Applying a singularity-inspired notion to the Birch and Swinnerton-Dyer conjecture

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Abstract

We apply *SING*, a notion inspired by singularity (viXra:1812.0480 [v1]), to the Birch and Swinnerton-Dyer conjecture to suggest that the conjecture is related to the twin prime conjecture.

1 Glossary

$A := B$: A is defined as B.

$a \in A$: a is an element of the set A .

$\overline{AB}$: line segment AB .

A^{T}: transpose of matrix A .

BSD: Birch and Swinnerton-Dyer .

$\mathbb{C}$: the set of complex numbers .

CHT: Cayley–Hamilton theorem .

D: discriminant .

$\det$: determinant .

EC: elliptic curve .

GCM: generalised Cartan matrix .

GIMP: GNU Image Manipulation Program .

$\imath$: imaginary unit .

I_n: $n \times n$ identity matrix .

LHS: left-hand side .

MR: multiple root .

MT: multiplication table .

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$\mathbb{N}$: $\{1, 2, 3, \ldots\}$.

O: the origin $(0, 0)$ or $(0, 0, 0)$.

O_n: $n \times n$ null matrix .

$\mathbb{R}$: the set of real numbers .

RHS: right-hand side .

SIM: singular matrix .

S^{n-1}: n-dimensional unit sphere .

SYM: symmetric matrix .

TPC: twin prime conjecture .

tr: trace .

2 Introduction

Being related to an EC , the BSD conjecture has been of some interest $[1, 2]$. Our idea is to consider the EC

$$y^2 = x^3 - d^2x \ \ [3, \ \mathbf{Figure\ 1.2}] \tag{1}$$

and apply *SING*, a singularity-inspired notion $[4]$, to it.

3 Applying *SING* to EC

First, let D of (1) be

$$-16\{4 \cdot (-d^2)^3 + 27 \cdot 0^2\} = 64d^6 \text{ [1]} . \tag{2}$$

We assume $D \neq 0$, that is, $d \neq 0$ in (2) [2] . Next, in order to compute *SING*, we write (1) as

$$x^3 - d^2x - y^2 = 0$$

and set this

$$\phi = x^3 - d^2x - y^2 = 0.$$

By computing

[1] *Cf*, [5].

[2] If $d = 0$, d^2 in (1) becomes 0. In this case, we deal with $y^2 = x^3$. *Cf.* [4, **7.1**]. We will also treat the cases d^2 in (1) are pure imaginary number s in **Discussion**.

$$\frac{d\phi}{dx} = \frac{d}{dx}(x^3 - d^2x - y^2) = 3x^2 - d^2 - 2y\frac{dy}{dx},$$

one gets the 1-form $\omega = d\phi = (3x^2 - d^2)dx - 2ydy$. And one solves

$$\begin{cases} 3x^2 - d^2 = 0, \\ -2y = 0 \end{cases}$$

to get $(x, y) = (\pm\frac{d}{\sqrt{3}}, 0)$. Thus, *SING*'s of (1) are the points

$$(\frac{d}{\sqrt{3}}, 0) \quad \text{and} \quad (-\frac{d}{\sqrt{3}}, 0). \tag{3}$$

Remark 3.1. Since $(\{\frac{d}{\sqrt{3}} + (-\frac{d}{\sqrt{3}})\}/2, (0+0)/2) = (0, 0)$, O is the midpoint of (3).

Example 3.2. $y^2 = x^3 - 25x$. *SING*'s are $(\pm\frac{5}{\sqrt{3}}, 0)$.

We visualise this example as follows:

