Mathematics for Incompletely Predictable Problems with Input-White Box-Output Modeling for Primes, Composites and Nontrivial zeros of Riemann zeta function

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Abstract. Utilizing Input-White Box-Output (I-WB-O) Modeling, we outline novel applications of Mathematics for Incompletely Predictable Problems (MIPP) to unique sets and subsets from prime and composite numbers, nontrivial zeros of Riemann zeta function, and selected number sequences from The On-Line Encyclopedia of Integer Sequences (OEIS). We show that MIPP is valid for mathematical functions, equations or algorithms containing an equality relationship between two expressions. When applied to OEIS number sequence A228186, MIPP is also valid for an inequality. Inclusion-exclusion (I-E) principle from combinatorics removes contributions from over-counted elements in sets and subsets. Invoking I-E principle, arising consequences from MIPP formulations containing I-WB-O Models provide necessary mathematical arguments for rigorously solving open problems Riemann hypothesis, Polignac's and Twin prime conjectures.

Contents

2010 *Mathematics Subject Classification.* 11M26, 11A41.

Key words and phrases. A228186, Birch and Swinnerton-Dyer conjecture, Carmichael numbers, Polignac's and Twin Prime conjectures, Restricted Primes, Riemann hypothesis.

Dedicated to the author's premature daughter Jelena born 13 weeks early on May 14, 2012. He contributes to all the work in this research paper. *Data Availability Statement*: All data supporting the findings of this study are available in this paper. *Ethical approval*: Not applicable. *Conflict of Interest Statement*: There is no conflict of interest.

Declarations on Funding: AUS $5,000 research grant was provided by Mrs. Connie Hayes and Mr. Colin Webb on January 20, 2020. An extra AUS $3,250 reimbursement was received from Q-Pharm on March 10, 2020. *Interesting Doctor of Philosophy (PhD) viewpoints*: With his level of working, short-term and long-term memory, the author possesses average level of *Concrete Mathematics* ability. Developmentally, he possesses above-average level of *Abstract Mathematics* ability.

1. Introduction

Equations and inequalities are mathematical sentences formed by relating two expressions to each other. In an equation, two expressions are deemed equal as indicated by symbol "=" [viz, an equation contains equality relationship]. Mathematics for Incompletely Predictable Problems (MIPP) is valid for various chosen functions, equations or algorithms that contain equality relationship[6]. The eligible functions or algorithms are literally quasi-equations containing this "analogical" equality relationship. E.g.: Origin point intercepts ≡ Gram[x=0, y=0] points in Analytically continued *proxy* Dirichlet eta function = {Set of All Nontrivial zeros in Riemann zeta function}. Algorithm *Sieve-of-Eratosthenes* ≡ All Integers greater than 1 with exactly two factors, 1 and the number itself = {Set of All Prime numbers}.

In an inequality, two expressions are not necessarily equal as indicated by symbols ">", "<", "≤" or "≥". As deductively shown in section 2 using proven mathematical arguments, MIPP is also valid for selected number sequences A100967 and A228186 from The On-Line Encyclopedia of Integer Sequences (OEIS) that are precisely defined by inequalities.

We advocate *Input* → $\boxed{Box}$ → *Output* Modeling should result in two descriptive types whereby the "→" must in general be replaced by the "⇌" to indicate bidirectional reversibility: Input-White Box-Output (I-WB-O) Model and Input-Black Box-Output (I-BB-O) Model. I-BB-O Model simply refers to I-WB-O Model when its "White Box" is unknown [not explicitly specified], which is traditionally labeled as "Black Box".

Basic questions regarding Riemann hypothesis, Polignac's and Twin prime conjectures, and Birch and Swinnerton-Dyer conjecture are *easy to state but difficult to resolve or reconcile.* Isaac Newton in 1675 wrote the expression: "If I have seen further [than others], it is by *standing on the shoulders of Giants* (Latin: *nani gigantum humeris insidentes*)". This famous metaphor meant *discovering truth by building on previous discoveries.* The inclusion-exclusion (I-E) principle from combinatorics remove all contributions from over-counted elements in sets and subsets. By appropriately invoking I-E principle, arising consequences from MIPP formulations using I-WB-O

Models provide all necessary correct and complete mathematical arguments for rigorously proving these intractable open problems in Number theory.

2. Hybrid integer sequence A228186, Prime-Composite identifier grouping and Inclusion-Exclusion principle

Binomial coefficients $\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2\cdot 1}$ $= \prod_{\ell=1}^{k} \frac{n-\ell+1}{\ell} = \prod_{\ell=0}^{k-1} \frac{n-\ell}{k-\ell}$. Despite having k factors in both numerator and denominator of the fraction, it is actually an integer. Denoted by binomial (n,k), it is equivalent to combination [order does not matter and *without repetition*] ${}^{n}C_k = C(n,k)$, which represents the number of ways to choose k items from a set of n distinct items such that the order of selection does not matter. Combination [order does not matter and *with repetition*] is given by $\frac{(k+n-1)!}{k!(n-1)!}$. Permutation [order matters and *without repetition*] ${}^{n}P_k = P(n,k) = \underbrace{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}_{k \text{ factors}}$ is 0 when $k > n$, and otherwise is equal to $\frac{n!}{(n-k)!}$. It represents the number of ways to choose k items from a set of n distinct items such that the order of selection does matter. Permutation [order matters and *with repetition*] is given by $n\times n\times n\times \cdots (k \text{ times}) = n^k$. We obtain $C(n,k) = \frac{P(n,k)}{P(k,k)} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{(n-k)!\,k!}$. The number of permutations are always greater than the number of combinations.

Paired [infinite-length] integer sequences with prestigious connections:

A100967+0, which is A100967[4], is precisely defined as "Least k such that binomial$(2k+1, k-n-1) \geq$ binomial$(2k,k)$ viz. $(2k+1)!k!k! \geq (2k)!(k-n-1)!(k+n+2)!$". The [infinite-length] terms commencing from Position n = 1, 2, 3, 4, 5,... of A100967+0 are 3, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 225, 263, 303, 347, 393, 442, 494, 549, 606, 667, 730, 797, 866, 938, 1013, 1091, 1172, 1255, 1342, 1431, 1524, 1619, 1717, 1818, 1922, 2029, 2138, 2251, 2366, 2485, 2606, 2730, 2857, 2987, 3119, 3255,....

A100967+1 is conveniently defined as "Add 1 to each and every terms from A100967+0". The [infinite-length] terms commencing from Position n = 1, 2, 3, 4, 5,... of A100967+1 are 4, 10, 19, 30, 45, 62, 82, 105, 131, 160, 192, 226, 264, 304, 348, 394, 443, 495, 550, 607, 668, 731, 798, 867, 939, 1014, 1092, 1173, 1256, 1343, 1432, 1525, 1620, 1718, 1819, 1923, 2030, 2139, 2252, 2367, 2486, 2607, 2731, 2858, 2988, 3120, 3256,....

A228186[5] is precisely defined as "Smallest natural number k such that $(k+n^*+1)!(k-n^*-2)! < 2k!(k-1)!$". It is alternatively defined as "Greatest natural number $k > n^*$ such that calculated peak values for ratio $R =$

$\dfrac{\text{Combinations With Repetition}}{\text{Combinations Without Repetition}} = \dfrac{(k+n^*-1)!(n^*-k)!}{n^*!(n^*-1)!}$ belong to the maximal rational numbers < 2". (Offset) [infinite-length] terms commencing from Position $n^* = 0$, 1, 2, 3, 4, 5,... of A228186 are 4, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 226, 263, 304, 347, 393, 442, 494, 549, 607, 667, 731, 797, 866, 938, 1013, 1091, 1172, 1256, 1342, 1432, 1524, 1619, 1717, 1818, 1922, 2029, 2139, 2251, 2367, 2485, 2606, 2730, 2857, 2987, 3120, 3255,....

As a mathematical curiosity using notation $n^* = n - 1 = 0$, 1, 2, 3, 4, 5... [where $n = n^*{+}1 = 1$, 2, 3, 4, 5...] and abbreviation CIS = Countably Infinite Set and CFS = Countably Finite Set; A228186 is an innovative [infinite-length ≡ CIS] "Hybrid integer sequence" that is identical to [infinite-length ≡ CIS] "non-Hybrid integer sequence" A100967+0 except for the interspersed [finite-length ≡ CFS] 21 'exceptional' terms located at Position $n^* = 0$, 11, 13, 19, 21, 28, 30, 37, 39, 45, 50, 51, 52, 55, 57, 62, 66, 70, 73, 77, and 81 with their associated 21 values exactly specified by "non-Hybrid integer sequence" A100967+1 at [corresponding] Position $n = 1$, 12, 14, 20, 22, 29, 31, 38, 40, 46, 51, 53, 56, 58, 63, 67, 71, 74, 78, and 82.

Definition 2.1. With "Entity X" forming a Countably Infinite Set and irrespective of whether "Entity X" are Completely or Incompletely Predictable entities, we consistently define "n^{th} Gap of Entity X" = "$(n+1)^{th}$ Entity X" − "$(n)^{th}$ Entity X". This [locational] definition is usually designated for Position $n = 1$, 2, 3, 4, 5... e.g. n^{th} Prime Gap = $(n+1)^{th}$ Prime number − $(n)^{th}$ Prime number with using Position $n = 1$, 2, 3, 4, 5... [where we arbitrarily denote *small Prime gaps* to be 2 and 4, and *large Prime gaps* to be ≥ 6]. This definition is equally valid when designated for Position $n^* =$ 0, 1, 2, 3, 4, 5... [where $n^* = n - 1$ or $n = n^* + 1$].

Then for $n^* = 0$, 1, 2, 3, 4, 5...; the formulation is there exist infinitely-many n^{*th} A228186 Gaps {5, 9, 11, 15, 17, 20, 23, 30, 29, 32, 35, 37, 41, 43, 46, 45, 52, 55, 58, 60, 64, 66, 69, 72, 75, 78, 81, 84, 86, 90, 92, 95, 98, 101, 104, 107, 110, 112, 116, 118, 121, 124, 127, 130, 133, 135, 139, 141, 144, 148, 150, 153,...}. Our n^{*th} A228186 Gaps are Incompletely Predictable entities seen to generally manifest fluctuating "cyclical" behavior and progressively increase in an unpredictable constant "linear" manner. E.g., when given a randomly selected $k = 14572$ value [an "unknown" $(n)^{*th}$ A228186 term that satisfies the inequality in A228186], we can only obtain its correct Position $n^* = 99$ by determining all preceding $n^* = 0$, 1, 2, 3, 4, 5,..., 97, 98 [total = 99] values that are associated with their corresponding k [total = 99] values.

For $n^* = 0$, 1, 2, 3, 4, 5... ≡ *Input*; we obtain infinite-length OEIS number sequence A228186 [based on an inequality ≡ *White Box*] with its $(n)^{*th}$ A228186 terms ≡ *Output* and associated n^{*th} A228186 Gaps ≡ *Output* that comply with both I-WB-O Modeling and MIPP from section 3.

Definition 2.2. Formal definition for *Prime-Composite identifier grouping*: We notationally use both i and $n = 1$, 2, 3, 4, 5,... to avoid ambiguity: Let E = even numbers, O = odd numbers, $\mathbb{P}$ = prime numbers, $\mathbb{C}$ = composite numbers, even Prime gap$_i$ = O-$\mathbb{P}_{i+1}$ − O-$\mathbb{P}_i$ = 2, 4, 6, 8, 10, 12..., Composite gap$_i$ = $\mathbb{C}_{i+1} - \mathbb{C}_i$ = 1, 2. For even Prime gaps 4, 6, 8, 10, 12..., we can generate the orderly consecutive numbers as sequence {Gap 2-E-$\mathbb{C}_1$, O-$\mathbb{P}_i$, Gap 1-E-$\mathbb{C}_2$, Gap 1-O-$\mathbb{C}_3$, Gap 1-E-$\mathbb{C}_4$, Gap 1-O-$\mathbb{C}_5$,..., Gap 1-E-$\mathbb{C}_{n-2}$, Gap 1-O-$\mathbb{C}_{n-1}$, Gap 2-E-$\mathbb{C}_n$, O-$\mathbb{P}_{i+1}$}. The cardinality of sub-sequence {Gap 1-E-$\mathbb{C}_2$, Gap 1-O-$\mathbb{C}_3$, Gap 1-E-$\mathbb{C}_4$, Gap 1-O-$\mathbb{C}_5$,..., Gap 1-E-$\mathbb{C}_{n-2}$, Gap 1-O-$\mathbb{C}_{n-1}$} = even Prime gap$_i$ − 2 = $n - 2$. However for twin primes; this sub-sequence [as an empty set or null set] do not exist with its cardinality = 0 since even Prime gap $2 - 2 = 0$. With cardinality of this sub-sequence given by the involved even Prime gap minus 2; we conveniently define $\mathbb{P}$-$\mathbb{C}$ **identifier grouping**[6] as {Gap 2-E-$\mathbb{C}_1$, O-$\mathbb{P}_i$, Gap 1-E-$\mathbb{C}_2$, Gap 1-O-$\mathbb{C}_3$, Gap 1-E-$\mathbb{C}_4$, Gap 1-O-$\mathbb{C}_5$,..., Gap 1-E-$\mathbb{C}_{n-2}$, Gap 1-O-$\mathbb{C}_{n-1}$} for Arbitrarily Large Number of even Prime gaps 4, 6, 8, 10, 12... with caveat $\mathbb{P}$-$\mathbb{C}$ identifier grouping for even Prime gap 2 is an exception given by Gap 2-E-$\mathbb{C}_1$, O-$\mathbb{P}_i$.

