On the Gravitational Entropy of Cantor Dust

Ervin Goldfain

Global Institute for Research, Education and Scholarship (GIRES), USA

E-mail ervingoldfain@gmail.com

Abstract

Cantor Dust (CD) is a hypothetical Dark Matter (DM) condensate of relic spacetime dimensions left over from the primordial Universe. A plausible assumption is that, in the complex dynamic regime of primordial cosmology, CD consists of continuous dimensional fluctuations acting as a *fractal mass distribution.* Starting from these premises, here we study the gravitational behavior of CD in the context of Rényi and Tsallis entropies. Our findings offer a fresh perspective on structure formation as a complex dynamical process.

Key words: Dark Matter, Cantor Dust, Rényi entropy, Tsallis entropy, cosmological structure formation, complex dynamics.

1. Structure Formation in Complex Dynamics Conditions

Traditional cosmological models are typically based on Gaussian random fields and linear perturbation theory. These models are bound to fail in *complex dynamic environments* for a variety of reasons:

- Structure formation is inherently *nonlinear*, involving gravitational collapse, void aggregation, and hierarchical clustering,
- Gravity is long-range, leading to *non-extensive energy exchange* and self-similarity across observation scales,
- Complex dynamics violates the assumptions of Boltzmann-Gibbs statistical physics, which applies to extensive thermodynamic systems having short-range interactions.

The so-called *generalized entropies* (Rényi and Tsallis) represent natural extensions of the entropy concept in equilibrium thermodynamics and are focused on:

- Fractal and multifractal cosmological structures (such as, but not limited to, the cosmic web),
- Non-Gaussian distributions of density fluctuations,
- Anomalous diffusion or Lévy flights in several representative DM models.

Recent years have shown that DM clustering is, in general, a long-range phenomenon whose thermodynamic description must follow the mathematics of *non-extensive statistics*, where total entropy is *not* equal to the sum of component entropies. In support of this view, there are several scenarios favoring the use of generalized entropies, for example,

- DM halos show power-law density profiles and self-similar substructure,
- N-body simulations of DM show Tsallis-like distributions in halo phase-space densities,

- Tsallis-based velocity distributions fit observed velocity dispersions of galactic dynamics better than Maxwellian distributions,
- Tsallis entropy predicts negative heat capacities and core–halo structures, found in simulations of self-gravitating systems.
- Rényi entropy is helpful in analyzing the anisotropies of the cosmic microwave background (CMB) and the multiscale texture of large-scale cosmic structures.

The paper is organized as follows: the next section delves into the gravitational field of fractal distributions of particles. Elaborating on this model, the time-dependent dimensional fluctuations $\varepsilon(t)=3-D(t)$ of the CD are interpreted as a *fractal mass distribution.* The entropic behavior of this distribution is subsequently studied in the context of Rényi and Tsallis entropies.

We caution upfront that our analysis is tentative in nature and needs to be further scrutinized, debunked or developed through independent evaluation.

2. Gravitational field in continuous dimensions

According to [1], the gravitational field of a homogeneous and spherically symmetric fractal distribution of particles in D spatial dimensions is given by

$$g_D(R) \propto \rho_0 G_N R^{D-2} \tag{1}$$

Here, ρ_0 represents the constant mass density, G_N is Newton's constant and R the radius of the mass distribution. To streamline the derivation, we proceed with the following set of simplifying assumptions:

A1) Dimensional fluctuations $\delta\varepsilon \propto \varepsilon \propto m$ in which $\varepsilon = 3 - D = O(m^2/\Lambda_{UV}^2)$ form a statistical ensemble of *pseudo-particles* of mass m. Here, Λ_{UV} stands for the ultraviolet cutoff matching the magnitude of the Planck scale, i.e., $\Lambda_{UV} \propto M_{Pl}$.

A2) We take the mass density and Newton's constant to be *scale-independent parameters*, that is, $\rho_0 \neq \rho_0(\varepsilon)$ and $G_N \neq G_N(\varepsilon)$.