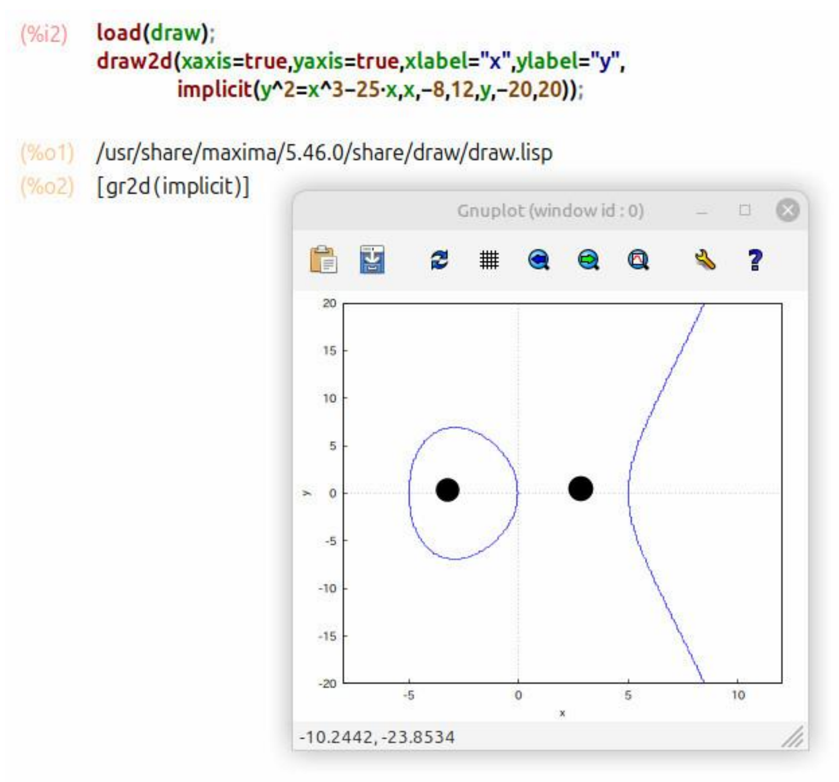


Fig. 1. Visualisation of *Example* 3.2. Black dots indicate its *SING*'s [3].

[3] wxMaxima ver. 24.02.1 is used for this kind of visualisation, and yielded images are sometimes retouched by appropriate software *e.g.*, GIMP ver. 2.10.36.

We classify this example into the categories **IN** and **OUT**, according to the whereabouts of its *SING*'s [4] .

4 Visualising 'non-EC's' with *SING*'s

Talking of the roots of the RHS of (1), when an EC is written as $y^2 = f(x)$, $f(x)$ doesn't have MR by definition . By the way, there seem some curves which are related to, but not regarded as EC's, since their RHS's have MR's. We call such curves 'non-EC's' tentatively, and for the sake of comparison, we visualise some of them with *SING*'s.

Example 4.1. $y^2 = (x+1)^2(x-1)$.

Remark 4.2. $x = -1$ is MR of the above example.

First, we try visualising the example.