The [decelerating] size of equally distributed Gap $2n$-O-$\mathbb{P}$ and Gap 2-E-$\mathbb{C}$ is "inversely proportional" to [accelerating] size of equally distributed Gap 1-E-$\mathbb{C}$ and Gap 1-O-$\mathbb{C}$. Gap 2-E-$\mathbb{C}_n$ is now acting as the new Gap 2-E-$\mathbb{C}_1$ for O-$\mathbb{P}_{i+1}$ in the following perpetually repeating cycles of O-$\mathbb{P}_i$ to O-$\mathbb{P}_{i+1}$ with a [usually] different even Prime gap$_i$ [except for rare recurring cases of two or more consecutive O-$\mathbb{P}$ having two or more identical consecutive even Prime gaps involving 6 and multiples of 6].

Abbreviations: CFS = Countably Finite Set, CIS = Countably Infinite Set, UIS = Uncountably Infinite Set. From section 3 on associating *"thin set"* with "decelerating CIS" and *"thick set"* with "accelerating CIS", we provide the following insightful deductions when analyzing $\mathbb{P}$-$\mathbb{C}$ identifier grouping. **Subset** {Gap 2-E-$\mathbb{C}$} and **Subsets** {Gap $2n$-O-$\mathbb{P}$} are identical "thin sets" and "decelerating CIS". With even Prime gaps {2, 4, 6, 8, 10,...} = (**cardinality** {Gap 1-E-$\mathbb{C}$} + **cardinality** {Gap 1-O-$\mathbb{C}$} + 2) and whereby **cardinality** {Gap 1-E-$\mathbb{C}$} [as null set] = **cardinality** {Gap 1-O-$\mathbb{C}$} [as null set] = 0 in even Prime gap = 2 for Twin primes; the combined **Subsets** {Gap 1-E-$\mathbb{C}$ + Gap 1-O-$\mathbb{C}$} are "thick sets" and "accelerating CIS". However as two CIS with exact same cardinality, **Subsets** {Gap 1-E-$\mathbb{C}$} and **Subsets** {Gap 1-O-$\mathbb{C}$} in isolation by themselves are neither "thin sets" nor "thick sets". There must be complete presence of both **Subsets** {Gap 1-E-$\mathbb{C}$} + **Subsets** {Gap 1-O-$\mathbb{C}$} as a [combined] "thick set" that contain n = 0, 2, 4, 6, 8, 10,... elements [≡ even Prime gaps $(n + 2)$ = 2, 4, 6, 8, 10, 12,...] with [combined] cardinality "accelerating CIS" $\implies$ Polignac's and Twin prime conjectures are true. Obeying Addition-Subtraction Laws of even ± even = even; even ± odd = odd; and odd ± odd = even and Multiplication Laws of even × even = even; even × odd = even; and odd × odd = odd for all n = 0, 2, 4, 6, 8, 10,... elements ≡ all even Prime gaps $(n + 2)$ = 2, 4, 6, 8, 10, 12,...: Corresponding consecutive $\sum n$ terms as $\frac{n}{2}$ Even numbers+$\frac{n}{2}$ Odd numbers = [alternating] even, odd, even, odd, even, odd,...; Corresponding consecutive $\prod n$ terms as $\frac{n}{2}$ Even numbers$\times\frac{n}{2}$ Odd numbers = [same] even, even, even, even, even, even,...

Synopsis on Product (Multiplication) of Integers and Complex numbers: An integer can be either zero, a nonzero natural number, or minus a nonzero natural number. The product of zero and another integer is always zero. The product of two nonzero integers is determined by product of their positive amounts, combined with the sign derived from following rule [which is a consequence of distributivity of multiplication over addition, and is not an additional rule]: A +ve number multiplied by a +ve number is +ve. A +ve number multiplied by a −ve number is −ve. A −ve number multiplied by a +ve number is −ve. A −ve number multiplied by a −ve number is +ve.

Rule for Product of two complex numbers is that two complex numbers can be multiplied by the distributive law and the fact that $i^2 = -1$:
$(a + bi) \cdot (c + di) = a \cdot c + a \cdot di + b\, i \cdot c + b \cdot d \cdot i^2 = (a \cdot c - b \cdot d) + (a \cdot d + b \cdot c)i$

Geometrically, complex multiplication is understood by rewriting complex numbers in polar coordinates: $a + b\,i = r \cdot (\cos(\varphi) + i \sin(\varphi)) = r \cdot e^{i\varphi}$
Furthermore, $c + d\,i = s \cdot (\cos(\psi) + i \sin(\psi)) = s \cdot e^{i\psi}$, from which one obtains $(a \cdot c - b \cdot d) + (a \cdot d + b \cdot c)i = r \cdot s \cdot e^{i(\varphi+\psi)}$. The geometric meaning is that the magnitudes are multiplied and the arguments are added.

Constructing two unique Subsets of Odd Primes that have ***'Even Parity'*** *or* ***'Odd Parity'*** *resulting in "Theorem of Nil Predilection for Even-Odd Parity in Odd Primes associated with Carmichael numbers":*

For n = 0, 2, 4, 6, 8, 10,...; Addition of consecutive $\frac{n}{2}$ E-$\mathbb{C}$ and $\frac{n}{2}$ O-$\mathbb{C}$ is dependent on n giving alternating even numbers manifesting "Even Parity" and odd numbers manifesting "Odd Parity" [but their Multiplication is not dependent on n]. For i = 1, 2, 3, 4, 5,...; Set of all Odd Primes are derived from Set of all even Prime gaps $2i$ = {2, 4, 6, 8, 10,...} whereby we classify Subset of Odd Primes derived from even Prime gaps $(4i - 2)$ = {2, 6, 10, 14, 18, 22, 26,...} to have *"Even Parity (P)"* and Subset of Odd Primes derived from even Prime gaps $(4i)$ = {4, 8, 12, 16, 20, 24, 28,...} to have *"Odd Parity (P)"*. Consecutive Odd Primes {3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41...} with Prime Gaps {2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2...} manifest {"Even P", "Even P", "Odd P", "Even P", "Odd P", "Even P", "Odd P", "Even P", "Even P", "Even P", "Odd P", "Even P"...} as, theoretically, ∼50% "Even P" and ∼50% "Odd P".

Theorem 1. *Theorem of Nil Predilection for Even-Odd Parity in Odd Primes associated with Carmichael numbers.*

Proof. A cyclic number is a natural number n such that n and $\varphi(n)$ are coprime. Here φ is Euler's totient function. An equivalent definition is that a number n is cyclic if and only if any group of order n is cyclic. Any prime number is clearly cyclic. All cyclic numbers are square-free. Let $n = p_1 p_2 ... p_k$ where the p_i are distinct primes, then $\varphi(n) = (p_1 - 1)(p_2 - 1)...(p_k - 1)$. If no p_i divides any $(p_j - 1)$, then n and $\varphi(n)$ have no common (prime) divisor, and n is cyclic. The first cyclic numbers are 1, 2, 3, 5, 7, 11, 13, 15, 17, 19, 23, 29, 31, 33, 35, 37, 41, 43, 47, 51, 53, 59, 61, 65, 67, 69, 71, 73, 77, 79, 83, 85, 87, 89, 91, 95, 97, 101, 103, 107, 109, 113, 115, 119, 123, 127, 131,....

Alternative equivalent definition of Carmichael numbers is the Korselt's criterion: *Theorem by Alwin Reinhold Korselt in 1899* "A positive composite integer n is a Carmichael number if and only if n is square-free, and for all prime divisors p of n, it is true that $p-1 \mid n-1$." It follows from this theorem all Carmichael numbers are odd, since any even composite number that is square-free (and hence has only one prime factor of two) will have at least one odd prime factor, and thus $p-1 \mid n-1$ results in an even dividing an odd, a contradiction. (The oddness of Carmichael numbers also follows from the fact -1 is a Fermat witness for any even composite number.) From the criterion it also follows Carmichael numbers are cyclic. Additionally, it follows there are no Carmichael numbers with exactly two prime divisors.

First 35 Carmichael numbers with their locations specified by Odd Primes with Prime gaps manifesting either "Even Parity (P)" or "Odd Parity (P)": 561 [⇔ Prime 557 Gap 6 "Even P"], 1105 [⇔ Prime 1103 Gap 6 "Even P"], 1729 [⇔ Prime 1723 Gap 10 "Even P"], 2465 [⇔ Prime 2459 Gap 8 "Odd P"], 2821 [⇔ Prime 2819 Gap 14 "Even P"], 6601 [⇔ Prime 6599 Gap 8 "Odd P"], 8911 [⇔ Prime 8893 Gap 30 "Even P"], 10585 [⇔ Prime 10567 Gap 22 "Even P"], 15841 [⇔ Prime 15823 Gap 36 "Odd P"], 29341 [⇔ Prime 29339 Gap 8 "Odd P"], 41041 [⇔ Prime 41039 Gap 8 "Odd P"], 46657 [⇔ Prime 46649 Gap 14 "Even P"], 52633 [⇔ Prime 52631 Gap 8 "Odd P"], 62745 [⇔ Prime 62743 Gap 10 "Even P"], 63973 [⇔ Prime 63949 Gap 28 "Odd P"], 75361 [⇔ Prime 75353 Gap 14 "Even P"], 101101 [⇔ Prime 101089 Gap 18 "Even P"], 115921 [⇔ Prime 115903 Gap 28 "Odd P"], 126217 [⇔ Prime 126211 Gap 12 "Odd P"], 162401 [⇔ Prime 162391 Gap 22 "Even P"], 172081 [⇔ Prime 159899 Gap 12 "Odd P"], 188461 [⇔ Prime 188459 Gap 14 "Even P"], 252601 [⇔ Prime 252589 Gap 18 "Even P"], 278545 [⇔ Prime 278543 Gap 6 "Even P"], 294409 [⇔ Prime 294403 Gap 28 "Odd P"], 314821 [⇔ Prime 314813 Gap 14 "Even P"], 334153 [⇔ Prime 334133 Gap 24 "Odd P"], 340561 [⇔ Prime 340559 Gap 14 "Even P"], 399001 [⇔ Prime 398989 Gap 34 "Even P"], 410041 [⇔ Prime 410029 Gap 34 "Even P"], 449065 [⇔ Prime 449051 Gap 26 "Even P"], 488881 [⇔ Prime 488879 Gap 14 "Even P"], 512461 [⇔ Prime 512443 Gap 24 "Odd P"], 530881 [⇔ Prime 530869 Gap 28 "Odd P"], 552721 [⇔ Prime 552709 Gap 22 "Even P"]. **P**("Even P") = 22/35 = 62.9%, **P**("Odd P") = 13/35 = 37.1%. **P**(last digit of Odd Prime is 1) = 4/35 = 11.4%, **P**(last digit of Odd Prime is 3) = 12/35 = 34.3%, **P**(last digit of Odd Prime is 7) = 2/35 = 5.7%, **P**(last digit of Odd Prime is 9) = 17/35 = 48.6%.

Respectively for ending digit {1, 3, 5, 7, 9} through 1^{st} 10,000 terms of Carmichael numbers, we see 80.3%, 4.1%, 7.4%, 3.8% and 4.3% apportionment. Bias towards ending digit 1: For any $m > 1$, remainders of Carmichael numbers modulo m are biased towards 1. Number of terms congruent to 1 modulo 4, 6, 8, ..., 24 among 1^{st} 10,000 terms: 9827, 9854, 8652, 8034, 9682, 5685, 6798, 7820, 7880, 3378 and 8518.

Prime gaps $(4i-2)$ = {2, 6, 10, 14, 18, 22, 26,...} manifest *"Even Parity"* (parity 0). Prime gaps $(4i)$ = {4, 8, 12, 16, 20, 24, 28,...} manifest *"Odd Parity"* (parity 1). Both types of Prime gaps have equal ∼20% (rotating) Probability of their last-digit ending in 0, 2, 4, 6 or 8. In keeping with Odd Primes-Prime gaps constraints from Axiom 3 [and its **List of eligible Last digit of Odd Primes**] on applying Prime number theorem for Arithmetic Progression to statistically confirm Polignac's and Twin prime conjectures to be true, we deduce "*Theorem of Nil Predilection for Even-Odd Parity in Odd Primes associated with Carmichael numbers*" must be true whereby "thin set" and "decelerating CIS" Carmichael numbers in the long run implies these rare but ubiquitous numbers are statistically associated with two kinds of Odd Primes that manifest ∼50% "Even Parity" and ∼50% "Odd Parity". "Thin set" and "decelerating CIS" Carmichael numbers ∈ "Thick set" and "accelerating CIS" Composite numbers but these Carmichael numbers ≪ "Thin set" and "decelerating CIS" Odd Primes manifesting either "Even Parity" or "Odd Parity". This provide strong (indirect) evidence for Polignac's and Twin prime conjectures to be true.

The proof is now complete for Theorem 1 □.

Useful (colloquial) mathematical statements: We simply have no choice but to accept "There is zero probability that appearances of ℙ-ℂ **identifier grouping** when computed as Cardinality 0 for Gap 2-Twin primes, Cardinality 2 for Gap 4-Cousin primes, Cardinality 4 for Gap 6-Sexy primes, etc should ever stop or terminate in a discriminatory manner over the large range of integer numbers, thus confirming Modified Polignac's and Twin prime conjectures to be true". Here the word *Modified* denote the use of more appropriate term "decelerating CIS" [that represent "thin sets" and "thin subsets"] instead of just "CIS". Similarly for Riemann zeta function via *proxy* Dirichlet eta function, we simply have no choice but to accept "The solitary $\sigma = \frac{1}{2}$-critical line location for all nontrivial zeros thus confirming Riemann hypothesis to be true".

