Remarks on a paper by K. Adegoke

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Abstract

Recently, K. Adegoke et al. (arXiv: 2505.11575v1, 16 May 2025) established a number of results on binomial series. In this note, we explore some particular cases of these series.

1 Introduction

In Reference [1], the following formulas are presented

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lceil \frac{n}{2} \rceil}}{2n+1} \binom{2n}{n} 2^{-2n} x^{2n+1} = \sqrt{2}\ \arctan\left(\frac{x}{\sqrt{1+\sqrt{1+x^4}}} \right) \ , \ |x| \leq 1 \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{2n+1} \binom{2n}{n} 2^{-2n} x^{2n+1} = \sqrt{2}\ \operatorname{arctanh}\left(\frac{x}{\sqrt{1+\sqrt{1+x^4}}} \right) \ , \ |x| \leq 1 \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lceil \frac{n}{2} \rceil}}{2^{2n}} \binom{2n}{n} x^{n} = \sqrt{\frac{\sqrt{1+x^2}-x}{1+x^2}} \ , \ |x| \leq 1 \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{2^{2n}} \binom{2n}{n} x^{n} = \sqrt{\frac{\sqrt{1+x^2}+x}{1+x^2}} \ , \ |x| \leq 1 \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lceil \frac{n}{2} \rceil}}{2n+1} \binom{2n}{n} 2^{-4n} = 2\ \sqrt{2}\ \arctan\left(\sqrt{\sqrt{17}-4} \right) \tag{5}$$

where $\lfloor \cdot \rfloor$ is the floor function and $\lceil \cdot \rceil$ is the ceiling function.

In this note, we derive many interesting identities involving the central binomial coefficients.

Recall that

$$\phi = \frac{1+\sqrt{5}}{2} \tag{6}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \tag{7}$$

$$F_{n+2} = F_{n+1} + F_n \ , \ F_0 = 0,\ F_1 = 1 \tag{8}$$

where F_n are the Fibonacci numbers.

2 Main Examples

Entry 1.

$$\pi = \frac{3\sqrt{3}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^{\lceil \frac{n}{2} \rceil}}{2n+1} \binom{2n}{n} \left(\frac{3}{16} \right)^n \tag{9}$$

Proof. setting $x = \frac{\sqrt{3}}{2}$ in (1) .

Entry 2.

$$\pi = 2 \sqrt{2 \sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil}}{2n+1} \binom{2n}{n} \left(\frac{1}{8\sqrt{2}} \right)^n \tag{10}$$

Proof. setting $x = 2^{-3/4}$ in (1) .

Entry 3.

$$\pi = 2 \sqrt{2 \sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} \binom{4n}{2n} 2^{-7n} - \frac{1}{2\sqrt{2\sqrt{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3} \binom{4n+2}{2n+1} 2^{-7n} \tag{11}$$

Proof. In (10), we use $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{2n} + \sum_{n=0}^{\infty} a_{2n+1}$.

Entry 4.

$$\pi = \sqrt{2} \sqrt{3\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil}}{2n+1} \binom{2n}{n} \left(\frac{1}{16\sqrt{3}} \right)^n \tag{12}$$

Proof. setting $x = \frac{1}{2} \cdot 3^{-1/4}$ in (1) .

Entry 5.

$$\pi = \sqrt{2} \sqrt{3\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} \binom{4n}{2n} 2^{-8n} \cdot 3^{-n} - \frac{\sqrt{\sqrt{3}}}{8\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3} \binom{4n+2}{2n+1} 2^{-8n} \cdot 3^{-n} \tag{13}$$

Proof. In (12), we use $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{2n} + \sum_{n=0}^{\infty} a_{2n+1}$.

Entry 6.

$$\pi = 2 \arctan\left(\frac{1}{\sqrt{\phi}} \right) + 2 \sum_{n=0}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil}}{2n+1} \binom{2n}{n} \left(\frac{1}{8} \right)^n \tag{14}$$

$$\pi + 4 \arctan\left(\frac{\sqrt{\phi}-1}{\sqrt{\phi}+1} \right) = 4 \sum_{n=0}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil}}{2n+1} \binom{2n}{n} \left(\frac{1}{8} \right)^n \tag{15}$$

Proof. setting $x = 1/\sqrt{2}$ in (1), and using the relations

$$2 \arctan\left(\sqrt{\sqrt{5}-2} \right) = \arctan\left(\sqrt{\phi} \right) = \frac{\pi}{2} - \arctan\left(\frac{1}{\sqrt{\phi}} \right) , \quad \phi = \frac{1+\sqrt{5}}{2} \tag{16}$$

$$\arctan\left(\frac{1}{\sqrt{\phi}} \right) = \frac{\pi}{4} - \arctan\left(\frac{\sqrt{\phi}-1}{\sqrt{\phi}+1} \right) \tag{17}$$