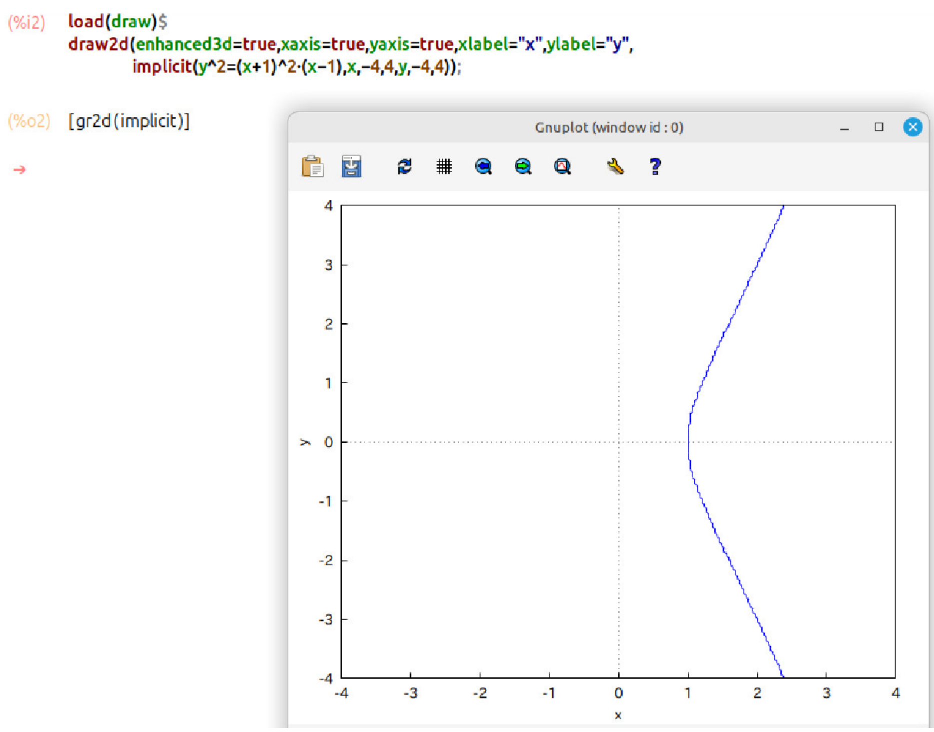


Fig. 2. Visualisation of *Example* 4.1

[4] *Cf.* [4, **Table**].

Clearly and unfortunately, the point $(-1, 0)$ [5] is missing for some reason in the above figure [6,7]. So we add it to the figure [8]:

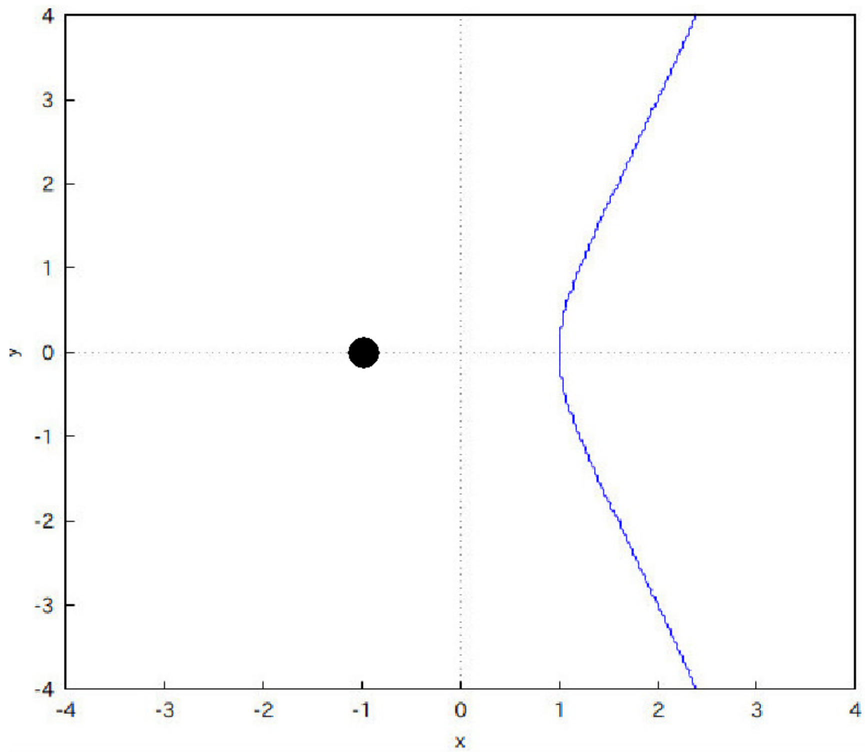


Fig. 3. Fig. 2 after the addition of the point $(-1, 0)$, which is indicated by a black dot.

Next, we compute *SING* as before. Since

$$\frac{d\phi}{dx} = \frac{d}{dx}\{(x+1)^2(x-1) - y^2\} = (x+1)(3x-1) - 2y\frac{dy}{dx},$$

we get the 1-form $\omega = d\phi = (x+1)(3x-1)dx - 2ydy$. Then, we set

$$\begin{cases} (x+1)(3x-1) = 0, \\ -2y = 0, \end{cases}$$

[5]See *Remark* 4.2.
[6]*Cf.* [6, Figure 12].
[7]There might be technical reason(s) behind this phenomenon.
[8]*Cf.* footnote 3.

which we solve to get $(x, y) = (-1, 0)$ and $(\frac{1}{3}, 0)$. We plot these points on the above figure, such points being indicated by red crosses [9] .