In combinatorics [that deals with counting and arrangements], inclusion-exclusion principle is a counting technique which generalizes the familiar method of obtaining number of elements in union of two or more sets when these sets may have overlaps. In essence, this principle removes all contributions from over-counted elements in sets and subsets. Instead of using raw *cardinality* from Pure Set theory when / if relevant, we should selectively use Measure theory such as length, area, probability (or proportion), Natural density (a.k.a. Asymptotic density, used when [for example] there is no uniform probability distribution over Natural numbers), and Dirichlet density (useful analytic tool for thin sets like set of Prime numbers that do not have well-defined Natural density; and with deep connections to Riemann zeta function, prime distribution and analytic number theory). Only under strict convergence conditions that any resultant infinite alternating series converge absolutely, this principle is valid for CFS, CIS or UIS [irrespective of whether there are finite or infinite number of these CFS, CIS or UIS]. We succinctly adapt or adopt this principle into mathematical arguments, lemmas, propositions, corollaries, axioms or theorems in this paper.

An example based on Measure theory: (Step 1) Define and measure two lengths as two UIS of Set A and Set B using two intervals of Real numbers on number line; viz, two individual "continuous lengths" are both [quantitatively] infinite $\mu(A) = 2 - 0 = 2$ and $\mu(B) = 4 - 1 = 3$. (Step 2) Measure the intersection as $A \cap B = [1, 2] \implies \mu(A \cap B) = 2 - 1 = 1$. (Step 3) Apply inclusion-exclusion principle $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) = 2+3-1 = 4 \implies$ total "discrete length" covered by both intervals is 4; viz, total "size" is [qualitatively] finite. Mainly based on Number theory, Mobius function $\mu(n)$ that connects deeply with Euler's totient function, zeta functions, and multiplicative number theory also gives a powerful compact formula for inclusion-exclusion principle over divisibility conditions.

Let A, B, C,... be finitely large sets or infinitely large sets, and $|S|$ indicates the cardinality of a set S ($\equiv$ 'number of elements' for set S). For CFS e.g. *Set* of even Prime number = $\{2\}$ with cardinality = 1, *Set* of odd Prime number with last-digit ending in 5 = $\{5\}$ with cardinality = 1; CIS e.g. *Set* of odd Prime numbers = $\{3, 5, 7, 11, 13, 17, 19...\}$ with cardinality = $\aleph_0$; and UIS e.g. *Set* of Real numbers with cardinality = $\mathfrak{c}$ (*cardinality of the continuum*). The inclusion-exclusion principle for three sets is given by $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. This formula expresses the fact that sum of sizes for these three sets may be too large since some elements may be counted twice (two times) or thrice (three times). General formula for a finite number of sets [with alternating signs +, −, +, −,... that depends on number of sets in the intersection] is $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$. In Probability theory, this formula for a finite number of sets is $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right)$. In closed form, this formula is $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} \mathbb{P}(A_I)\right)$, where

the last sum runs over all subsets I of indices 1, ..., n which contain exactly k elements, and $A_I := \bigcap_{i \in I} A_i$ denotes intersection of all those A_i with index in I. This formula for an infinite number of sets [strict convergence for infinite alternating series] is $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{k=1}^{\infty}\left((-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,\infty\} \\ |I|=k}} \mathbb{P}(A_I)\right)$.

For a general measure space (S, Σ, μ) and measurable subsets $A_1, A_2, ..., A_n$ [or $A_1, A_2, ..., A_n, A_{n+1}, A_{n+2}, ..., A_\infty$] of finite [or infinite measure], the above identity also hold when probability measure $\mathbb{P}$ is replaced by the measure μ.

3. Mathematics for Incompletely Predictable Problems and Input-White Box-Output Modeling

Definition 3.1. Where all infinitely-many prime [and composite] numbers are classified as Pseudo-random entities, so must all nontrivial zeros be classified as such. Pseudo-random entities are Incompletely Predictable entities. Largely based on p. 18 of [6], we provide formal definitions for three types of [infinitely-many] entities as Countably Infinite Sets in a succinct manner.

Completely Unpredictable (non-deterministic) entities are [the statistically] defined as entities that are actually random and DO behave like one e.g. [true] random number generator that supply sequences of entities (as non-distinct Sets of numbers) that are not reproducible; viz, these entities do not contain any repeatable spatial or temporal patterns. We work in the base-10 system (a.k.a. decimal system) that represent numbers using ten unique digits {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. For n = {1, 2, 3, 4, 5,...}, and with our [true] random number generator also utilizing these ten unique digits to supply n^{th} Entity as $n \to \infty$; then Probability (P) of independently obtaining each digit is P(0) = P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = P(7) = P(8) = P(9) $\simeq \frac{1}{10} \simeq 0.1 \simeq 10\%$.

Completely Predictable (deterministic) entities are defined as entities that are actually not random and DO NOT behave like one e.g. non-overlapping distinct Set of Even numbers {0, 2, 4, 6, 8, 10,...} and Set of Odd numbers {1, 3, 5, 7, 9, 11,...}; viz, these entities are reproducible. Chosen "Even [or Odd] Gap", as [non-varying] integer number value 2 between any two adjacent Even [or Odd] numbers, always consist of a fixed value. The distinct Sets of trivial zeros from various L-functions [as infinitely-many negative integers] are other examples of Completely Predictable entities. Both Riemann zeta function and its *proxy* Dirichlet eta function have simple zeros at each even negative integer $s = -2n$ where n = 1, 2, 3, 4, 5,...; viz, $s = -2, -4, -6, -8, -10, ...$. In addition, the factor $1 - 2^{1-s}$ in Dirichlet eta function adds an infinite number of [Completely Predictable] complex simple zeros, located at equidistant points on the line $\Re(s) = 1$, at $s_n = 1 + \dfrac{2n\pi i}{\ln(2)}$ whereby $n = ..., -3, -2, -1, 1, 2, 3, ...$ is any nonzero integer and i is the imaginary unit satisfying equation $i^2 = -1$.

Incompletely Predictable [or *Pseudo-random*] (deterministic) entities are defined as entities that are actually not random but DO behave like one e.g. non-overlapping distinct Set of Prime numbers {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31...} and Set of Composite numbers {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28...}; viz, these entities are reproducible. Chosen "Prime [or Composite] Gaps", as [varying] integer number values between any two adjacent Prime [or Composite] numbers, will never consist of a fixed value. Examples: Set of Prime Gaps = {1, 2, 2, 4, 2, 4, 2, 4, 6, 2,...} and Set of Composite Gaps = {2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 1, 1, 1, 1, 2,...}. The (odd) Prime Gap 1 indicates the only (solitary) even Prime number 2. All infinitely many (odd) Prime numbers have (even) Prime Gaps 2, 4, 6, 8, 10,... [to infinitely large size or, more precisely, to an arbitrarily large number] at sufficiently large integer range. Again, one can conveniently and arbitrarily classify small Prime Gaps to be 2 and 4, and large Prime Gaps to be ≥ 6. Only two finite integer number values {1, 2} represent Composite Gaps. Occurrences of (even) Composite Gap 2 in the specific Composite even number that always precede an (odd) Prime number are thus associated with appearances of all (odd) Prime numbers. The cardinality of consecutive (odd) Composite Gap 1 $\propto$ size of (even) Prime Gaps; viz, cardinality of consecutive Gap 1-Composite numbers = even Prime Gap $-$ 2 with cardinality of Gap 1-Composite even numbers $= \dfrac{\text{even Prime Gap} - 2}{2}$ and cardinality of Gap 1-Composite odd numbers $= \dfrac{\text{even Prime Gap} - 2}{2}$. Note that Gap 2-Prime numbers (twin primes) do not have Gap 1-Composite numbers. The inclusion-exclusion principle for two sets $|A \cup B| = |A| + |B| - |A \cap B|$. |All Even numbers $\cap$ All Prime numbers| = 1, which represent the only even Prime number 2. All Prime numbers [with exception of even Prime number 2] are (almost totally) constituted by Odd numbers. All odd Prime numbers are (totally) constituted by Odd numbers [although the majority of Odd numbers are not odd Prime numbers].

Apart from integers, Incompletely Predictable entities are also constituted from other number systems e.g. distinct Sets of t-valued transcendental numbers that faithfully represent infinitely-many nontrivial zeros (spectrum) of dual or self-dual L-functions. Geometrically, all nontrivial zeros of L-functions are simply the "Origin point intercepts" or Gram[x=0,y=0] points. L-functions [e.g. from the Genus 1 elliptic curves representing self-dual L-functions] can have Analytic rank 0, 1, 2, 3, 4, 5,... [to an arbitrarily large number]; viz, have "solitary" (zero) Analytic rank and "all other" (nonzero) Analytic rank. Thus, it seems that most L-functions should "qualitatively" have MORE (nonzero) Analytic rank = 1, 2, 3, 4, 5,... and LESS (zero)

Analytic rank = 0. Only (zero) Analytic rank L-functions, such as from Genus 0 (non-elliptic) Riemann zeta function [and its *proxy* Dirichlet eta function] and selected (Analytic rank 0) Genus 1 elliptic curves, DO NOT HAVE first nontrivial zeros with t value = 0 [viz, an algebraic number]. Then, all (nonzero) Analytic rank L-functions DO HAVE first nontrivial zeros with t value = 0 [viz, an algebraic number].

In Riemann hypothesis or Generalized Riemann hypothesis, all nontrivial zeros are conjecturally *only* located on $\Re(s) = \frac{1}{2}$-Critical line or Analytically normalized $\Re(s) = \frac{1}{2}$-Critical line. "Nontrivial Zero Gaps", as [varying] transcendental number values, between any two adjacent nontrivial zeros never consist of a fixed value. All infinitely-many nontrivial zeros are Incompletely Predictable entities. Note the infinitely-many digits after decimal point of each (algebraic) or (transcendental) irrational number are also Incompletely Predictable entities whereby individual irrational number has greater precision or accuracy when it is computed as having increasing number of digits.

We use abbreviations: CP = Completely Predictable, IP = Incompletely Predictable, CFS = Countably finite sets, CIS = Countably infinite sets, UIS = Uncountably infinite sets.

Remark 3.1. We compare and contrast Sets, Subsets, Even k-tuple and Prime k-tuple when derived from CP entities versus IP entities. There is only one mathematical possibility for CIS having CP or IP entities: Cardinality of *different or changing values* denoting the "Gaps" between any two adjacent elements in CIS with CP entities must be CFS. Cardinality of *different or changing values* denoting the "Gaps" between any two adjacent elements in CIS with IP entities must be CIS. Broadly applying inclusion-exclusion principle to two or more [mutually exclusive] cardinalities:

We can never obtain CIS having both CP entities and IP entities. In a similar manner, irrespective of having CP entities or IP entities, a given set must simply be UIS, CIS or CFS [and cannot be a mixture of UIS, CIS and/or CFS]. Subsets of CP entities are "non-unique and overlapping" e.g. Derived from Set of Gap 2-Even numbers (Twin Even numbers) = {0, 2, 4, 6, 8, 10,...}: Subset of Gap 4-Even numbers (Cousin Even numbers) = {0, 4, 8, 12, 16, 20,...}, Subset of Gap 6-Even numbers (Sexy Even numbers) = {0, 6, 12, 18, 24, 30,...}, etc. Subsets of IP entities are "unique and non-overlapping" e.g. Derived from Set of All Prime numbers = {2, 3, 5, 7, 11, 13,...}: Subset of Gap 2-Prime numbers (Twin Primes) = {3, 5, 11, 17, 29, 41,...}, Subset of Gap 4-Prime numbers (Cousin Primes) = {7, 13, 19, 37, 43, 67,...}, Subset of Gap 6-Prime numbers (Sexy Primes) = {23, 31, 47, 53, 61, 73,...}, etc.

For k = 2, 3, 4, 5, 6,... and n = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29,...; the diameter of a Prime k-tuple is difference of its largest and smallest elements. Note the special case of $k = 2$ simply corresponds to Gap 2-prime numbers (Twin primes). An admissible Prime k-tuple with smallest possible diameter d (among all admissible Prime k-tuples) is a Prime constellation $\equiv$ Prime k-tuplet. Prime constellations manifest the Incompletely Predictable property whereby certain prime numbers are "non-unique and overlapping" represented e.g. When $k = 3$, $d = 6$: Constellation (0, 2, 6) $\equiv$ [smallest] prime numbers (5, 7, 11) with chosen $n = 5$; Constellation (0, 4, 6) $\equiv$ [smallest] prime numbers (7, 11, 13) with chosen $n = 7$. When $k = 4$, $d = 8$: Constellation (0, 2, 6, 8) $\equiv$ [smallest] prime numbers (5, 7, 11, 13) with chosen $n = 5$. For all $n \geq k$ this will always produce consecutive Primes. Recall from above that all n are integers for which values $(n+a, n+b, n+c, ...)$ are prime numbers. This means that, for large n: $p_{n+k-1} - p_n \geq d$ where p_n is the n^{th} prime number.

We intuitively infer from above synopsis in previous two paragraphs that only by analyzing non-overlapping Subsets of even Prime gaps 2, 4, 6, 8, 10,... [instead of analyzing overlapping Prime k-tuples or Prime k-tuplets] would we obtain the rigorous proofs for Polignac's and Twin prime conjectures.

For $k = 2, 3, 4, 5, 6,...$ and $n = 0, 2, 4, 6, 8, 10, 12, 14, 16,...$; the diameter of an Even k-tuple is difference of its largest and smallest elements. An admissible Even k-tuple with smallest possible diameter d (among all admissible Even k-tuples) is an Even constellation $\equiv$ Even k-tuplet. Even constellations manifest the Completely Predictable property whereby even numbers are "unique and non-overlapping" represented e.g. When $k = 4$, $d = 2(k-1) = 6$: Constellation (0, 2, 4, 6) $\equiv$ [smallest] even numbers (0, 2, 4, 6) with chosen $n = 0$ or [using larger] even numbers (102, 104, 106, 108) with arbitrarily chosen $n = 102$. For all n [as fully obtained from $n < k$ and $n \geq k$], this will always produce consecutive even numbers. Recall from above that all n are integers for which values $(n+2, n+4, n+6, ...)$ are even numbers. This means that, for all n: $E_{n+k-1} - E_n = d$ where E_n is the n^{th} even number. Observe we could instead use odd numbers that will also produce the same equally valid deductions.