A3) The radial coordinate R is expressed in dimensionless form and it is commensurate with the magnitude of the corresponding Compton wavelength, that is, $R \propto O(\Lambda_{UV}/m)$. For reference, note that an ultralight boson mass of 10^{-22} eV corresponds to a Compton wavelength of 2×10^{15} m, indicating matter clustering on large cosmic scales.

Under A1)-A3), the gravitational field (1) in $3-\varepsilon$ dimensions turns into

$$\boxed{g(\varepsilon) \propto \frac{\varepsilon^{\varepsilon/2}}{\sqrt{\varepsilon}} = \varepsilon^{(\varepsilon-1)/2}} \tag{2}$$

whose graph is plotted in Fig. 1. The interaction strength of the fractal mass distribution diverges at $\varepsilon = 0$, drops fast and reaches a minimum at $\varepsilon = 0.278$ then grows strong again as ε approaches and goes past $\varepsilon = 1$.

The natural interpretation of this behavior is that, depending on the flow of dimensional parameter ε, a spherical CD mass distribution reproduces *either the effects of Strongly Interacting Dark Matter (SIDM) or of Weakly Interacting Dark Matter (WIDM).*

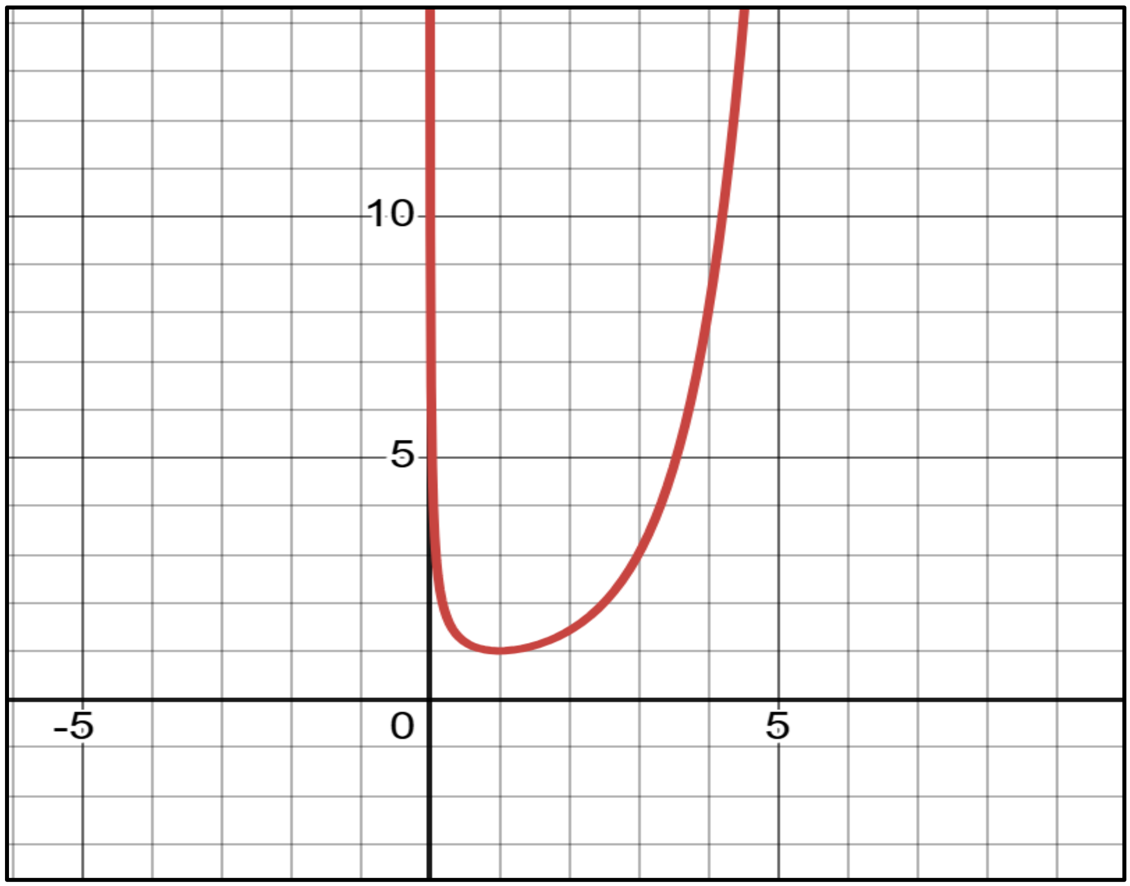

Fig. 1: Gravitational field (2) as a function of $\varepsilon = 3 - D$ dimensions