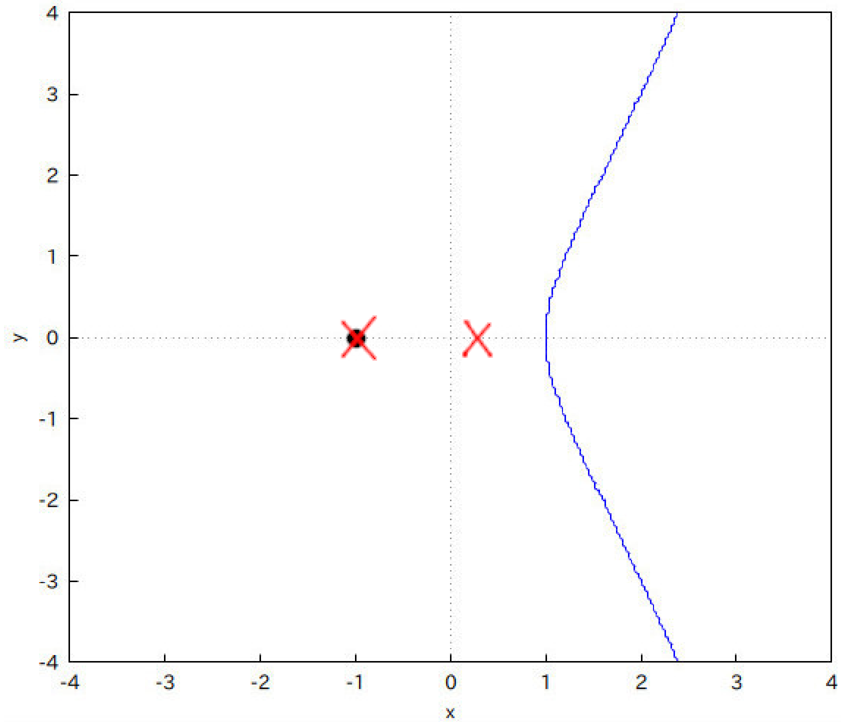


Fig. 4. Designating *SING*'s in Fig. 3

We classify this example into the categories **ON** and **OUT**, according to the whereabouts of its *SING*'s [10] .

Remark 4.3. A straightforward computation yields $(\frac{1\cdot(-1)+2\cdot 1}{2+1}, \frac{1\cdot 0+2\cdot 0}{2+1}) = (\frac{1}{3}, 0)$. So the right *SING* $(\frac{1}{3}, 0)$ can be regarded as the point dividing the line segment that joins the points $(-1, 0)$ and $(1, 0)$ in the ratio $2 : 1$ [11] .

Example 4.4. $y^2 = (x+1)(x-1)^2$.

Remark 4.5. $x = 1$ is MR of the above example.

[9] Ditto.
[10] See footnote 4.
[11] *Cf.* here .

Likewise, we visualise this example.