Gram's rule says there is exactly one nontrivial zero (NTZ) $\equiv$ Gram[x=0, y=0] point in Riemann zeta function between any two Gram points $\equiv$ Gram[y=0] points. A Gram block is an interval bounded by two "good" Gram points such that all Gram points between them are "bad". Rosser's rule says Gram blocks often have the expected number of NTZ in them [viz, NTZ is "conserved" and is the same as the number of Gram intervals], even though some individual Gram intervals in the block may not have exactly one NTZ in them [viz, some of the individual Gram intervals in the block violate Gram's rule]. Both Gram's rule and Rosser's rule say in some sense NTZ do not stray too far from their expected positions. Violations of Gram's rule equate to intermittently observable geometric variants of two consecutive (+ve first and then −ve) Gram points [$\equiv$ missing NTZ] that is alternatingly followed by two consecutive NTZ [$\equiv$ extra NTZ]. The rarer violations of Rosser's Rule equate to intermittently observable geometric variants of reduction in expected number of x-axis intercept points. They both fail infinitely many times in a +ve proportion of cases. We expect in ∼66% one NTZ is enclosed by two successive Gram points, but in ∼17% no NTZ and in ∼17% two NTZ are in such a Gram interval on the long run.

The success and failures of both Gram's rule and Rosser's rule occur in Dirichlet eta function [*proxy* for Riemann zeta function] on $\sigma = \frac{1}{2}$-Critical line. An insightful inference with deep connection to Riemann hypothesis: Only by analyzing non-overlapping Subset of "One NTZ" = ∼66%, Subset of "Zero NTZ" = ∼17%, and Subset of "Two NTZ" = ∼17% as precisely derived from Set of "All NTZ" = "conserved" 100% [instead of analyzing various overlapping Gram blocks and Gram intervals containing "good" or "bad" Gram points, missing NTZ or extra NTZ] can we rigorously prove Riemann hypothesis.

$INPUT \rightarrow$ $\boxed{\textit{White Box}}$ or $\boxed{\textit{Black Box}}$ $\rightarrow OUTPUT$. White Box (or Black Box) is a system where its [unique] inner components or logic are (or are not) available for inspection. Key ideas for computer & mathematical systems as White Box or Black Box $INPUT \rightarrow$ (unique) CP vs IP $\boxed{\textit{Information processor}}$ & $\boxed{\textit{Mathematical function, equation or algorithm}}$ $\rightarrow$ (reproducible) CP vs IP $OUTPUT$. Examples of Mathematical function, equation or algorithm:

CP CIS k^{th} Even numbers are Integers {0, ±2, ±4, ±6, ±8, ±10,...} [≡ $OUTPUT$] faithfully given by equation $n = \pm 2k$ [≡ *White Box*], where k are Integers {0, 1, 2, 3, 4, 5,...} [≡ $INPUT$]. Since Even n are integrally divisible by 2, congruence $n = 0 (\text{mod } 2)$ holds for Even n. The generating function of Even numbers is $\frac{2x}{(x-1)^2} = 2x^1 + 4x^2 + 6x^3 + 8x^4 + \cdots$.

CP CIS k^{th} Odd numbers are Integers {±1, ±3, ±5, ±7, ±9, ±11,...} [≡ $OUTPUT$] faithfully given by equation $n = \pm(2k-1)$ [≡ *White Box*], where k are Integers {1, 2, 3, 4, 5,...} [≡ $INPUT$]. Since Odd n when divided by 2 leave a remainder 1, congruence $n = 1 (\text{mod } 2)$ holds for Odd n. The generating function of Odd numbers is $\frac{x(1+x)}{(x-1)^2} = 1x^1 + 3x^2 + 5x^3 + 7x^4 + \cdots$. The oddness of a number is called its parity, so an Odd number has parity 1 (Odd Parity), while an Even number has parity 0 (Even Parity). The product of an Even number and an Odd number is always Even, as can be seen by writing $(2k)(2l+1) = 2[k(2l+1)]$, which is divisible by 2 and hence is Even.

IP "decelerating"-CIS k^{th} Prime numbers are Integers {±2, ±3, ±5, ±7, ±11, ±13, ±17, ±19, ±23, ±29, ±31, ±37,...} [≡ $OUTPUT$] faithfully given by algorithm ±*"Sieve-of-Eratosthenes"* [≡ *White Box*], where k are Integers {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,...} [≡ $INPUT$]. A Prime number is an Integer greater than 1 with exactly two factors, 1 and the number itself.

IP "accelerating"-CIS k^{th} Composite numbers are Integers {±4, ±6, ±8, ±9, ±10, ±12, ±14, ±15, ±16, ±18, ±20, ±21,...} [≡ $OUTPUT$] faithfully given by algorithm ±*"Complement-Sieve-of-Eratosthenes"* [≡ *White Box*], where k are Integers {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,...} [≡ *INPUT*]. A Composite number is an Integer greater than 1 with more than two factors (including 1 and the number itself).

IP CIS k^{th} Nontrivial zeros are Complex numbers $s = \sigma \pm it = \frac{1}{2} \pm it$ that are traditionally denoted by t-valued Transcendental numbers {±14.13, ±21.02, ±25.01, ±30.42, ±32.93, ±37.58,...} [≡ *OUTPUT*] as faithfully satisfied by equation *"Riemann zeta function* $\zeta(s) = 0$*" / "Dirichlet eta function* $\eta(s) = 0$*"* [≡ *White Box*], where k are Integers {1, 2, 3, 4, 5, 6,...} [≡ *INPUT*]. All nontrivial zeros are proposed in 1859 Riemann hypothesis to be only located on $\sigma = \frac{1}{2}$-Critical Line.

For i = 1, 2, 3,..., ∞; let i^{th} Even number = E_i and i^{th} Odd number = O_i. We can precisely, easily and independently calculate e.g. $E_5 = (2 \times 5) = 10$ and e.g. $O_5 = (2 \times 5) - 1 = 9$. A generated CP number is *locationally defined* as a number whose i^{th} position is independently determined by simple calculations without needing to know related positions of all preceding numbers - this is a "reproducible" Universal Property. The congruence n ≡ 0 (mod 2) holds for positive even numbers (n). The congruence n ≡ 1 (mod 2) holds for positive odd numbers (n). Then the zeroeth Even number $E_0 = (2 \times 0) = 0$ must exist.

For i = 1, 2, 3,..., ∞; let i^{th} Prime number = P_i and i^{th} Composite number = C_i. We can precisely, tediously and dependently compute e.g. $C_6 = 12$ and $P_6 = 13$: 2 is 1^{st} prime, 3 is 2^{nd} prime, 4 is 1^{st} composite, 5 is 3^{rd} prime, 6 is 2^{nd} composite, 7 is 4^{th} prime, 8 is 3^{rd} composite, 9 is 4^{th} composite, 10 is 5^{th} composite, 11 is 5^{th} prime, 12 is 6^{th} composite, 13 is 6^{th} prime, etc. Our desired integer 12 is the 6^{th} composite and integer 13 is the 6^{th} prime. A generated IP number is *locationally defined* as a number whose i^{th} position is dependently determined by complex calculations with needing to know related positions of all preceding numbers - this is a "reproducible" Universal Property. Observe that integers {0, 1} are neither prime nor composite.

Natural and Dirichlet density in Thin and Thick set:

Natural density of a Set $A \subseteq \mathbb{N}$ is: $d(A) = \lim_{n\to\infty} \frac{|A \cap \{1,2,3,...,n\}|}{n}$. If this limit exist, it measure how "large" a subset of the set of natural numbers is. It relies chiefly on the probability of encountering members of the desired subset when combing through the interval [1, n] as n grows large. We have $0 \le d(A) \le 1$: If $d(A) = 1$, the set is thick or co-dense (almost everything is in A). If $d(A) = 0$, the set is thin or sparse.

Let $A \subseteq \mathbb{P}$ be a subset of prime numbers. Dirichlet density of A is defined as: $\delta(A) = \lim_{s\to 1^+} \frac{\sum_{p\in A} \frac{1}{p^s}}{\sum_{p} \frac{1}{p^s}}$ provided this limit exist. Since the prime zeta function [an analogue of Riemann zeta function] $\sum_{p} \frac{1}{p^s} \sim \log_e(\frac{1}{s-1})$ as $s \to 1^+$, we also have $\delta(A) = \lim_{s\to 1^+} \frac{\sum_{p\in A} \frac{1}{p^s}}{\log_e(\frac{1}{s-1})}$. This expression is usually the order of the "pole" of $\prod_{p\in A} \frac{1}{1-\frac{1}{p^s}}$ at $s = 1$, (though in general it is not really a pole as it has non-integral order), at least if this function is a holomorphic function times a (real) power of $s-1$ near $s = 1$. Dirichlet density is useful when Natural density is undefined or hard to compute. It is especially effective for sets of primes in arithmetic progression. It bridges combinatorics with complex analysis through the zeta and L-functions. If Natural density exists, Dirichlet density also exists, and they are equal [but not the opposite way]. If A is the set of all primes, it is the Riemann zeta function which has a pole of order 1 at $s = 1$, so the set of all primes has Dirichlet density 1.

Set A is a *thin set* if it has zero Natural density, and is sparse or rare among Natural numbers; viz, it becomes vanishingly small compared to Natural numbers as you go to infinity. Example: The [earliest] ancient Euclid's Proof of the infinitude of $\mathbb{P}$ (c. 300 BC) utilize *reductio ad absurdum.* Set of all (odd) Prime numbers $\mathbb{P}$ is a "decelerating CIS" and a "thin set". Let $\mathbb{P}$-$\pi(n)$ be the Prime counting function defined as number of primes $\leq n$. Prime number theorem tells us $\mathbb{P}\text{-}\pi(n) \sim \frac{n}{\log_e n}$. With Prime Gaps = Set of $\mathbb{E} = \{2, 4, 6, 8, 10...\}$ being Arbitrarily Large in Number as you go to infinity, the Natural density for All odd $\mathbb{P}$ Set [$= \sum_{n\in\mathbb{E}}$ Gap n-$\mathbb{P}$] that "decelerates to an infinitesimal small number value just above zero" is given by $\lim_{n\to\infty} \frac{\mathbb{P}\text{-}\pi(n)}{n} = \lim_{n\to\infty} \frac{1}{\log_e n} = 0$. We recognize that Gap 2-$\mathbb{P}$ Subset, Gap 4-$\mathbb{P}$ Subset, Gap 6-$\mathbb{P}$ Subset, Gap 8-$\mathbb{P}$ Subset, Gap 10-$\mathbb{P}$ Subset,... being proposed to all consist of "decelerating CIS" and "thin sets" would imply the 1849-dated Polignac's conjecture [regarding all even Prime Gaps 2, 4, 6, 8, 10...] and the 1846-dated Twin prime conjecture [regarding "subset" even Prime Gaps 2] to both be true. We ultimately observe *Dimensional analysis homogeneity* when "decelerating CIS" and "thin set" properties are uniformly applicable to all quantities from both sides of the equation: All odd $\mathbb{P}$ Set = Gap 2-$\mathbb{P}$ Subset + Gap 4-$\mathbb{P}$ Subset + Gap 6-$\mathbb{P}$ Subset + Gap 8-$\mathbb{P}$ Subset + Gap 10-$\mathbb{P}$ Subset + $\cdots$.

A thick set is a set of integers that contains arbitrarily long intervals; viz, long blocks of consecutive integers [even if it also skips large chunks elsewhere]. Given a thick set A, for every $p \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that $\{n, n+1, n+2, ..., n+p\} \subset A$. Trivially Natural numbers $\mathbb{N}$, as Completely Predictable entities having "Natural Gap" = non-varying integer number value 1, is a thick set with Natural density being exactly 1. Other well-known sets that are thick include non-primes and non-squares. Thick sets can also be sparse, e.g. $\bigcup_{n\in\mathbb{N}} \{x : x = 10^n + m : 0 \leq m \leq n\}$. Thus a thick set has Natural density which can be 0 or > 0; viz, can be sparse or dense overall. It must always have long intervals (large chunks) but its sparsity can be low [or can be high]; viz, having Natural density close to 1 [or close to 0]. Set of Incompletely Predictable Composite numbers $\mathbf{C}$ is both thick and dense. Let $\mathbf{C}$-$\pi(n)$ be the Composite counting function defined as number of composites $\leq n$. Analogical "Composite number theorem" tells us $\mathbf{C}$-$\pi(n) \approx n - \mathbb{P}$-$\pi(n)$. Since $\mathbb{P}$-$\pi(n) \sim \frac{n}{\log_e n}$, we get $|\mathbf{C} \cap [1, n]| \approx n - \frac{n}{\log_e n}$ $\implies \frac{|\mathbf{C} \cap [1,n]|}{n} \to 1$ as $n \to \infty$; viz, Composite numbers as an "accelerating CIS" and "thick set" that "accelerate to an infinitesimal small number value just below one" have Natural density 1. An exception is specific subset of Gap 2-Composite even numbers that precede, and are associated with, every odd Prime numbers: This unique subset is "decelerating CIS", and is a "thin set" with Natural density 0.

Both the Completely Predictable sets of Even numbers $\mathbb{E}$ and Odd numbers $\mathbb{O}$ are neither *thin set* nor *thick set*. There are never any arbitrarily long blocks of consecutive $\mathbb{E}$ or $\mathbb{O}$ $\implies$ both sets are not thick set. Natural density of both $\mathbb{E}$ or $\mathbb{O}$ is $\frac{1}{2}$ [viz, $\neq 0$] $\implies$ both sets are not thin set.