3. Entropy of CD in Evolving Spacetime Dimensions

As previously pointed out, Rényi and Tsallis entropies are generalized measures of entropy that extend the classical Boltzmann–Gibbs entropy to systems exhibiting *non-extensivity*, *long-range correlations*, *multifractal behavior*, or *anomalous diffusion* [2-3]. Per the summary table below, these features are highly relevant in cosmology and DM physics, where traditional thermodynamic assumptions often break down.

Summary Table

Feature	Why Renyi/Tsallis Matter
Long-range interactions	Capture non-extensive thermodynamics in gravitational systems
Multifractal cosmic web	Describe scaling, hierarchy, and clustering beyond BGS entropy
Dark Matter dynamics	Model velocity distributions and density profiles with q-statistics
Early Universe thermodynamics	Account for anomalous entropy production during reheating or inflation
Black hole entropy	Incorporate corrections to Bekenstein-Hawking via generalized entropies
Modified gravity models	Support emergent gravity frameworks with entropy-based forces

Tab. 1: Relevance of generalized entropies in cosmology and DM physics

Generalized entropies take the following form,

$$S_q^{(T)} = \frac{1-\sum_i p_i^q}{q-1} \qquad \text{(Tsallis)} \qquad (3)$$

$$S_q^{(R)} = \frac{1}{1-q}\log(\sum_i p_i^q) \qquad \text{(Rényi)} \tag{4}$$

where $q \in R$ is the non-extensivity parameter (Shannon entropy corresponds to $q = 1$) and p_i are probabilities normalized as in

$$p_i^{norm} = \frac{p_i}{\sum_{j=1}^{N} p_j} \tag{5}$$

with

$$\sum_{j=1}^{N} p_j^{norm} = 1 \tag{6}$$

Following the remarks detailed in the first section and the beginning of this section, we proceed by assuming that a) the gravitational field (2) takes on the role of a *probability*, and b) the dimensional deviation ε evolves in time towards its classical value $\varepsilon = 0$. For a statistical ensemble of gravitational fields $i = 1, 2, ..., N$, these assumptions amount to,

$$p_i(t) \propto (\varepsilon_i(t))^{\frac{\varepsilon_i(t)-1}{2}} \qquad (7)$$

where, mimicking structure formation and DM clustering, the dimensional deviation drops in time according to,

$$\varepsilon_i(t) = \frac{\varepsilon_i}{t^{\beta}}, \qquad \beta > 0 \qquad (8)$$

Based on (7) and (8), figs 2 – 6 below illustrate, respectively, the time evolution of generalized entropies, entropy behavior vs. β, entropy behavior vs. q, spectrum of multifractal dimensions D_q over q and the singularity spectrum $f(\alpha)$ computed from the Rényi multifractal dimensions.

These plots support the idea that generalized entropies are sensitive to the structure of the distribution and the scaling dynamics, making them suitable for studying systems like DM/CD clustering and cosmological structure formation. Also note that the dimension spectra D_q quantifies how entropy and information are distributed across scales and probabilities—which is a helpful tool for characterizing complex, scale-invariant systems such as the cosmic web, DM halos, or voids.