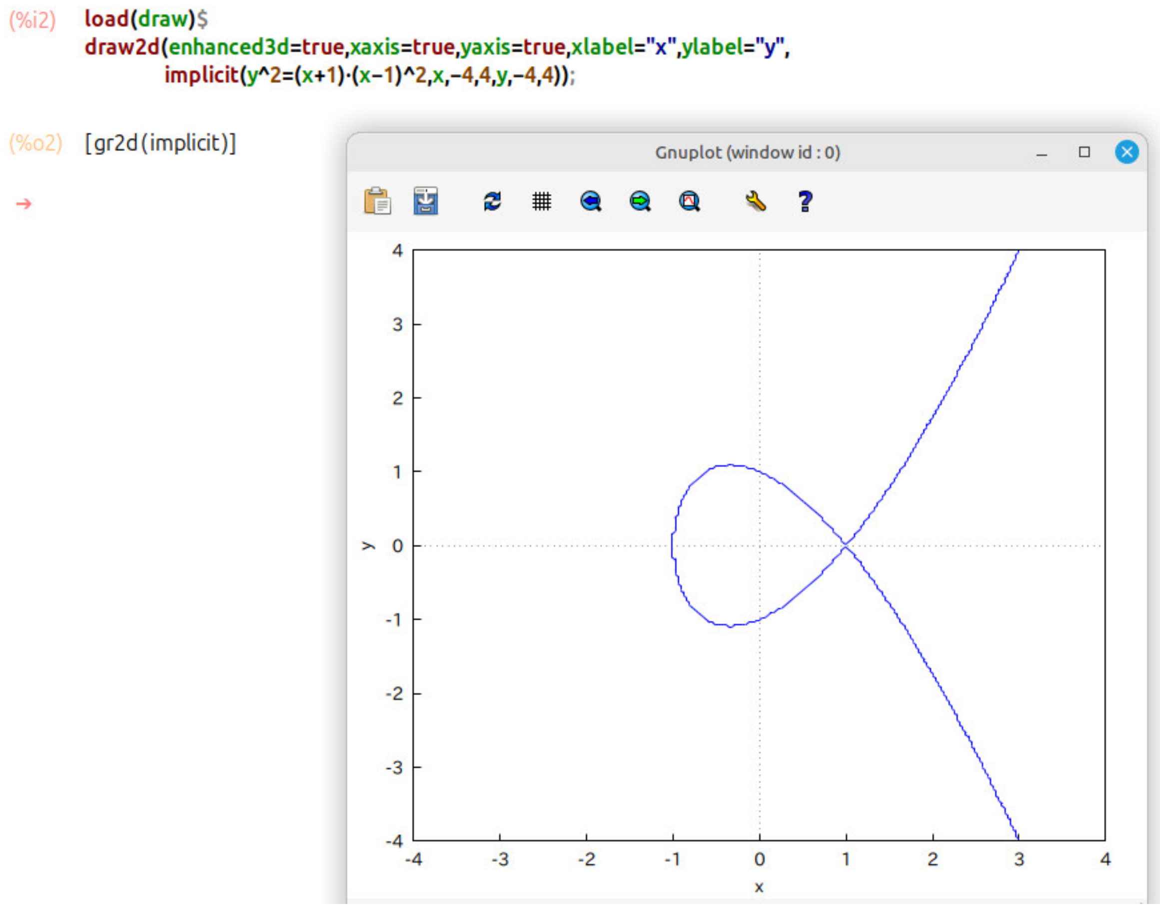


Fig. 5. Visualisation of *Example* 4.4

We compute *SING* as before. First,

$$\frac{d\phi}{dx} = \frac{d}{dx}\{(x+1)(x-1)^2 - y^2\} = (x-1)(3x+1) - 2y\frac{dy}{dx}.$$

Then, we get the 1-form $\omega = d\phi = (x-1)(3x+1)dx - 2ydy$. We set

$$\begin{cases} (x-1)(3x+1) = 0, \\ -2y = 0, \end{cases}$$

which we solve to get the points $(x, y) = (-\frac{1}{3}, 0)$ and $(1, 0)$. Indicating these points by black crosses, we plot them on the above figure:

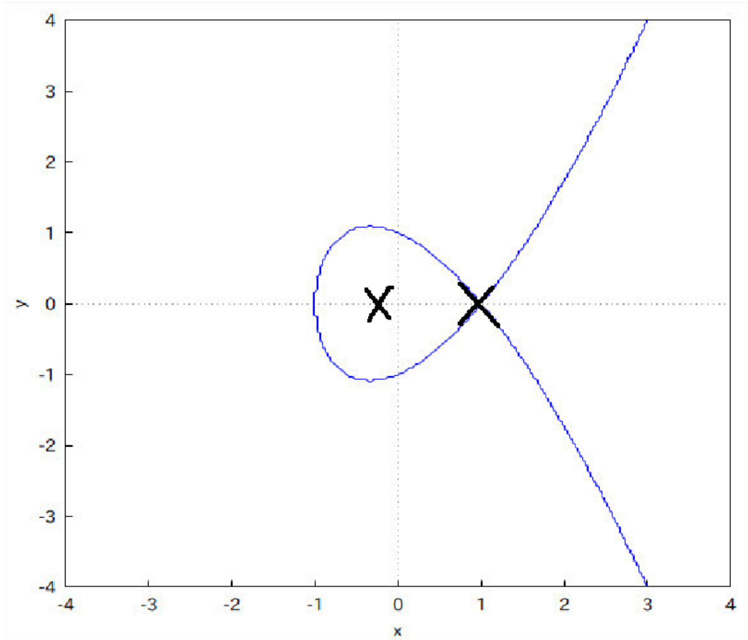


Fig. 6. Showing *SING*'s in Fig. 5

We classify this example into the categories **ON** and **IN**, according to the whereabouts of its *SING*'s [12].