Remark 3.2. Yitang Zhang proved a landmark result [announced on April 17, 2013]: There are infinitely many pairs of (odd) Prime numbers that differ by unknown even number $N \leq 70$ million[7]; viz, there is a "decelerating CIS" and "thin set" of Gap N-Prime numbers with unknown even number $N \leq$ 70 million. This solitary N value as an existing "privileged" but unknown even Prime gap must, without exception, comply with the imposed Odd Prime-Prime Gap constraint on "eligible last digit of Odd Primes" as per the itemized List from Axiom 3. Aesthetically, this N value by itself is insufficient since its generated "decelerating CIS" Odd Primes simply cannot exist alone amongst the large range of prime numbers. Always as finite [but NOT infinite] length, we observe as a side note that two or more consecutive Odd Primes can validly and rarely be constituted by [same] even Prime gap of 6 or multiples of 6. Hence there must be at least two, if not three, existing even Prime gaps generating their corresponding "decelerating CIS" Odd Primes. Polignac's and Twin prime conjectures refers to all even Prime gaps 2, 4, 6, 8, 10... generating corresponding "decelerating CIS" Odd Primes [which are, *by default*, all "thin sets"].

Polymath8a "Bounded gaps between primes" (4 June 2013 – 17 November 2014) was a project to improve N by developing the techniques of Zhang [viz, constructing an "admissible k-tuple" whose diameter was bounded by 70 million]. This project concluded with obtaining N = 4,680.

Polymath8b "Bounded intervals with many primes" (19 November 2013 – 19 June 2014) was a project to further improve N by combining Polymath8a results with the techniques of James Maynard [viz, introducing a refinement of GPY sieve method for studying prime k-tuples and small gaps between primes which establishes that "a positive proportion of admissible m-tuples satisfy the prime m-tuples conjecture for every m"]. This project concluded with a bound of N = 246; and by assuming Elliott-Halberstam conjecture and its generalized form further lower N to 12 and 6, respectively. Regarded as "Zhang's optimized result", these lowering of N involve studying *overlapping* k-tuples. But maximally lowering N to 2 requires clever breakthrough concepts that involve studying *non-overlapping* even Prime gaps.

Remark 3.3. The notion of *thin set* and *thick set* typically apply to subsets of [discrete] $\mathbb{N}$ (Natural numbers), or more generally, [discrete] $\mathbb{Z}$ (integer numbers). Set of Incompletely Predictable [discrete] nontrivial zeros from e.g. Riemann zeta function $\zeta(s)$ are derived from complex solutions to $\zeta(s)$ = 0, whereby $s = \sigma \pm it$. Traditionally given as $\pm t$-valued transcendental numbers; nontrivial zeros conceptually form a "discrete" and "sparse" ("small") set in [continuous] 1-dimensional $\sigma = \frac{1}{2}$-Critical Line [viz, constituted by $\mathbb{R}$ of infinite length] or in [continuous] 2-dimensional Complex plane [viz, constituted by $\mathbb{C}$ of infinite area]. CIS nontrivial zeros has zero Natural density in UIS $\mathbb{R}$ or UIS $\mathbb{C}$, and do not form dense clusters or intervals. We intuitively and meaningfully interpret Set of nontrivial zeros as a *thin set*.

4. Incompletely Predictable Carmichael numbers in Base-10

In number theory, an n-Knodel number for a given positive integer n is a composite number m with the property that each $i < m$ coprime to m satisfies $i^{m-n} \equiv 1 \pmod{m}$. The set of all n-Knodel numbers is denoted K_n. The special case K_1 is the Carmichael numbers. There are infinitely many n-Knodel numbers for a given n. Due to Euler's theorem every composite number m is an n-Knodel number for $n = m - \varphi(m)$ where φ is Euler's totient function.

Fermat's little theorem states that if p is a prime number, then for any integer a, the number $a^p - a$ is an integer multiple of p. In the notation of modular arithmetic, this is expressed as $a^p \equiv a \pmod{p}$. For example, if a = 2 and p = 7, then 27 = 128, and 128 − 2 = 126 = 7 × 18 is an integer multiple of 7. If a is not divisible by p, that is, if a is coprime to p, then Fermat's little theorem is equivalent to the statement that $a^{p-1} - 1$ is an integer multiple of p, or in symbols: $a^{p-1} \equiv 1 \pmod{p}$. For example, if $a = 2$ and $p = 7$, then $26 = 64$, and $64 - 1 = 63 = 7 \times 9$ is a multiple of 7.

A Carmichael number can be defined as a composite number n which in modular arithmetic satisfies congruence relation $b^n \equiv b \pmod{n}$ for all integers b. The relation may also be expressed in the form $b^{n-1} \equiv 1 \pmod{n}$ for all integers b that are relatively prime to n. They are infinitely many Carmichael numbers. They constitute the comparatively rare instances where the strict converse of Fermat's Little Theorem does not hold. This fact precludes the use of that theorem as an absolute test of primality.

Derived from prime and composite numbers, the following are well-defined Incompletely Predictable entities that faithfully comply with inclusion-exclusion principle: All Odd Primes [as a subset] derived from All Prime numbers [as a set] are selected prime numbers that constitute a "decelerating CIS" and "thin set" with Natural density 0. Gap 2-Composite even numbers always precede Odd Primes. All Gap 2-Composite even numbers [as a subset] derived from All Composite numbers [as a set] are highly selective composite numbers that constitute a "decelerating CIS" and "thin set" with Natural density 0. The n-Carmichael numbers [constituting the entire smaller subsets of All Carmichael numbers] will always be Gap 1-Composite odd numbers that have n prime factors with $n \geq 3$ e.g. 3-Carmichael numbers [as a subset] have three prime factors, 4-Carmichael numbers [as a subset] have four prime factors, etc. All Carmichael numbers [as a subset] derived from All Composite numbers [as a set] are highly selective composite numbers that constitute a "decelerating CIS" and "thin set" with Natural density 0.

For Carmichael numbers with exactly three prime factors, once one of the primes has been specified, there are only a finite number of Carmichael numbers which can be constructed. Indeed, for Carmichael numbers with n prime factors, there are only a finite number with the least $n-2$ specified.

Chernick's construction of Carmichael numbers is an extended way to obtain even smaller subsets of n-Carmichael numbers: $M_k(m) = (6m+1)$ $(12m+1) \prod_{i=1}^{k-2} (9 \cdot 2^i m + 1)$, $k \geq 3$, with the condition that each of the factors are prime and that m is divisible by 2^{k-4}. For example, Chernick Carmichael numbers as a subset of 3-Carmichael numbers: If, for a natural number m, the three numbers $6m+1$, $12m+1$ and $18m+1$ are prime numbers, the product $M_k(m) = (6m+1)\,(12m+1)\,(18m+1)$ is a 3-Carmichael number. This condition can only be satisfied if the number m ends with digits 0, 1, 5 or 6 in base 10 (i.e. m is congruent to 0 or 1 modulo 5). The first few 3-Carmichael numbers that correspond to $m = 1, 6, 35, 45, 51, 55, 56, 100, 121, ...$ are 1729, 294409, 56052361, 118901521, 172947529, 216821881, 228842209, 1299963601, 2301745249,.... An equivalent formulation of Chernick's construction is that if p, $2p-1$ and $3p-2$ are prime numbers congruent to 1 modulo 6, then the product p $(2p-1)$ $(3p-2)$ is a 3-Carmichael number. Incidentally, the Hardy-Ramanujan number, e.g. $1729 = 7{\cdot}13{\cdot}19$, is the third Carmichael number and the first Chernick Carmichael number.

Let $C(x)$ denote the number of Carmichael numbers less than x. Then, for all sufficiently large x, $C(x) > x^{2/7}$, which proves that there are infinitely many Carmichael numbers. The upper bound $C(x) < x \exp\left(-\frac{\log x \log\log\log x}{\log\log x}\right)$ has also been proved.

Bertrand's postulate is the theorem that for any integer $n > 3$, there exists at least one prime number p with $n < p < 2n-2$. A less restrictive formulation is: for every $n > 1$, there is always at least one prime p such that $n < p < 2n$. Another formulation, where p_n is the n-th prime, is: for $n \geq 1$ $p_{n+1} < 2p_n$.

Daniel Larsen in 2021[1] proved an analogue of Bertrand's postulate for Carmichael numbers first conjectured by Alford, Granville, and Pomerance in 1994. Using techniques developed by Yitang Zhang and James Maynard [see Remark 3.2] to establish results concerning small gaps between primes, his work yielded the much stronger statement that, for any $\delta > 0$ and sufficiently large x in terms of δ, there will always be at least $\exp\left(\frac{\log x}{(\log\log x)^{2+\delta}}\right)$ Carmichael numbers between x and $x + \frac{x}{(\log x)^{\frac{1}{2+\delta}}}$.

Carmichael numbers have the following properties:
1. If a prime p divides Carmichael number n, then $n \equiv 1 \pmod{p-1}$ implies that $n \equiv p \pmod{p(p-1)}$.
2. Every Carmichael number is squarefree.
3. An odd composite squarefree number n is a Carmichael number *iff* n divides the denominator of Bernoulli number B_{n-1}.

Theorem 1 in section 2 contains the required proof for "*Theorem of Nil Predilection for Even-Odd Parity in Odd Primes associated with Carmichael numbers*". The "thin set" and "decelerating CIS" of very rare Carmichael numbers are statistically associated with Odd Primes manifesting ∼50% "Even Parity" and ∼50% "Odd Parity". This deduction validly lend strong (indirect) support for Polignac's and Twin prime conjectures to be true.

5. Incompletely Predictable Primes with restricted digits in Base-10

Let $a_0 \in \{0, 1, 2, ..., 7, 8, 9\}$. James Maynard in 2016 show there are infinitely many Primes which do not have digit a_0 in their decimal expansion; viz, there are infinitely many Primes with restricted digits "as missing one digit". The proof is an application of the Hardy-Littlewood circle method to a binary problem, and rests on obtaining suitable 'Type I' and 'Type II' arithmetic information for use in Harman's sieve to control the minor arcs.

Throughout in his paper[3] that publish the required proof for this problem, $f \asymp g$ means that there are absolute constants c_1, $c_2 > 0$ such that $c_1 f < g$ $< c_2 f$. In relation to (i) Set of All Prime numbers {2, 3, 5, 7, 11, 13, 17, 19, 23, 29,...}, (ii) Subset of All Odd Primes {3, 5, 7, 11, 13, 17, 19, 23, 29,...}, and (iii) ten Subsets of Restricted Primes missing one digit {0 or 1 or 2 or 3 or 4 or 5 or 6 or 7 or 8 or 9} [when written in base 10]; relevant Set and Subsets will consist of "decelerating CIS" and "thin set" whereby the phrase "infinitely many" should more concisely be replaced by "Arbitrarily Large Number (ALN)".

Maynard then consider the same problem in bases other than 10, and with more than one excluded digit. The set of numbers less than X missing s digits in base q has $\asymp X^c$ elements, where $c = \log(q-s)/\log q$. For fixed s, the density becomes larger as q increases, and so the problem becomes easier. His methods are not powerful enough to show the existence of infinitely many primes with two digits not appearing in their decimal expansion, but they can show that there are infinitely many primes with s digits excluded in base q provided q is large enough in terms of s. Moreover, if the set of excluded digits possesses some additional structure this can apply to very thin sets formed in this way.

Primes having one digit $a_0 \in$ {0, 1, 2,..., 7, 8, 9} in their decimal expansion			
Digit a_0	Prime numbers	No. of Primes	Properties
0	nil	0	Not applicable
1	11, 1111111111111111111, 11111111111111111111111,...	?ALN	Repunit Primes (extremely rare)
2	2 [Only even Prime number]	1	odd Prime gap 1
3	3	1	As 1-digit Prime
4	nil	0	Always composite
5	5	1	As 1-digit Prime
6	nil	0	Always composite
7	7	1	As 1-digit Prime
8	nil	0	Always composite
9	nil	0	Always composite

TABLE 1. Restricted Primes with 9 digits excluded in base 10.

A repdigit, or sometimes monodigit, is a natural number composed of repeated instances of the same digit in a positional number system (that is often implicitly decimal). A repunit in base-10 is a number like 1, 11, 111, 1111,... that contains only the digit 1 – a more specific type of repdigit. A repunit prime is a repunit that is also a prime number.

The base-b repunits are defined as (where this b can be either positive or negative) $R_n^{(b)} \equiv 1 + b + b^2 + \cdots + b^{n-1} = \dfrac{b^n - 1}{b - 1}$ for $|b| \geq 2, n \geq 1$. Thus, the number $R_n^{(b)}$ consists of n copies of the digit 1 in base-b representation. The first two repunits base-b for $n = 2$ are $R_1^{(b)} = \dfrac{b-1}{b-1} = 1$ and $R_2^{(b)} = \dfrac{b^2-1}{b-1}$ $= b+1$ for $|b| \geq 2$. In particular, decimal (base-10) repunits that are often referred to as simply repunits are defined as $R_n \equiv R_n^{(10)} = \dfrac{10^n-1}{10-1} = \dfrac{10^n-1}{9}$ for $n \geq 1$. The number $R_n = R_n^{(10)}$ consists of n copies of digit 1 in base-10 representation. The sequence of repunits base-10 starts with 1, 11, 111, 1111, 11111, 111111, 1111111,.... For decimal (base-10) repunit primes, R_n is prime for $n = 2$, 19, 23, 317, 1031, 49081, 86453, 109297,.... Prime repunits are a trivial subset of permutable primes, i.e., primes that remain prime after any permutation of their digits.