Entry 7.

$$\pi = 4 \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor}}{2n+1} \binom{2n}{n} 2^{-2n} \sum_{k=0}^{n} \binom{2n+1}{2k+1} (-1)^k u^{2n-2k} v^{2k+1} \tag{18}$$

where

$$u = \frac{1}{4} \sqrt{-\frac{9}{5} + \sqrt{10}} , \quad v = \frac{1}{52} \sqrt{-\frac{9}{10} + \sqrt{\frac{5}{2}} \left(9\sqrt{2} + 10\sqrt{5} \right)} \tag{19}$$

Proof. setting $x = u + iv$ in (2) and performing some algebraic manipulations.

Entry 8.

$$\pi = 12 \arctan\left(\frac{2 - \sqrt{3} - \sqrt{\sqrt{82} - 9}}{1 + (2 - \sqrt{3})\sqrt{\sqrt{82} - 9}}\right) + 2\sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil}}{2n+1} \binom{2n}{n} 6^{-2n} \tag{20}$$

Proof. setting $x = 1/3$ in (1), and use the relation $\tan(\pi/12) = 2 - \sqrt{3}$.

Entry 9. for $|x| \le 1$ we have

$$x\sqrt{\frac{\sqrt{1+x^4} - x^2}{1+x^4}} = \sqrt{2} \arctan\left(\frac{x}{\sqrt{1 + \sqrt{1+x^4}}}\right) + 2 \sum_{n=1}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil} n}{2n+1} \binom{2n}{n} 2^{-2n} x^{2n+1} \tag{21}$$

$$x\sqrt{\frac{\sqrt{1+x^4} + x^2}{1+x^4}} = \sqrt{2} \operatorname{arctanh}\left(\frac{x}{\sqrt{1 + \sqrt{1+x^4}}}\right) + 2 \sum_{n=1}^{\infty} \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} n}{2n+1} \binom{2n}{n} 2^{-2n} x^{2n+1} \tag{22}$$

Proof of (21). Using (3), we find that

$$x\sqrt{\frac{\sqrt{1+x^4} - x^2}{1+x^4}} = \sum_{n=1}^{\infty} (-1)^{\left\lceil \frac{n}{2} \right\rceil} \binom{2n}{n} 2^{-2n} x^{2n+1} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil} n}{2n+1} \binom{2n}{n} 2^{-2n} x^{2n+1} + \sum_{n=1}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil}}{2n+1} \binom{2n}{n} 2^{-2n} x^{2n+1} \tag{23}$$

Using (1), we obtain (21).

Proof of (22). Using (4), we find that

$$x\sqrt{\frac{\sqrt{1+x^4} + x^2}{1+x^4}} = \sum_{n=1}^{\infty} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{2n}{n} 2^{-2n} x^{2n+1} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} n}{2n+1} \binom{2n}{n} 2^{-2n} x^{2n+1} + \sum_{n=1}^{\infty} \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor}}{2n+1} \binom{2n}{n} 2^{-2n} x^{2n+1} \tag{24}$$

Using (2), we obtain (22).

Entry 10. for $\frac{1 - \sqrt{\sqrt{2} - 1}}{1 + \sqrt{\sqrt{2} - 1}} \le x \le \sqrt{\sqrt{2} - 1}$ we have

$$\pi = 2\sqrt{2} \sum_{n=1}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil}}{2n+1} \binom{2n}{n} 2^{-2n} \left(\left(\frac{\sqrt{2}\,x}{\sqrt{1-x^4}}\right)^{2n+1} + \left(\frac{1-x^2}{\sqrt{x}\,\sqrt{4+4x^2}}\right)^{2n+1}\right) \tag{25}$$

Proof. In (1), we use the relation $\pi/4 = \arctan(x) + \arctan((1-x)/(1+x))$.

Entry 11.

$$\pi = 8 \arctan\left(\sqrt{\sqrt{17} - 4}\right) + \sqrt{2} \sum_{n=0}^{\infty} \frac{2^{-3n-3}}{2n+3} \binom{2n+2}{n+1} \sum_{k=0}^{n} 2^{-k} (-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor} (2 + (-1)^k) \tag{26}$$

Proof. we use Abel's summation formula in (5), and $\pi/4 = \arcsin(1/\sqrt{2})$.