Remark 4.6. A straightforward computation yields $(\frac{2\cdot(-1)+1\cdot 1}{1+2}, \frac{2\cdot 0+1\cdot 0}{1+2}) = (-\frac{1}{3}, 0)$. So the left *SING* $(-\frac{1}{3}, 0)$ can be regarded as the point dividing the line segment which joins the points $(-1, 0)$ and $(1, 0)$ in the ratio $1 : 2$ [13].

5 Implication of $\frac{1}{\sqrt{3}}$?

When we once dealt with *SING*'s, we encountered the points $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, etc [7, Fig. 4]. Taking this opportunity, we try to know whether such $\frac{1}{\sqrt{3}}$'s play some role in our current interest.

First, we notice that the points in (3) contain two $\frac{1}{\sqrt{3}}$'s, which leads us to imagine the following right triangle , where $\overline{\mathrm{AB}} = \sqrt{3}$, $\overline{\mathrm{BC}} = 1$, $\overline{\mathrm{CA}} = 2$.

[12] *Cf.* footnote 4.
[13] *Cf. Remark* 4.3.

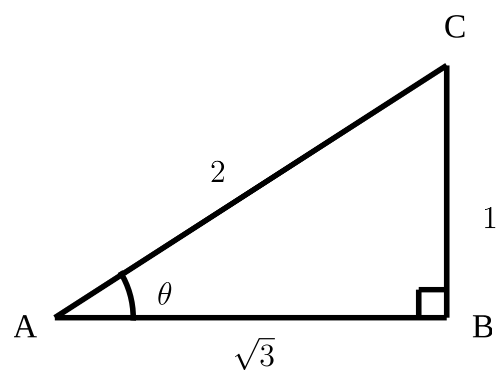


Fig. 7. A right triangle which is related to the value $\frac{1}{\sqrt{3}}$

Actually, $\tan\theta = \frac{\overline{\mathrm{BC}}}{\overline{\mathrm{AB}}} = \frac{1}{\sqrt{3}}$. And due to the Pythagoras' theorem , we have $\overline{\mathrm{AB}}^{\,2} + \overline{\mathrm{BC}}^{\,2} = \overline{\mathrm{CA}}^{\,2}$. Next, we rewrite $\overline{\mathrm{AB}}, \overline{\mathrm{BC}}$, and $\overline{\mathrm{CA}}$ as x, y, and z, respectively. Then, we have

$$x^2 + y^2 = z^2. \tag{4}$$

(4) is a kind of the double cone , a three-dimensional object. That is, such rewriting resulted in 'higher-dimensionalisation' of the two-dimensional object in Fig. 7. So we leave the Cartesian plane and mention *e.g.*,

$$(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \tag{5}$$

a point on $S^2 : x^2 + y^2 + z^2 = 1$. Multiplying (5) by $\sqrt{3}$ yields

$$(1, 1, 1). \tag{6}$$

We get inspired by (6) to write vector $\vec{\mathbf{n}} = (1, 1, 1)$ and let α, β, γ be angles between the vector and x, y, z-axes, respectively. Then,

$$\cos\alpha = \tfrac{1\cdot1+1\cdot0+1\cdot0}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+0^2+0^2}} = \tfrac{1}{\sqrt{3}}, \qquad \cos\beta = \tfrac{1\cdot0+1\cdot1+1\cdot0}{\sqrt{1^2+1^2+1^2}\sqrt{0^2+1^2+0^2}} = \tfrac{1}{\sqrt{3}},$$

$$\cos\gamma = \tfrac{1\cdot0+1\cdot0+1\cdot1}{\sqrt{1^2+1^2+1^2}\sqrt{0^2+0^2+1^2}} = \tfrac{1}{\sqrt{3}}\ ^{14}\ .$$

We have thus seen $\frac{1}{\sqrt{3}}$'s again and again. At this point, these values might seem a miscellany of coincidence, but they will prove to have some implication in the following arguments (hopefully).

[14] *Cf.* here .