Particular properties of decimal (base-10) repunit primes:

- The remainder of R_n modulo 3 is equal to the remainder of n modulo 3. Using $10^a \equiv 1 \pmod 3$ for any $a \geq 0$, $n \equiv 0 \pmod 3 \Leftrightarrow R_n \equiv 0 \pmod 3 \Leftrightarrow R_n \equiv 0 \pmod{R_3}$, $n \equiv 1 \pmod 3 \Leftrightarrow R_n \equiv 1 \pmod 3 \Leftrightarrow R_n \equiv R_1 \equiv 1 \pmod{R_3}$, $n \equiv 2 \pmod 3 \Leftrightarrow R_n \equiv 2 \pmod 3 \Leftrightarrow R_n \equiv R_2 \equiv 11 \pmod{R_3}$. Therefore, $3 \mid \mathrm{n} \Leftrightarrow 3 \mid R_n \Leftrightarrow R_3 \mid R_n$.
- The remainder of R_n modulo 9 is equal to the remainder of n modulo 9. Using $10^a \equiv 1 \pmod 9$ for any $a \geq 0$, $n \equiv r \pmod 9 \Leftrightarrow R_n \equiv r \pmod 9 \Leftrightarrow R_n \equiv R_r \pmod{R_9}$, for $0 \leq r < 9$. Therefore, $9 \mid \mathrm{n} \Leftrightarrow 9 \mid R_n \Leftrightarrow R_9 \mid R_n$.

The repunits base-2 are defined as $R_n^{(2)} = \dfrac{2^n-1}{2-1} = 2^n - 1$ for $n \geq 1$. The number $R_n^{(2)}$ consists of n copies of digit 1 in base-2 representation. These repunits are the well-known Mersenne numbers $M_n = 2^n - 1$ {1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535,...}. Primes that are repunits in base-2 are Mersenne primes.

Known properties of Repunits:

- Any repunit in any base having a composite number of digits is necessarily composite. Only repunits (in any base) having a prime number of digits can be prime [*whereby there are an ALN or decelerating CIS or thin set of prime numbers*]. This is a necessary but not sufficient condition as discussed next. It is easy to show that if n is divisible by a, then $R_n^{(b)}$ is divisible by $R_a^{(b)}$: $R_n^{(b)} = \dfrac{1}{b-1}\prod_{d|n}\Phi_d(b)$, where $\Phi_d(x)$ is the d^{th} cyclotomic polynomial and d ranges over the divisors of n. For p prime, $\Phi_p(x) = \sum_{i=0}^{p-1} x^i$, which has the expected form of a repunit when x is substituted with b. For example, 9 is divisible by 3, and thus R_9 is divisible by R_3 – in fact, $111111111 = 111 \cdot 1001001$. The corresponding cyclotomic polynomials $\Phi_3(x)$ and

$\Phi_9(x)$ are x^2+x+1 and x^6+x^3+1, respectively. Thus, for R_n to be prime, n must necessarily be prime, but it is not sufficient for n to be prime. For example, $R_3 = 111 = 3 \cdot 37$ is not prime. Except for this case of R_3, p can only divide R_n for prime n if $p = 2kn+1$ for some k.

- If p is an odd prime, then every prime q that divides $R_p^{(b)}$ must be either 1 plus a multiple of $2p$, or a factor of $b-1$. For example, a prime factor of R_{29} is $62003 = 1 + 2 \cdot 29 \cdot 1069$. The reason is that the prime p is the smallest exponent greater than 1 such that q divides $b^p - 1$, because p is prime. Therefore, unless q divides $b-1$, p divides the Carmichael function of q, which is even and equal to $q-1$.
- Any positive multiple of the repunit $R_n^{(b)}$ contains at least n nonzero digits in base-b. Any number x is a two-digit repunit in base $x-1$. The only known numbers that are repunits with at least 3 digits in more than one base simultaneously are 31 (111 in base-5, 11111 in base-2) and 8191 (111 in base-90, 1111111111111 in base-2). Goormaghtigh conjecture says there are only these two cases.
- Using the pigeon-hole principle it can be easily shown that for relatively prime natural numbers n and b, there exists a repunit in base-b that is a multiple of n. To see this consider repunits $R_1^{(b)}$,..., $R_n^{(b)}$. Because there are n repunits but only $n-1$ non-zero residues modulo n there exist two repunits $R_i^{(b)}$ and $R_j^{(b)}$ with $1 \le i < j \le n$ such that $R_i^{(b)}$ and $R_j^{(b)}$ have the same residue modulo n. It follows that $R_j^{(b)} - R_i^{(b)}$ has residue 0 modulo n, i.e. is divisible by n. Since $R_j^{(b)} - R_i^{(b)}$ consists of $j-i$ ones followed by i zeroes, $R_j^{(b)} - R_i^{(b)} = R_{j-i}^{(b)} \times b^i$. Now n divides the left-hand side of this equation, so it also divides the right-hand side, but since n and b are relatively prime, n must divide $R_{j-i}^{(b)}$.
- The Feit-Thompson conjecture is that $R_q^{(p)}$Rq(p) never divides $R_p^{(q)}$ for two distinct primes p and q.
- Using the Euclidean Algorithm for repunits definition: $R_1^{(b)} = 1$; $R_n^{(b)} = R_{n-1}^{(b)} \times b + 1$, any consecutive repunits $R_{n-1}^{(b)}$ and $R_n^{(b)}$ are relatively prime in any base-b for any n.
- If m and n have a common divisor d, $R_m^{(b)}$ and $R_n^{(b)}$ have the common divisor $R_d^{(b)}$ in any base-b for any m and n. That is, the repunits of a fixed base form a strong divisibility sequence. As a consequence, If m and n are relatively prime, $R_m^{(b)}$ and $R_n^{(b)}$ are relatively prime. The Euclidean Algorithm is based on $\gcd(m,n) = \gcd(m-n,n)$ for $m > n$. Similarly, using $R_m^{(b)} - R_n^{(b)} \times b^{m-n} = R_{m-n}^{(b)}$, it can be easily

shown that $\gcd(R_m^{(b)}, R_n^{(b)}) = \gcd(R_{m-n}^{(b)}, R_n^{(b)})$ for $m > n$. Therefore, if $\gcd(m, n) = d$, then $\gcd(R_m^{(b)}, R_n^{(b)}) = R_d^{(b)}$.

Theorem 2. *With nine excluded digits in decimal expansion of Primes [in base 10 with $a_0 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$], there is a theoretical inherent (maximum) limitation on number eligibility of these particular Primes.*

We concentrate on Repunit Primes in base-10: In keeping with Odd Primes-Prime gaps constraints from Axiom 3 [and its **List of eligible Last digit of Odd Primes**] on applying Prime number theorem for Arithmetic Progression to statistically confirm Polignac's and Twin prime conjectures to be true, we deduce that all Repunit Primes must be associated with their Prime gaps with last digit ending in 0 [viz, Gap 10, Gap 20, Gap 30, Gap 40... $= \sim$25%] or 2 [viz, Gap 2, Gap 12, Gap 22, Gap 32... $= \sim$25%] or 6 [viz, Gap 6, Gap 16, Gap 26, Gap 36... $= \sim$25%] or 8 [viz, Gap 8, Gap 18, Gap 28, Gap 38... $= \sim$25%] but NOT 4 [viz, Gap 4, Gap 14, Gap 24, Gap 34... $=$ 0%]. Repunit Primes have probabilistically an ALN of Prime gaps to choose from. Therefore, we can statistically state that the extremely rare $\geq$ 2-digit Repunit Primes belong to "decelerating CIS" and "thin set", whereby they seem to occur roughly as often as the Prime number theorem would predict: the exponent of the N^{th} repunit prime is generally around a fixed multiple of the exponent of the $(N-1)^{th}$.

Table 1 depict "Primes having only one digit a_0 in their decimal expansion" to overall be rarest: (i) non-existing when $a_0 = \{0, 4, 6, 8 \text{ or } 9\}$, (ii) solitary when $a_0 = \{2, 3, 5 \text{ or } 7\}$, and (iii) ALN as "decelerating CIS" or "thin set" when $a_0 = \{1\}$. By logical extrapolation, Repunit Primes with 9 digits excluded in base-10 should statistically be much rarer than any other Restricted Primes with lesser {8 or 7 or 6 or 5 or 4 or 3 or 2 or 1} digits excluded in base-10 $\implies$ In (*similar and parallel*) keeping with Odd Primes-Prime gaps constraints from Axiom 3 [and its **List of eligible Last digit of Odd Primes**] on applying Prime number theorem for Arithmetic Progression to statistically confirm Polignac's and Twin prime conjectures to be true, these Restricted Primes with all possible combinations and permutations of lesser digits excluded in base-10 must inevitably also be "decelerating CIS" and "thin sets" that validly lend strong (direct) support for Polignac's and Twin prime conjectures to be true. For example, in analyzing Primes having two digits $a_0 = \{1, 3\}$ as "decelerating CIS" and "thin sets" of Restricted Primes with 8 digits excluded in base-10; one must consider two (mutually exclusive) subsets of Odd Primes with its last digit ending in 1 or 3 but whereby these two subsets [again as "decelerating CIS" and "thin sets"] must simultaneously always contain both digits 1 and 3 in their decimal expansion.
The proof is now complete for Theorem 2$\square$.

6. Prime number theorem for Arithmetic Progressions

Solitary even prime number 2 has odd Prime gap 1. Abbreviations: ALN = Arbitrarily Large Number, CFS = Countably Finite Set, CIS = Countably Infinite Set, CP = Completely Predictable, IP = Incompletely Predictable, FL = Finite Length, IL = Infinite Length. For i = 1, 2, 3, 4, 5,..., n; algorithm and sub-algorithms from Sieve of Eratosthenes for even Prime gaps 2 (Twin primes), 4 (Cousin primes) and 6 (Sexy primes):

(a) For IP IL algorithm [Gap 2, 4, 6, 8, 10...]-Sieve of Eratosthenes $p_{n+1} = 3 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd $\mathbb{P}$ {3, 5, 7, 11, 13, 17, 19...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd $\mathbb{P}$ is denoted by $g_n = (n+1)^{st}$ Odd $\mathbb{P}$ − $(n)^{th}$ Odd $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 2, 2, 4, 2, 4, 2....

(b) For CP FL sub-algorithm [Gap 1]-Sieve of Eratosthenes $p_{n+1} = 2+\sum_{i=1}^{n} g_i$ [where n = 1 and not ALN] that faithfully generates the first and only Even $\mathbb{P}$ {2} ≡ first and only paired Even $\mathbb{P}$ {(2,3)} with cardinality CFS of 1, the solitary n^{th} odd prime gap between two successive primes is denoted by $g_n = (n+1)^{st}$ Odd $\mathbb{P}$ − $(n)^{th}$ Even $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n = 3 - 2 = 1$.

(c) For IP IL sub-algorithm [Gap 2]-Sieve of Eratosthenes $p_{n+1} = 3 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd twin $\mathbb{P}$ {3, 5, 11, 17, 29, 41, 59...} ≡ all paired Odd twin $\mathbb{P}$ {(3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61)...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd twin $\mathbb{P}$ is denoted by $g_n = (n+1)^{st}$ Odd twin $\mathbb{P}$ − $(n)^{th}$ Odd twin $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 2, 6, 6, 12, 12, 18....

(d) For IP IL sub-algorithm [Gap 4]-Sieve of Eratosthenes $p_{n+1} = 7 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd cousin $\mathbb{P}$ {7, 13, 19, 37, 43, 67...} ≡ all paired Odd cousin $\mathbb{P}$ {(7,11), (13,17), (19,23), (37,41), (43,47), (67,71)...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd cousin $\mathbb{P}$ is denoted by $g_n = (n+1)^{st}$ Odd cousin $\mathbb{P}$ − $(n)^{th}$ Odd cousin $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 6, 6, 8, 6, 24....

(e) For IP IL sub-algorithm [Gap 6]-Sieve of Eratosthenes $p_{n+1} = 23 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd sexy $\mathbb{P}$ {23, 31, 47, 53, 61, 73, 83...} ≡ all paired Odd sexy $\mathbb{P}$ {(23,29), (31,37), (47,53), (53,59), (61,67), (73,79), (83,89)...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd sexy $\mathbb{P}$ is denoted by $g_n = (n+1)^{st}$ Odd sexy $\mathbb{P}$ − n^{th} Odd sexy $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 8, 16, 6, 8, 12, 10....

A number base, consisting of any whole number > 0, is the number of digits or combination of digits that a number system uses to represent numbers e.g. decimal number system or base 10, binary number system or base2, octal number system or base 8, hexa-decimal number system or base 16. As $x \to \infty$, various derived properties of Prime counting function, Prime-$\pi(x)$ [= number of primes up to x] occur in, for instance, Prime number theorem for Arithmetic Progressions, Prime-$\pi(x; b, a)$ [= number of primes up to x with last digit of primes given by a in base b]. For any choice of digit a in base b with $\gcd(a, b) = 1$: Prime-$\pi(x; b, a) \sim \dfrac{\text{Prime-}\pi(x)}{\phi(b)}$. Here, Euler's totient function $\phi(n)$ is defined as the number of positive integers $\leq n$ that are relatively prime to (i.e., do not contain any factor in common with) n, where 1 is counted as being relatively prime to all numbers. Then each of the last digit of primes given by digit a in base b as $x \to \infty$ is equally distributed between the permitted choices for digit a with this result being valid for, and is independent of, any chosen base b.

In base-10: Numbers with their last digit ending in (i) 1, 3, 7 or 9 [which can be either primes or composites] constitute 40% of all integers; and (ii) 0, 2, 4, 5, 6 or 8 [which must be composites] constitute 60% of all integers. We validly ignore the only single-digit even prime number 2 and odd prime number 5. We note $\geq$2-digit Odd Primes can only have their last digit ending in 1, 3, 7 or 9 but not in 0, 2, 4, 5, 6 or 8.

