An inconsistency with hidden variable theories based on a single projection operator

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Abstract

We propose no hidden variable theorem for the quantum measurement outcomes by measuring an observable that is obtained through a single projection operator. We are in the inconsistency with hidden variable theories when the first predetermined hidden result is $+1$ by measuring a single projection operator $|\uparrow\rangle\langle\uparrow|$ in the quantum state $|\uparrow\rangle$, the second predetermined hidden result is the same $+1$ as by measuring the same projection operator $|\uparrow\rangle\langle\uparrow|$ in the same quantum state $|\uparrow\rangle$. It turns out that we cannot assign the predetermined hidden result for quantum measurement outcome as $+1$ when measuring a single projection operator.

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I. INTRODUCTION

Quantum mechanics (cf. [1–8]) gives explanations for the microscopic behaviors of the nature. We see researches concerning the mathematical formulations of quantum mechanics. For example, the mathematical foundations of quantum mechanics are discussed by Mackey [9]. On the quantum logic approach to quantum mechanics is also discussed by Gudder [10]. Conditional probability and the axiomatic structure of quantum mechanics are also reported by Guz [11].

von Neumann's mathematical model for quantum mechanics is logically successful [5]. The axiomatic system for the mathematical model is a consistent one. Thus, we cannot say that von Neumann's mathematical model has the hidden variable inconsistency. What is the hidden variable inconsistency to be discussed in this paper? We cannot expand von Neumann's mathematical model more in handling real experimental data [12, 13]. Mathematically, von Neumann's model is logically consistent, which fact is true. However, von Neumann's theory is questionable in the sense that the mathematical model does not always expand to real experimental data. And there is the hidden variable inconsistency if we apply von Neumann's model to expanding even a simple physical situation. In short, von Neumann's mathematical model might not be useful in that case.

The hidden variable inconsistency to be discussed in this paper is significant. von Neumann's mathematical model has the qualification to be true axiomatic system for quantum mechanics. Therefore, we cannot modify the axioms based on the nature of Matrix theory. Nevertheless, we encounter the hidden variable inconsistency, probably due to the nature of Matrix theory, within von Neumann's theory, that is, his mathematical model is perfect but his physical thought might be a little bit questionable today.

Of course our analyses rely on von Neumann's model and the possible positions of the hidden variable inconsistency lie outside von Neumann's model. Thus we cannot sometimes expand von Neumann's model to experimental situations. In short, we might have to limit in some sense his model for quantum measurement theories.

On the other hand, the incompleteness argument to quantum mechanics itself is discussed by Einstein, Podolsky, and Rosen [14]. A hidden variable interpretation of quantum mechanics is a topic of research [3, 4] and the no hidden variable theorem is discussed by Bell, Kochen, and Specker [15, 16]. A strengthened Kochen–Specker theorem, i.e., the free will theorem is discussed by Conway and Kochen [17].

Recently, Nagata, Diep, and Nakamura discuss a novel inconsistency with hidden variable theories, using two symmetric measurements. They are independent of the order of measurements themselves [18].

In this paper, we propose no hidden variable theorem for the quantum measurement outcomes by measuring an observable that is obtained through a single projection operator. We are in the inconsistency with hidden variable theories when the first predetermined hidden result is $+1$ by measuring a single projection operator $|\uparrow\rangle\langle\uparrow|$ in the quantum state $|\uparrow\rangle$, the second predetermined hidden result is the same $+1$ as by measuring the same projection operator $|\uparrow\rangle\langle\uparrow|$ in the same quantum state $|\uparrow\rangle$. It turns out that we cannot assign the predetermined hidden result for quantum measurement outcome as $+1$ when measuring a single projection operator.

The essence of our deep desire to describe our honest view is a new insight into hidden variable theories for future studies over the academic world because we cannot assign the predetermined hidden result for quantum measurement outcome as +1 when measuring a single projection operator.

II. THE SUM RULE IS EQUIVALENT TO THE PRODUCT RULE FOR COMMUTING PROJECTION OPERATORS

We would discuss that the sum rule is equivalent to the product rule for commuting projection operators. First, we define the functional rule as follows:

$$f(g(O)) = g(f(O)), \tag{1}$$

where O is a Hermitian operator and f, g are appropriate functions to be used later. We introduce the above assumption (the functional rule) for hidden variable theories. We want to investigate if the algebraic structure of quantum mechanical operators could be mirrored in the algebraic structure of the predetermined hidden result of the operators. Second, the sum rule is defined as follows:

$$f(A_1 + A_2) = f(A_1) + f(A_2), \tag{2}$$

where A_1, A_2 are Hermitian operators. Finally, the product rule is defined as follows:

$$f(A_1 \cdot A_2) = f(A_1) \cdot f(A_2). \tag{3}$$

This fact above (the sum rule is equivalent to the product rule) is based on the property of these two Hermitian operators themselves. This leads to the propositions that they are valid even for the real numbers of the diagonal elements of the two Hermitian operators.

