k}-\sum_{k=n+1}^{2n}\frac{1}{k}=0.$$

First sum represents a partial sum of alternating harmonic series (AH series), and we will mark it as $A_h(2n)$ while the second sum represents the sum of the *n* consecutive elements of harmonic series that starts with 1/(*n*+1) and we mark it with Š(*n*). We will denote that series as a Tail of Harmonic series, or ToH series. It is simple to conclude that Š(*n*) represents an infinite series that is monotonically increasing and value of it is not smaller than ½ and is smaller than 1, which means that it is converging to some number between ½ and 1. It is easy to understand that the sum of Š(*n*) tends toward ln(2) as *n* →∞ (it can be written as difference of 2 harmonic series). Since we also know that the sum of AH series is ln(2) it can be concluded that sum of B(*n*) as *n* →∞ has to be 0, so it actually will not be changed by rearrangement of the elements of the series. We can write it as

$$B=\lim_{n\to\infty}B(n)=\lim_{n\to\infty}A_h(2n)-\lim_{n\to\infty}\check{S}(n)=0.$$

The wrong conclusion supported by the Riemann's theorem comes from the fact that infinitely many elements of the initial ZSH series have not been included in the AH series.

Now, it should be easy for the reader to understand where is the mistake in the original proof of the theorem that led to the wrong conclusion that rearrangement of the elements of the infinite series can lead to a different sum. Here, a short and sinplified exaplanation is going to be presened.

From previous analysis it can be seen that ToH series is convergent. From the example that was analyzed we can see that the sum of the second half of the harmonic series is finite and equal to ln(2). It is not difficult to understand that the last third, forth or *p*/*q* part of the ZSH series are convergent, too. So, it is possible to find last part of the tail of harmonic series that is smaller than some arbitrary value. Now, we are going to quote part of the original proof (from translated version) and show where is the flaw.

" … Indeed if we denote the positive terms of a series in the second class by a_1, a_2, a_3, … and the negative terms by $-b_1$, $-b_2$, $-b_3$, ...then it is clear that $\sum a$ as well as $\sum b$ must be infinite. For, if they were both finite, the series would still be convergent after making all the signs the same. If only one were infinite, then the series would diverge. Clearly now an arbitrary given values C can be obtained by a suitable reordering of the terms. We take alternatively the positive terms of the series until the sum is greater than C, and then the negative terms until the sum is less than C. The deviation from C never amounts to more than the size of the last place the signs were switched. Now, since the number a as well as numbers b become infinitely small with increasing index, so also are the deviations from C. If we proceed sufficiently far in the series, the deviation becomes arbitrarily small, that is, the series converges to C. "

However, we can see that there is no proof that after the switching of the sign at term b_k, the sum of the rest of the elements in a will be sufficient to achieve again a sum that is greater than C (when we switch at a_k we do not have a proof that the rest of the elements in b will be sufficient to achieve a sum smaller than C), although it can come arbitrarily close to C. From our previous analysis we know that it is exactly what happens – after some value of b_k we will never use the rest of the elements in b since sum of the rest of elements in a will not again become grater than C. Of course, in the literature you can find the proofs that claim of the proof is correct for finite k. *However, since* *we are talking about infinite series we can talk also about the infinite indexes of the elements of the series and it has been explained that the sum of all elements of the tail of harmonic series can become arbitrarily small, so also smaller than some member of the series.* Detailed proof of this claim in the case of ZSH series or or for any other series, will not be presented in this paper.

5. Conclusion

In this paper a counterexample of the Riemann series (rearrangement) theorem has been presented. Presented material gives reader enough information to detect incompleteness of the proof of the theorem published by Bernhard Riemann - that is related to the part of the proof that all elements of the initial series are contained in the rearranged series. Author also hopes that presented analysis will help reader to understand that Ross-Littlewood paradox is not a paradox at all, that it is not possible to generate bijection between natural numbers and rational numbers or between natural numbers and even natural numbers, and so on. In all those cases, an infinite number of elements is "ignored" due to the fact that not a single one of them can be specified. And it should not be the case, since it leads to the wrong conclusions.

References

[1] B. Riemann (2004). On the representation of a function by a trigonometric series. *Collected papers*. Translated by Baker, Rodger; Christenson, Charles; Orde, Henry. Translation of 1892 German edition, Heber City, UT, Kendrick Press.