List of eligible Last digit of Odd Primes:

- The last digit of Odd Primes having their Prime gaps with last digit ending in 2 [viz, Gap 2, Gap 12, Gap 22, Gap 32...] can only be 1, 7 or 9 [but not (5) or 3] as three choices.
- The last digit of Odd Primes having their Prime gaps with last digit ending in 4 [viz, Gap 4, Gap 14, Gap 24, Gap 34...] can only be 3, 7 or 9 [but not (5) or 1] as three choices.
- The last digit of Odd Primes having their Prime gaps with last digit ending in 6 [viz, Gap 6, Gap 16, Gap 26, Gap 36...] can only be 1, 3 or 7 [but not (5) or 9] as three choices.
- The last digit of Odd Primes having their Prime gaps with last digit ending in 8 [viz, Gap 8, Gap 18, Gap 28, Gap 38...] can only be 1, 3 or 9 [but not (5) or 7] as three choices.
- The last digit of Odd Primes having their Prime gaps with last digit ending in 0 [viz, Gap 10, Gap 20, Gap 30, Gap 40...] can only be 1, 3, 7 or 9 [but not (5)] as four choices.

Axiom 3. ***Applying Prime number theorem for Arithmetic Progressions confirm Modified Polignac's and Twin prime conjectures, and support generalized and ordinary Riemann hypothesis.***

Proof. We use decimal number system (base $b = 10$), and ignore the only single-digit even prime number 2 and odd prime number 5. For $i = 1, 2, 3,$

4, 5...; the last digit of all Gap $2i$-Odd Primes can only end in 1, 3, 7 or 9 that are each proportionally and equally distributed as $\sim$25% when $x \to \infty$, whereby this result is consistent with Prime number theorem for Arithmetic Progressions. The 100%-Set of, and its derived four unique 25%-Subsets of, Gap $2i$-Odd Primes based on their last digit being 1, 3, 7 or 9 must all be decelerating CIS. "Different Prime numbers literally equates to different Prime gaps" is a well-known intrinsic property. Since the ALN of Gap $2i$ as fully represented by all Prime gaps with last digit ending in 0, 2, 4, 6 or 8 are associated with various permitted combinations of last digit in Gap $2i$-Odd Primes being 1, 3, 7 and/or 9 as three or four choices [as per **List of eligible Last digit of Odd Primes**]; then these ALN unique subsets of Prime gaps based on their last digit being 0, 2, 4, 6 or 8 together with their correspondingly derived ALN unique subsets constituted by Gap $2i$-Odd Primes having last digit 1, 3, 7 or 9 must also all be decelerating CIS. The Probability (any Gap $2i$ abruptly terminating as $x \to \infty$) = Probability (any Gap $2i$-Odd Primes abruptly terminating as $x \to \infty$) = 0. Thus Modified Polignac's and Twin prime conjectures is confirmed to be true. With ordinary Riemann hypothesis being a special case, generalized Riemann hypothesis formulated for Dirichlet L-function holds once $x > b^2$, or base $b < \frac{1}{2}$ as $x \to \infty$. ***The ["statistical"] proof is now complete for Axiom 3***.

7. Conclusions

From Remark 3.2 and Axiom 3, Polignac's and Twin prime conjectures are proven to be true by predominantly using statistical arguments.

With having *Analytic rank 0* as common overlapping "component" between them, both Riemann hypothesis (RH), and Birch and Swinnerton-Dyer (BSD) conjecture involve proving the unexpected presence of certain [overall] "macro-properties". Profound Statement: *Irrespective of L-function sources and always with the one [unique] set of nontrivial zeros as OUTPUTS from each L-function, all the infinitely-many nontrivial zeros as [well-defined] Incompletely Predictable entities are ONLY located on (Analytically normalized)* $\sigma = \frac{1}{2}$*-Critical Line.*

With respecting Remark A.1, the above statement insightfully describes intractable open problem in Number theory of (Generalized) RH. Graphs of Z-function from LMFDB[2] on Genus 1 elliptic curves with nonzero Analytic rank 1, 2, 3, 4, 5... have trajectories that intersect Origin point. Graphs of Z-function from LMFDB[2] on Genus 1 elliptic curves with Analytic rank 0 [viz, having zero independent basis point (with infinite order) associated with either finitely many or zero $E(\mathbb{Q})$ solutions] DO NOT have trajectories that intersect Origin point. *Ditto* for Graph of Z-function on Genus 0 (non-elliptic) Riemann zeta function / Dirichlet eta function [in Figure 1] with Analytic rank 0 [viz, it DOES NOT have trajectory that intersect Origin point]. *This implies "simplest version" of BSD conjecture to be true*;

and simultaneously *implies "simplest version" of RH to also be true (with its Geometrical-Mathematical proof in* [6] *and its Algebraic-Transcendental proof in Appendix* A*).*

Acknowledgements
The author is grateful to Editors and Reviewers, LMFDB Managing Editor Prof. Andrew Sutherland, Associate Editor Dr. Edgar Costa, Prof. David Platt and Jonathan Bober for previous constructive feedbacks.

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Appendix A. Algebraic-Transcendental proof for Riemann hypothesis using Algebraic-Transcendental theorem

Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots=\prod_{p\text{ prime}}\frac{1}{1-p^{-s}}$ for $\text{Re}(s)>1$. $\zeta(s)$ [via its attached Euler product] is deeply connected to prime numbers [and also, *by default*, "complementary" composite numbers]. Dirichlet eta function $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\cdots$ for $\text{Re}(s)>0$. Where $\Gamma(s)$ is gamma function, $\zeta(s)$ and $\eta(s)$ will satisfy their respective functional equations $\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s)$ and $\eta(-s)=2\frac{1-2^{-s-1}}{1-2^{-s}}\pi^{-s-1}s\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\eta(s+1)$. Complex variable $s=\sigma\pm it$ where σ and t are real numbers. Critical Line: $\sigma=\frac{1}{2}$. Critical strip: $0<\sigma<1\equiv(0<\sigma<\frac{1}{2})+(\frac{1}{2}<\sigma<1)$.

Proposed by German mathematician Bernhard Riemann (17 September 1826 – 20 July 1866) in 1859, Riemann hypothesis states that all infinitely-many nontrivial zeros (NTZ), as a "thin set", of $\zeta(s)$ are located on its $\sigma = \frac{1}{2}$-Critical line. L-function associated to Genus 0 (non-elliptic) curve of $\zeta(s)$ is known to admit an analytic continuation and satisfy a functional equation via its *proxy* $\eta(s)$; viz, we do not need to assume Hasse-Weil conjecture.

The infinitely-many success / failures of Gram's rule and Rosser's rule only occur in Dirichlet eta function [*proxy* for Riemann zeta function] on $\sigma = \frac{1}{2}$-Critical line. To solve Riemann hypothesis, one must analyze non-overlapping Subset of "One NTZ" = ∼66%, Subset of "Zero NTZ" = ∼17%, and Subset of "Two NTZ" = ∼17% as precisely derived from Set of "All NTZ" = ("conserved") 100% [instead of analyzing various overlapping Gram blocks and Gram intervals containing "good" or "bad" Gram points, missing NTZ or extra NTZ].

Transcendental functions ≫ Algebraic functions with the Uncountably Infinite Set of Transcendental irrational numbers ≫ Countably Infinite Set of Algebraic irrational numbers. From selected mathematical arguments, we formally derive Algebraic-Transcendental theorem which supports the Statement: *Algebraic functions must form a subset of the broader class of Transcendental functions.* We now supply a non-exhaustive list of Algebraic-Transcendental links. This will suffice for our purpose to create Algebraic-Transcendental theorem required to complete the deceptively simple Algebraic-Transcendental proof for Riemann hypothesis.

Lemma 4. *We outline relevant Algebraic-Transcendental connections when based on algebraic functions and algebraic numbers, and transcendental functions and transcendental numbers.*

Proof. An algebraic function is a function often defined as root of an irreducible polynomial equation. The algebraic functions are usually algebraic expressions using a *finite number of terms*, involving only algebraic operations addition (+), subtraction (−), multiplication (×), division (÷), and raising to a fractional power. Examples of pure algebraic function are: $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, $f(x) = \frac{\sqrt{1+x^3}}{x^{3/7} - \sqrt{7}x^{1/3}}$, Golden ratio $\phi = \frac{1+\sqrt{5}}{2} =$ 1.6180339887... [that is the most irrational number because it's hard to approximate with a rational number], etc. Algebraic functions usually cannot be defined as finite formulas of elementary functions, as shown by the example of Bring radical $f(x)^5 + f(x) + x = 0$ (this is the Abel-Ruffini theorem).

A transcendental function is an analytic function that does not satisfy a polynomial equation whose coefficients are functions of independent variable written using only basic operations of addition, subtraction, multiplication, and division (without need of taking limits). Pure transcendental functions e.g. logarithm function $\ln x$ or $\log_e x$, exponential function e^x, trigonometric functions $\sin x$ and $\cos x$, hyperbolic functions $\sinh x$ and $\cosh x$, generalized hypergeometric functions, class of numbers called Liouville numbers [that can be more closely approximated by rational numbers than can any irrational algebraic number]. Equations over these expressions are called transcendental equations. A transcendentally transcendental function or hypertranscendental function is transcendental analytic function which is not the solution of an algebraic differential equation with coefficients in integers $\mathbb{Z}$ and with algebraic initial conditions; e.g. zeta functions of algebraic number fields, in particular, Riemann zeta function $\zeta(s)$ and gamma function $\Gamma(s)$ (cf. Holder's theorem).

The indefinite integral of many algebraic functions is transcendental. For example, integral $\int_{t=1}^{x} \frac{1}{t} dt$ turns out to equal logarithm function $log_e(x)$. Similarly, the limit or the infinite sum of many algebraic function sequences is transcendental. Example, $\lim_{n\to\infty}(1 + x/n)^n$ converges to exponential function e^x, and infinite sum $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ turns out to equal hyperbolic cosine function $\cosh x$. In fact, it is impossible to define any transcendental function in terms of algebraic functions without using some such "limiting procedure" (integrals, sequential limits, and infinite sums are just a few).

A function that is not a transcendental function must logically be an algebraic function. This implies every algebraic function is algebraic solution to a polynomial equation but transcendental functions are not solutions to any such equation. Alternatively stated: The output values of an algebraic function (for specific input values of x) are algebraic numbers. This is because algebraic function itself is defined as a solution to an algebraic equation, and any solution to such an equation is [and must be] an algebraic number.

While transcendental functions often produce transcendental numbers as outputs, they also have solutions as algebraic numbers. The composition of transcendental functions in $f(x) = \cos \arcsin x = \sqrt{1 - x^2}$ give an algebraic function. Outputs from transcendental functions as algebraic numbers: Equation $e^x = 1$ has solution $x = 0$, an algebraic number (since 0 is algebraic). Equation $\sin(x) = 0$ has solutions $x = n\pi$, where $n = 0, 1, 2, 3, 4, 5...$ are algebraic numbers (since integers are algebraic). Equation $\ln(x) = 0$ has solution $x = 1$, an algebraic number (since 1 is algebraic). Outputs from transcendental function as transcendental numbers: Equation $e^x = 2$ has solution $x = \ln(2)$, which is transcendental, since $\ln(2)$ is a transcendental number.

Two trigonometric functions in equation $\sin(x) = \cos(x) = \frac{\pi}{4}$ have identical solution $x = \frac{1}{\sqrt{2}}$. This "sweet-spot" property is due to sine cosine complementary angle relationship for isosceles triangle. $\frac{1}{\sqrt{2}} \approx 0.70710678$ is (algebraic) irrational number and $\frac{\pi}{4}$ is (transcendental) irrational number. Then $\frac{\pi}{4}$ radian ≈ 0.785398 radian $\equiv 45^o$.

The proof is now complete for Lemma 4□.

Proposition 5. *Algebraic functions will never give rise to transcendental numbers as outputs unless we start involving transcendental functions.*

Proof. Algebraic numbers = {Integers + Rational numbers + Roots of Integers (or Algebraic irrational numbers) as Algebraic (non-complex) numbers} + {$z = a + bi$ as Algebraic (complex) numbers where a, b must be Integers or Rational numbers}. Thus certain algebraic functions may involve more complex operations such as roots or radicals, giving complicated outputs that are still algebraic. We deduce from mathematical arguments in Lemma 4: While a given *de novo* function itself is algebraic [viz, a pure algebraic function], it will never give rise to transcendental numbers unless we involve transcendental functions [viz, create a mixed algebraic-transcendental function]. ***The proof is now complete for Proposition 5□.***

Corollary 6. *Any outputs as transcendental numbers from a given function must involve transcendental functions, which can be given as either pure transcendental functions or mixed algebraic-transcendental functions.*

Proof. Pure algebraic functions always give outputs that are algebraic [but never transcendental]. Both pure transcendental functions and mixed algebraic-transcendental functions give outputs as transcendental numbers ± algebraic numbers [but never as outputs that are all algebraic numbers].

Examples of mixed algebraic-transcendental functions: $f(x) = x^2 + e^x$ that involves both algebraic and transcendental terms; and $f(x) = e^{\sqrt{2}}$ that involves transcendental operation on algebraic number.

The proof is now complete for Corollary 6□.

Axiom 7. *Nontrivial zeros (spectrum) computed [e.g. using Hardy Z-function as Z(t) plots] for any L-function involve transcendental functions in one form or another, and are inherently given as t-valued transcendental numbers.*

Proof. It is precisely the case that since all infinitely-many nontrivial zeros (spectrum) computed [e.g. using Hardy Z-function as Z(t) plots] for any given L-function will involve transcendental functions in one form or another [often as mixed algebraic-transcendental functions]; then it is simply a mathematical impossibility that nontrivial zeros as outputs will not be given as t-valued transcendental numbers. This deduction is completely consistent with our Proposition 5 and Corollary 6.