We have [4] the following relations between the three rules for commuting projection operators:

$$\begin{aligned}&\text{The functional rule} \Rightarrow \text{The sum rule}\\&\text{The functional rule} \Rightarrow \text{The product rule}\end{aligned} \tag{4}$$

For example, let us derive the sum rule and the product rule from the functional rule. Suppose now that A and B are two commuting Hermitian operators. Since A and B commute they can be diagonalized simultaneously. This means that there exists a basis $\{P_i\}$ by which we can expand $A = \sum_i a_i P_i$ and such that B can also be expanded in the form $B = \sum_i b_i P_i$. Now construct a Hermitian operator $O := \sum_i o_i P_i$ with real values o_i, which are all different. Here O is assumed to be nondegenerate by construction. Let us define respectively functions j and k by $j(o_i) := a_i$ and $k(o_i) := b_i$. Then we can see that if A and B commute, there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. Thus we have

$$\begin{aligned}&f(A \cdot B) = f(h(O)) = h(f(O)) = j(f(O)) \cdot k(f(O))\\&f(j(O)) \cdot f(k(O)) = f(A) \cdot f(B),\end{aligned} \tag{5}$$

where we use the functional rule. We can introduce also a function l such that $A + B = l(O)$ where $l := j + k$. Thus we have

$$\begin{aligned}&f(A + B) = f(l(O)) = l(f(O)) = j(f(O)) + k(f(O))\\&f(j(O)) + f(k(O)) = f(A) + f(B),\end{aligned} \tag{6}$$

where we use the functional rule. Additionally, we may introduce a function m, if the following conditions are satisfied, such that $m(l(O)) := h(O)$, $m(j(O)) := j(O)$, and $m(k(O)) := k(O)$. For example, we may have

$$A = B = |\uparrow\rangle\langle\uparrow| \equiv \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \tag{7}$$

and let us take the function m as $m(2) = 1$, $m(0) = 0$, and $m(1) = 1$. Thus we can introduce the function m that satisfies the requirement conditions. Then we have

$$\begin{aligned}&f(A) \cdot f(B) = f(A \cdot B) = f(h(O)) = f(m(l(O)))\\&= f(l(m(O))) = l(f(m(O)))\\&= j(f(m(O))) + k(f(m(O)))\\&= f(m(j(O))) + f(m(k(O)))\\&= f(A) + f(B) = f(A + B).\end{aligned} \tag{8}$$

In fact, the sum rule is equivalent to the product rule for commuting projection operators. Thus we may introduce a supposition that the operation Addition is equivalent to the operation Multiplication when $A = B = |\uparrow\rangle\langle\uparrow|$ and the quantum state is $|\uparrow\rangle$ (commuting projection operators).

III. AN INCONSISTENCY WITH HIDDEN VARIABLE THEORIES BASED ON A SINGLE PROJECTION OPERATOR

Let $P \equiv |\uparrow\rangle\langle\uparrow|$ be a single projection operator. Let $|\uparrow\rangle$ be an eigenstate of P such that $P|\uparrow\rangle = +1|\uparrow\rangle$. The predetermined hidden results for quantum measurement outcomes are only +1 in the ideal case.

We might be in the inconsistency with hidden variable theories when the first predetermined hidden result is +1 by measuring the projection operator P in the quantum state $|\uparrow\rangle$, the second predetermined hidden result is the same +1 as by measuring the same projection operator P in the same quantum state $|\uparrow\rangle$.

We consider a value V which is the sum of the two predetermined hidden results in a thought experiment. The predetermined hidden results of outcomes are only +1. If the number of outcomes is two, then we have

$$V = (+1) + (+1) = +2. \tag{9}$$

We derive a general and natural necessary condition of the value V.

On the other hand, we can depict the predetermined hidden results using v_1, v_2 as follows: $v_1 = +1$ and $v_2 = +1$. Let us write V as follows:

$$V = v_1 + v_2. \tag{10}$$

In the following, we may introduce a supposition that the operation Addition is equivalent to the operation Multiplication and we evaluate a value V and derive a specific and unnatural necessary condition.

Then, we have

$$V = (v_1 + v_2) = (v_1 \times v_2) = +1. \tag{11}$$

This is possible for the specific and unnatural case.

We cannot assign simultaneously the same two truth values ("1" and "1") or ("0" and "0") for the two propositions (9) and (11). We derive the inconsistency with hidden variable theories based on a single projection operator.

In summary, we have been in the inconsistency with hidden variable theories when the first predetermined hidden result is +1 by measuring the projection operator P in the quantum state $|\uparrow\rangle$, the second predetermined hidden result is the same +1 as by measuring the same projection operator P in the same quantum state $|\uparrow\rangle$.

IV. CONCLUSIONS

In conclusions, we have proposed no hidden variable theorem for the quantum measurement outcomes by measuring an observable that is obtained through a single projection operator. We have been in the inconsistency with hidden variable theories when the first predetermined hidden result is +1 by measuring a single projection operator $|\uparrow\rangle\langle\uparrow|$ in the quantum state $|\uparrow\rangle$, the second predetermined hidden result is the same +1 as by measuring the same projection operator $|\uparrow\rangle\langle\uparrow|$ in the same quantum state $|\uparrow\rangle$. It has turned out that we cannot assign the predetermined hidden result for quantum measurement outcome as +1 when measuring a single projection operator.

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The authors are in an applicable thought to ethical approval.

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The authors state that there is no conflict of interest.

Author contributions

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