Entry 12. for $0 \le x \le \sqrt{\sqrt{2} - 1}$ we have

$$x \arctan(x) - \frac{1}{2} \ln(1 + x^2) = \sum_{n=0}^{\infty} \frac{(-1)^{\left\lceil \frac{n}{2} \right\rceil} 2^{-n}}{(2n+1)(2n+2)} \binom{2n}{n} x^{2n+2} \, {}_2F_1\left(n + \frac{1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, x^4\right) \tag{27}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Proof. setting $x = \frac{\sqrt{2}\,y}{\sqrt{1-y^4}}$ in (1) and integrating both sides of equation.

Entry 13. for $|x| < 1$ we have

$$\frac{1}{2} x \sqrt{x^2 + \sqrt{1+x^4}} + \frac{1}{2\sqrt{2}} \arctan\left(\sqrt{2}\,x\sqrt{x^2 + \sqrt{1+x^4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor}}{2n+1} \binom{2n}{n} 2^{-2n} x^{2n+1} \, {}_2F_1\left(-\frac{1}{2}, \frac{2n+1}{4}, \frac{2n+5}{4}, -x^4\right) \tag{28}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Proof. In (4): $x \mapsto x^2$, and use the integrals $\int x^{2n} \sqrt{1 + x^4} \, dx$, $\int \sqrt{\sqrt{1 + x^4} + x^2} \, dx$.

Entry 14.

$$\pi + 6 \arctan\left(\frac{\sqrt{3}}{\sqrt{4\,\phi^4 + \sqrt{9 + 16\,\phi^8}}}\right) = 3\,\sqrt{10}\,\sum_{n=0}^{\infty}\frac{(-1)^{\lceil n/2 \rceil}\,2^{-2n}}{2\,n + 1}\binom{2\,n}{n}\left(\frac{\sqrt{3}}{2\,\phi}\right)^{2n+1} F_{2\,n+1} \tag{29}$$

$$\pi + 8 \arctan\left(\frac{1}{\sqrt{2\,\sqrt{2}\,\phi^4 + \sqrt{1 + 8\,\phi^8}}}\right) = 4\,\sqrt{10}\,\sum_{n=0}^{\infty}\frac{(-1)^{\lceil n/2 \rceil}\,2^{-2n}}{2\,n + 1}\binom{2\,n}{n}\left(\frac{1}{\sqrt[4]{8}\,\phi}\right)^{2n+1} F_{2\,n+1} \tag{30}$$

Proof. In (1), we use the relation $\sqrt{5}\,F_{2n+1} = \phi^{2n+1} - (-1/\phi)^{2n+1}$.

Entry 15.

$$\pi - 8 \operatorname{arctanh}\left(\frac{1}{\sqrt{2\,\sqrt{2}\,\phi^2 + \sqrt{1 + 8\,\phi^4}}}\right) = 2\,\sqrt{2\,\sqrt{2}}\,\sqrt{5}\,\sum_{n=0}^{\infty}\frac{(-1)^{\lceil n/2 \rceil}\,2^{-2n}}{2\,n + 1}\binom{2\,n}{n}\left(\frac{1}{2\,\sqrt{2}\,\phi}\right)^{n} F_{n} \tag{31}$$

Proof. In (1), we use the relation $\sqrt{5}\,F_n = \phi^n - (-1/\phi)^n$.

3 Endnote

$$\pi - \frac{8}{3} - 4\sum_{n=1}^{\infty}\frac{(4\,n + 2)\,2^{4n+2}}{\binom{8\,n + 4}{4\,n + 2}} = \sum_{n=1}^{\infty}\left((-1)^{\lceil n/2 \rceil} + (-1)^{\lfloor n/2 \rfloor}\right)\frac{n\,2^n}{\binom{2\,n}{n}} \tag{32}$$

$$\pi = 2\,\sqrt{2} + 2\sum_{n=0}^{\infty}\frac{(-1)^{\lceil n/2 \rceil}\,2^{-3n}}{2\,n + 1}\binom{2\,n}{n}\left(\frac{\cos\left(\sqrt{2}\right)}{\sqrt{\sin\left(\sqrt{2}\right)}}\right)^{2n+1} \tag{33}$$

$$\pi = \sum_{n=0}^{\infty}\frac{(-1)^{\lceil \frac{n}{2} \rceil}}{2\,n + 1}\binom{2\,n}{n}2^{-3n}\left(4 - 2^{-n}\,\sqrt{2}\,\left(\sqrt[4]{\phi}\right)^{-18n-9}\right) \tag{34}$$

$$\pi = 12\sum_{n=0}^{\infty}\frac{(-1)^{\lceil n/2 \rceil}\,2^{-3n}}{2\,n + 1}\binom{2\,n}{n}\left(\frac{1}{\sqrt{5\,\sqrt{2} + 3\,\sqrt{6}}}\right)^{2n+1} \tag{35}$$

4 References

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