A particular solution with '0' (zero) value from a given function may imply that function to be a pure algebraic function, a pure transcendental function or a mixed algebraic-transcendental function. Integer 0 is an algebraic number (since 0 is algebraic). L-functions are usually mixed algebraic-transcendental functions: [1] Analytic rank 0 L-functions will never have

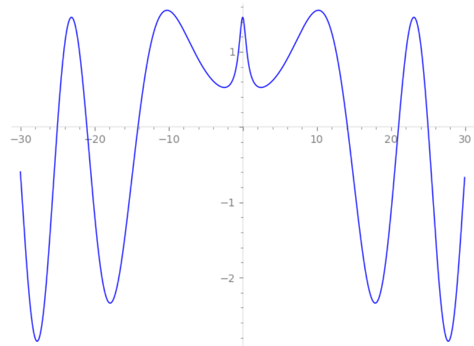

FIGURE 1. Graph of Z-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet eta function $\eta(s)$ of degree 1 over $K = \mathbb{Q}$ as *Analytic continuation* of Riemann zeta function $\zeta(s)$. Line Symmetry vertical y-axis, trajectory DO NOT intersect Origin point, and manifest Z(t) positivity as part of Sign normalization by L-functions and modular forms database[2]. Integral basis 1. An integral basis of a number field K is a $\mathbb{Z}$-basis for ring of integers of K. It is a $\mathbb{Q}$-basis for K. Initial $+ve$ nontrivial zeros: 14.13, 21.02, 25.01, 30.42, 32.93, 37.58,.... "Nontrivial Zero Gaps" between any two adjacent nontrivial zeros never consist of a fixed value $\implies$ all infinitely-many nontrivial zeros must be Incompletely Predictable entities.

their 1^{st} nontrivial zero being endowed with algebraic 0 value. [2] Analytic rank 1, 2, 3, 4, 5... (viz, non-zero ≥ 1) L-functions will always have their 1^{st} nontrivial zero being endowed with algebraic 0 value.

The proof is now complete for Axiom 7$\square$.

Theorem 8. *We can categorically formulate Algebraic-Transcendental theorem which states that all infinitely-many nontrivial zeros (spectrum) from Riemann zeta function must be located on its* $\sigma = \frac{1}{2}$*-Critical Line [as was originally proposed by the 1859-dated Riemann hypothesis].*

Proof. Being a self-dual L-function, Riemann zeta function as a Genus 0 curve admits an analytic continuation and satisfy a functional equation via *proxy* Dirichlet eta function. By its very definition, geometrical $\sigma = \frac{1}{2}$-Origin point $\equiv$ mathematical $\sigma = \frac{1}{2}$-Critical line. All infinitely-many Origin intercept points $\equiv$ All infinitely-many Nontrivial zeros are proposed to lie on this Critical line $\implies$ Geometrical-Mathematical proof for Riemann hypothesis as outlined in [6]. Consistent with Axiom 7 is the fact that all

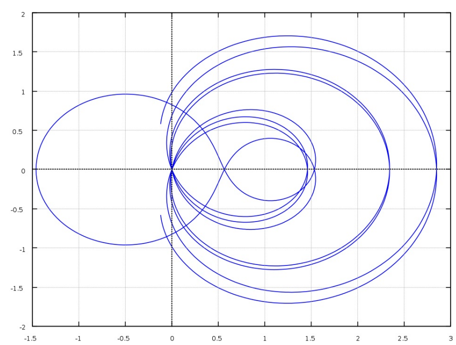

FIGURE 2. OUTPUT at $\sigma = \frac{1}{2}$-Critical Line. Polar graph of $\zeta(\frac{1}{2}+\imath t)$ / $\eta(\frac{1}{2}+\imath t)$ plotted for real values t between -30 and $+30$ from $s = \sigma \pm \imath t$. Horizontal axis: $Re\{\eta(\frac{1}{2}+\imath t)\}$. Vertical axis: $Im\{\eta(\frac{1}{2} + \imath t)\}$. *Indicating Riemann hypothesis*, Origin intercept points $\equiv$ nontrivial zeros are present. Manifesting perfect Mirror (Line) symmetry about horizontal x-axis.

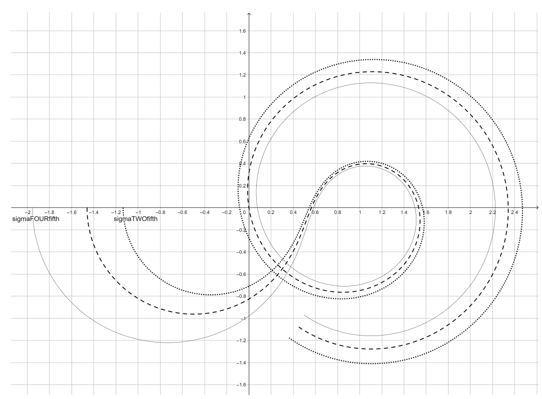

FIGURE 3. Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{2}{5}$ & $\sigma = \frac{4}{5}$. Horizontal axis: $Re\{\zeta(\sigma + \imath t)\}$ / $Re\{\eta(\sigma+\imath t)\}$ & vertical axis: $Im\{\zeta(\sigma+\imath t)\}$ / $Im\{\eta(\sigma+\imath t)\}$. Presence of Origin intercept points at [static] Origin point. Presence of virtual Origin intercept points as additional $-$ve virtual Gram[y=0] points on x-axis (e.g. using $\sigma = \frac{2}{5}$ value) at [infinitely-many varying] virtual Origin points; viz, these $-$ve virtual Gram[y=0] points on x-axis cannot exist at Origin point since two trajectories form co-linear lines (or co-lines) [*two parallel lines that never cross over near Origin point*].

infinitely-many nontrivial zeros from Dirichlet eta function [*proxy* function for Riemann zeta function] are always given as t-valued transcendental (irrational) numbers. From previous mathematical arguments in section 3 on properties for Incompletely Predictable entities, there are two occurrences of these entities in nontrivial zeros: (i) The integer numbers representing each and every one of infinitely-many nontrivial zeros, and (ii) Infinitely-many digital numbers after decimal point in each and every one of infinitely-many nontrivial zeros.

$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ [as example involving infinite sum of infinitely-many algebraic functions] turns out to equal hyperbolic cosine function $\cosh x$ [which is a transcendental function]. We can now conceptually represent a transcendental function as infinite sum of infinitely-many algebraic function sequences [viz, an 'infinite series']. Thus when based on inclusion-exclusion principle, we validly deduce that $\sigma = \frac{1}{2}$-Dirichlet eta function is an unique $\sigma = \frac{1}{2}$-mixed-algebraic-transcendental function [that contains all nontrivial zeros] AND $\sigma \neq \frac{1}{2}$-Dirichlet eta functions are infinitely-many non-unique $\sigma \neq \frac{1}{2}$-mixed-algebraic-transcendental functions [that cannot contain nontrivial zeros]. We now have the mutually exclusive statement based on $\sigma = \frac{1}{2}$-Dirichlet eta function and $\sigma \neq \frac{1}{2}$-Dirichlet eta functions being completely different 'infinite series': *{It is a mathematical impossibility for any nontrivial zeros to be located away from Critical line.}* $\equiv$ *{It is a mathematical certainty for all nontrivial zeros to be located on Critical line.}*

Euler formula can be stated as $e^{\imath n} = \cos n + \imath \cdot \sin n$. Applying this famous formula to $\sigma = \frac{1}{2}$-Dirichlet eta function results in simplified $\sigma = \frac{1}{2}$-Dirichlet eta function that faithfully contains all t-valued nontrivial zeros [whereby this simplified function will clearly identify itself as representing a mixed-algebraic-transcendental function involving both algebraic and transcendental functions]: The simplified $\sigma = \frac{1}{2}$-Dirichlet eta function

$$= \sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n) + \frac{1}{4}\pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n-1) + \frac{1}{4}\pi)$$

The proof is now complete for Theorem 8$\square$.

Remark A.1. Hasse-Weil zeta function is a global L-function defined as an Euler product of local zeta functions. Hasse-Weil conjecture states that Hasse-Weil zeta function attached to an algebraic variety V defined over an algebraic number field K should *admit an meromorphic continuation* for all complex s and *satisfy a functional equation*. In, for instance, Genus 2 curves over totally real fields, they have non-regular Hodge numbers and the Taylor-Wiles method that was successful in proving this conjecture for Genus 1 curves (for example) breaks down in several places. Many of the

L-functions we consider in this paper (including those associated to curves of Genus > 1), are not known to admit an analytic continuation or satisfy a functional equation. To properly discuss nontrivial zeros on Critical Line and in Hardy Z-function; we therefore need to, at least, assume this conjecture.

Taking Remark A.1 into perspective consideration; all correct & complete mathematical arguments are assumed to comply with two conditions in this paper [that have "Analytic rank 0" component present in both]:

Condition 1. Generalized Riemann hypothesis (RH): All the nontrivial zeros (spectrum) of General [or Generic] L-functions from Genus 0, 1, 2, 3, 4, 5... curves with Analytic rank 0, 1, 2, 3, 4, 5... lie on the $\sigma = \frac{1}{2}$-Critical Line or the Analytically normalized $\sigma = \frac{1}{2}$-Critical Line. The 'special case' (*simplest*) RH[6] refers to the [Analytic rank 0] Genus 0 non-elliptic curve called Riemann zeta function / Dirichlet eta function.

Condition 2. Generalized Birch and Swinnerton-Dyer (BSD) conjecture: All Generic L-functions from Genus 0, 1, 2, 3, 4, 5... curves satisfy Algebraic (Mordell-Weil) rank = Analytic rank [for even Analytic rank 0, 2, 4, 6, 8, 10... and odd Analytic rank 1, 3, 5, 7, 9, 11...]. The 'special case' (*simplest*) BSD conjecture refers to Genus 1 elliptic curves; expressed as *weak form* and *strong form* of BSD conjecture.

Analogy for (Generalized) Riemann hypothesis: Let $\delta = \frac{1}{\infty}$ [which represents an infinitesimal small number value], Geometrical 0-dimensional $\sigma = \frac{1}{2}$-Origin point $\equiv$ Mathematical 1-dimensional $\sigma = \frac{1}{2}$-Critical Line, and Origin intercept points $\equiv$ nontrivial zeros. Using sine-cosine complementary angle relationship $\sin(\theta) = \cos(\theta - \frac{\pi}{2}) \equiv \cos(\theta) = \sin(\theta - \frac{\pi}{2}) \equiv$ "always having complete set of nontrivial zeros" as alternative analogical explanation: Riemann hypothesis is uniquely $\leftrightarrow \theta = \frac{\pi}{4}$ with $\sin(\theta) = \cos(\theta) = \frac{1}{\sqrt{2}}$ where all (100%) nontrivial zeros are "conserved". Generalized Riemann hypothesis are non-uniquely $\leftrightarrow \theta \neq \frac{\pi}{4}$ with $\sin(\theta) \neq \cos(\theta) \neq \frac{1}{\sqrt{2}}$.

Proposition: Always having Origin point intercept $\Leftrightarrow \sin x = \cos(Ax - \frac{C\pi}{2})$ uniquely when $C = 1$. *Corollary*: Never having Origin point intercept $\Leftrightarrow \sin x \neq \cos(Ax - \frac{C\pi}{2})$ non-uniquely when $C = 1 \pm \delta$. Assigned values for A is "inconsequential" in the sense that solitary $A = 1$ value $\implies$ 'special case' Riemann hypothesis [on Genus 0 curve], and multiple $A \neq 1$ values $\implies$ Generalized Riemann hypothesis [on Genus 1, 2, 3, 4, 5... curves].

Geometrical-Mathematical proof[6] for Riemann hypothesis is exemplified by Figure 1, Figure 2 and Figure 3. Let $\delta = \frac{1}{\infty}$ [viz, an infinitesimal small number value] in reference to Figure 3. Then the plotted trajectories arising from inputting $\sigma = \frac{1}{2}+\delta$ and $\sigma = \frac{1}{2}-\delta$ into Riemann zeta function/Dirichlet eta function will always result in two co-linear lines being located (approximately) an infinitesimal small δ distance, respectively, just to right and left of Origin point [but never touching Origin point $\equiv$ Critical line].

Proof by induction for Riemann hypothesis using plotted co-linear lines [that conceptually comply with the inclusion-exclusion principle]. For $n = 0, 1, 2, 3, 4, 5, ..., \infty$ in reference to $-\infty < t < +\infty$ when inputting $\sigma = \frac{1}{2} + n\delta$ [$\equiv$ "To Right of Origin Point"] and $\sigma = \frac{1}{2} - n\delta$ [$\equiv$ "To Left of Origin Point"] into Riemann zeta function/Dirichlet eta function, there are infinitely-many [self-similar] plotted trajectories as co-linear lines using Polar graph in, and to cover, entire $0 < \sigma < 1$-Critical strip.

Proving the Base case when $n = 1$: At $n = 0$ [$\equiv$ "On the Origin Point"] in Figure 2 using either $\sigma = \frac{1}{2} + n\delta$ or $\sigma = \frac{1}{2} - n\delta$, this always represent Polar graph at $\sigma = \frac{1}{2}$-Critical line with having all (100%) nontrivial zeros, thus implying Riemann hypothesis to be true. At $n = 1$ [$\equiv$ "To Right of Origin Point"] using $\sigma = \frac{1}{2} + n\delta$, this always represent Polar graph at $\sigma \neq \frac{1}{2}$-Non-critical line without having any (0%) nontrivial zeros. At $n = 1$ [$\equiv$ "To Left of Origin Point"] using $\sigma = \frac{1}{2} - n\delta$, this always represent Polar graph at $\sigma \neq \frac{1}{2}$-Non-critical line without having any (0%) nontrivial zeros.

Induction step: Suppose $\sigma = \frac{1}{2} + k\delta \equiv$ "To Right of Origin Point" or $\sigma = \frac{1}{2} - k\delta \equiv$ "To Left of Origin Point" for some $k > 0$ [viz, $k = 1, 2, 3, 4, 5, ..., \infty$]. Based on deviation property "Increasing distance away from Origin Point as k becomes larger", we correctly claim both scenario are valid for next case $k + 1$ that always represent Polar graph at $\sigma \neq \frac{1}{2}$-Non-critical line without having any (0%) nontrivial zeros. We now establish the truth of this statement for all natural numbers $k \geq 1$, thus implying Riemann hypothesis to be true.

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