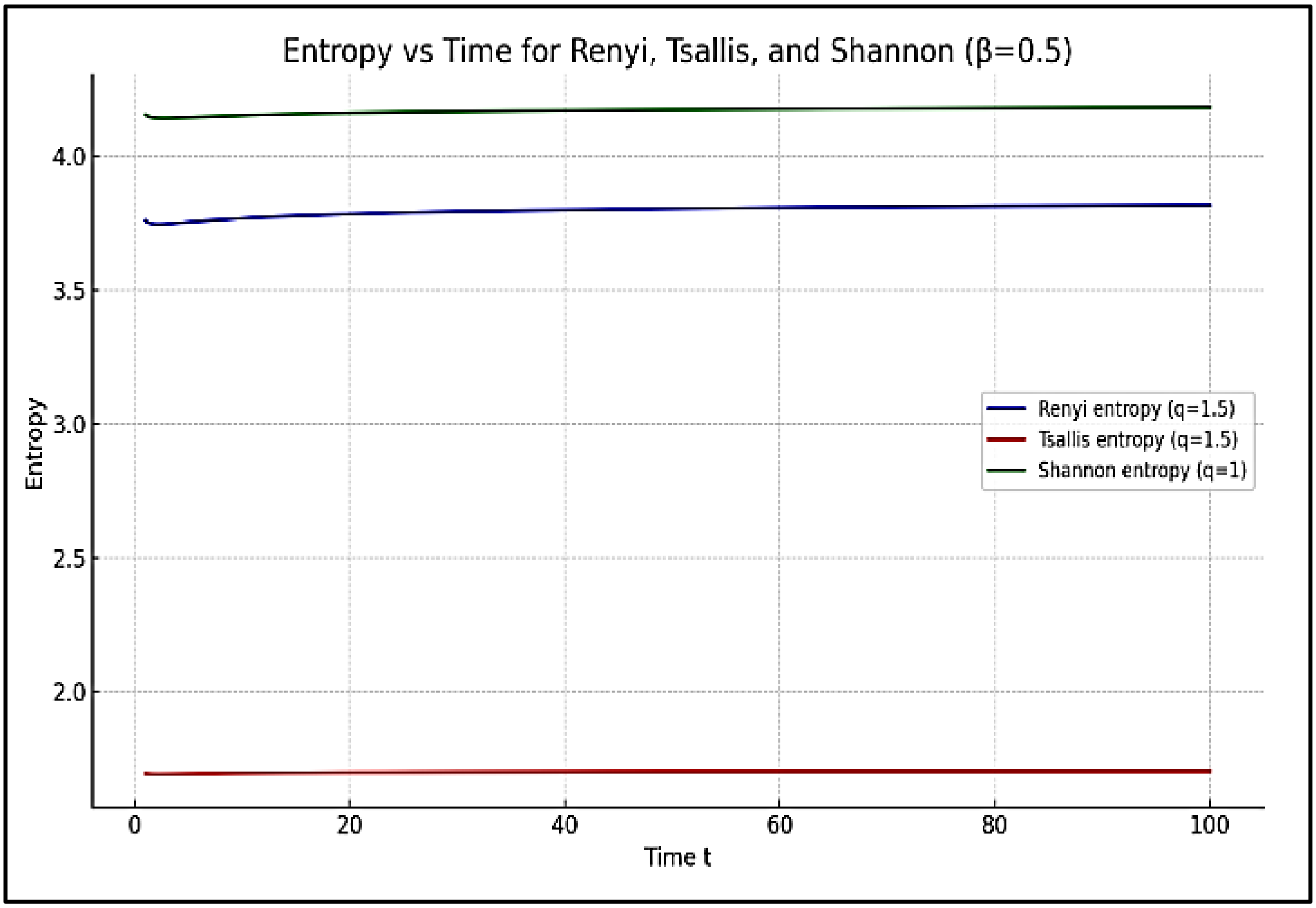

Fig. 2: Entropy evolution over time for $\beta = 0.5$

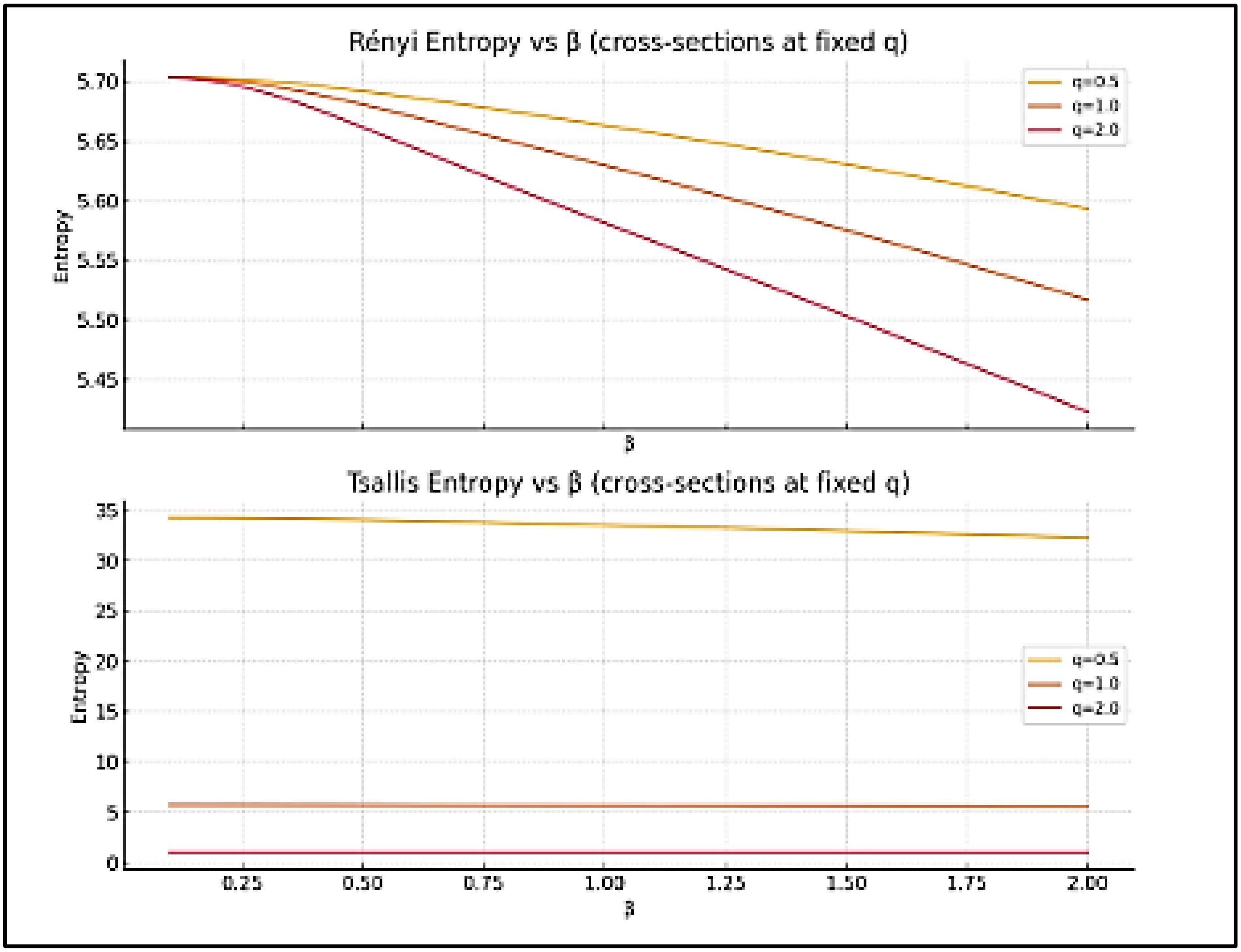

Fig. 3: Entropy behavior vs. β for various index parameters q

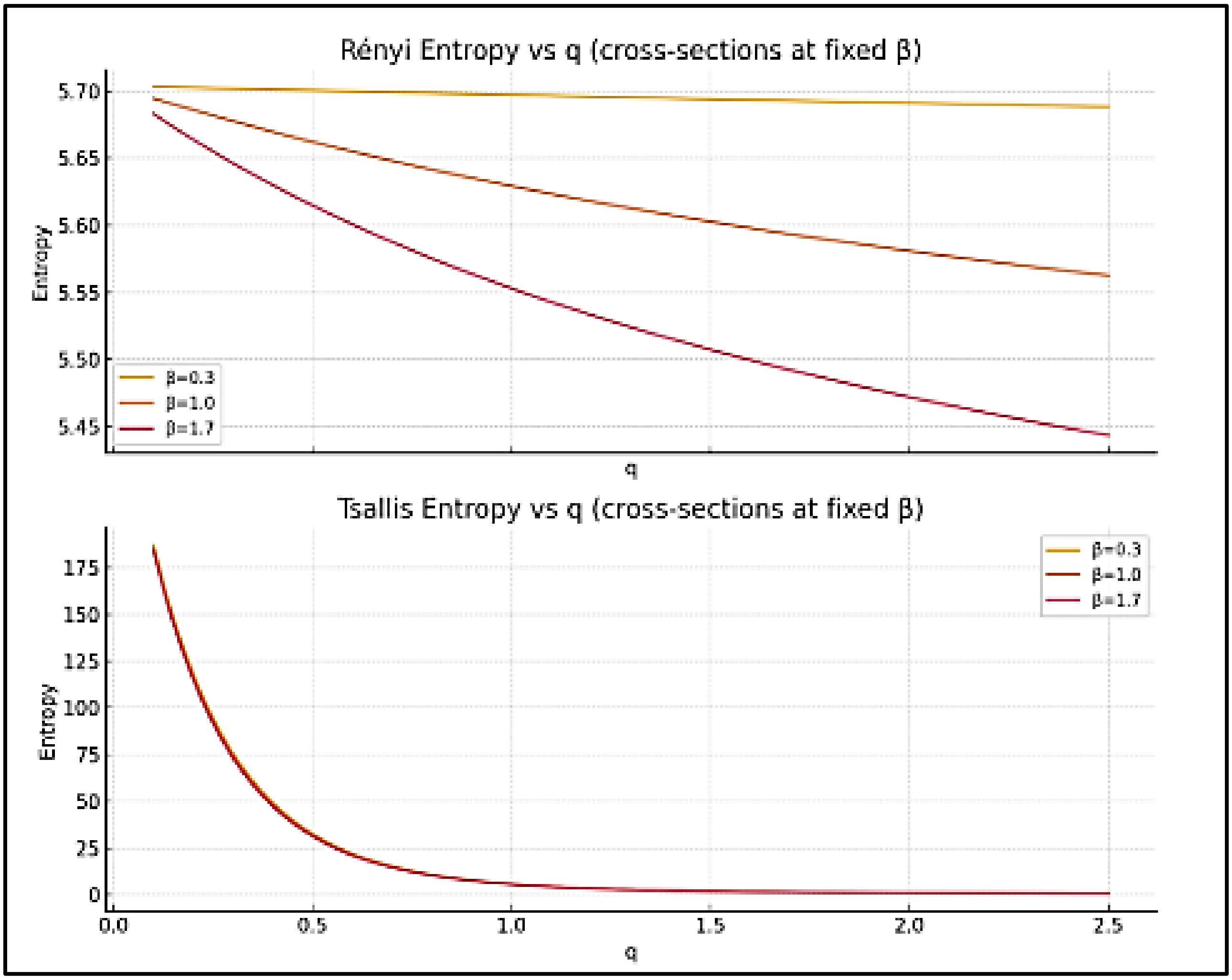

Fig. 4: Entropy behavior vs. q for various values of β

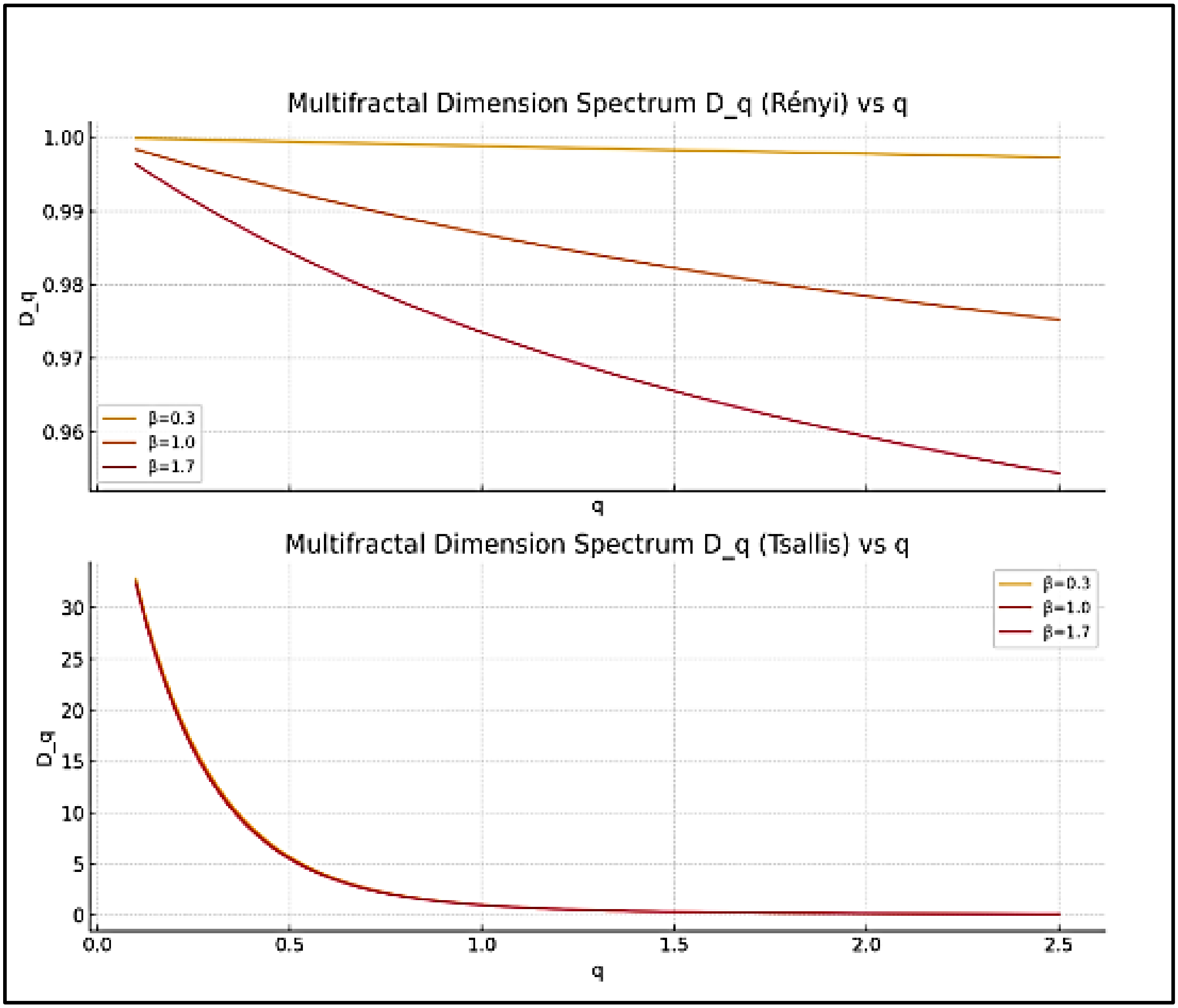

Fig. 5: Multifractal dimension spectrum D_q for various β

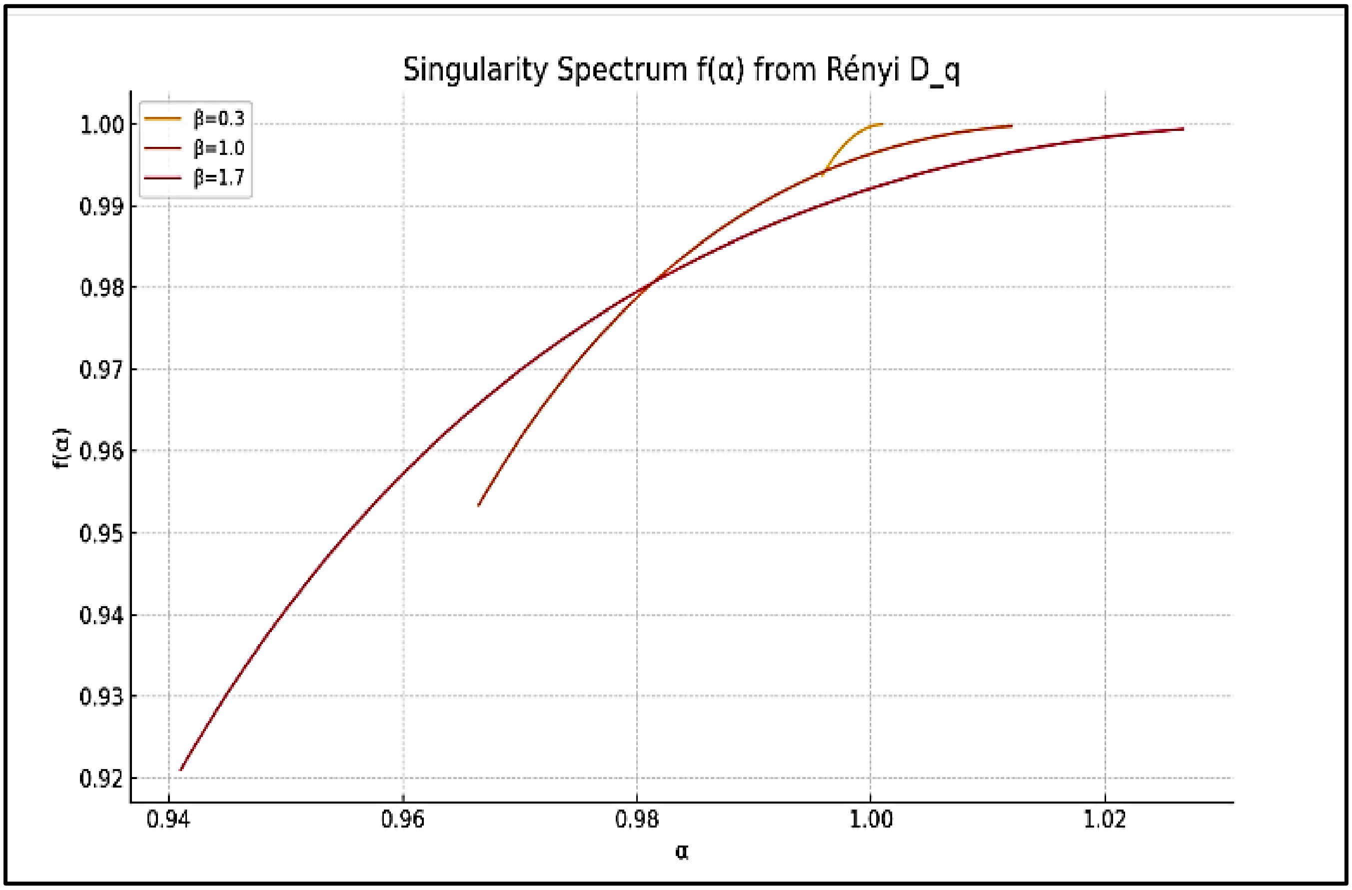

Fig. 6: Singularity spectrum computed from Rényi multifractal dimensions

The boost of generalized entropy in a cosmological context means *less homogeneity (higher heterogeneity)* and enhanced clumping/clustering. The summary table below highlights the meaning of monotonic entropy growth in cosmological structure formation.

Summary: What Does Monotonic Entropy Growth Mean in Cosmology?

Aspect	Interpretation
Entropy ↑	Emergence of cosmic structure, increase in gravitational clumping
$\varepsilon(t)$ ↓	Phase-space becomes more fine-grained or multifractal
Dark Matter	Clustering, virialization, halo formation, substructure
Multifractal view	IR evolution, complexity across scales, dimensional flow
Entropy formula	Encodes phase-space complexity and evolving probability geometry

Tab. 2: Consequences of monotonic entropy growth in cosmology

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