6 Identity-based decomposition of GCM

Inspired by (6) again, we imagine the line $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$, whose direction vector is $(1,1,1)$. This line passes through O and can be obtained from

$$(x-y)^2+(y-z)^2+(z-x)^2=0 \tag{7}$$

$[4$, **Discussion**$]$. Using the LHS of (7), we introduce the following identity.

$$2x^2+2y^2+2z^2-2xy-2yz-2zx=(x-y)^2+(y-z)^2+(z-x)^2. \tag{8}$$

Each side of (8) is rewritten as follows.

$$\text{LHS}= x(2x-y-z)+y(-x+2y-z)+z(-x-y+2z)=(x\ \ y\ \ z)\left(\begin{array}{c} 2x-y-z \\ -x+2y-z \\ -x-y+2z \end{array}\right)$$

$$=(x\ \ y\ \ z)\left(\begin{array}{ccc} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{array}\right)\left(\begin{array}{c} x \\ y \\ z \end{array}\right),$$

and

$$\text{RHS}= (x-y\ \ y-z\ \ z-x)\left(\begin{array}{c} x-y \\ y-z \\ z-x \end{array}\right)=(x\ y\ z)\left(\begin{array}{ccc} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right)\left(\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{array}\right)\left(\begin{array}{c} x \\ y \\ z \end{array}\right).$$

Since LHS amounts to RHS, we have

$$\left(\begin{array}{ccc} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{array}\right)=\left(\begin{array}{ccc} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right)\left(\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{array}\right).$$

We notice that the LHS of the above is a GCM. So use of a certain identity resulted in the decomposition of GCM $A=\left(\begin{array}{ccc} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{array}\right)$, which is also an L-matrix , into the product of $B=\left(\begin{array}{ccc} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right)$ and $C=\left(\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{array}\right)$. And we point out the following.

Property 6.1. $\det A = 0$, so A is a SIM.

Property 6.2. $\mathrm{tr}A = 6$.

Property 6.3. $A^2 = 3A$.

Property 6.4. A is a SYM, since $A^{\mathrm{T}} = A$.

Property 6.5. B is a SIM, since $\det B = 0$.

Property 6.6. $\mathrm{tr}B = 3$.

Property 6.7. $B^{\mathrm{T}} = C$.

Property 6.8. Since $\det$ of a square matrix is the same as that of its transpose , $\det B = \det B^{\mathrm{T}} = \det C = 0$ [15] . So C is a SIM.

Property 6.9. Since a matrix and its transpose have the same tr , $\mathrm{tr}B = \mathrm{tr}B^{\mathrm{T}} = \mathrm{tr}C = 3$ [16] .

Property 6.10. $CB = BC(= A)$.

Property 6.11. $B + C = A$.

Property 6.12. $\det(B + C) = \det B + \det C$.

Here we get interested in whether other ways of decomposition *e.g.*, $A = DE$, $A = FG$, $\ldots$, and so on are possible. We note that from $(x-y)^2+(y-z)^2+(z-x)^2 = (y-x)^2+(y-z)^2+(z-x)^2$

$$= (y-x \quad y-z \quad z-x)\begin{pmatrix} y-x \\ y-z \\ z-x \end{pmatrix} = (x \;\; y \;\; z)\begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

we obtain the decomposition

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

[15] See *Properties* 6.5 and 6.7.

[16] See *Properties* 6.6 and 6.7.

Likewise, from $(x-y)^2+(y-z)^2+(z-x)^2=(x-y)^2+(y-z)^2+(x-z)^2$

$$= (x-y \quad y-z \quad x-z)\begin{pmatrix} x-y \\ y-z \\ x-z \end{pmatrix} = (x \;\; y \;\; z)\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

we obtain the decomposition

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix},$$

and from $(x-y)^2+(y-z)^2+(z-x)^2=(y-z)^2+(x-z)^2+(x-y)^2$, the decomposition

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = PP^{\mathrm{T}}.$$

And we point out the following.

Property 6.13. $P+P^{\mathrm{T}} \neq A$ [17] .

Property 6.14. $\det P = \det P^{\mathrm{T}} = 0$ [18] . So P and P^{T} are SIM's.

Property 6.15. $\mathrm{tr}P = \mathrm{tr}P^{\mathrm{T}} = 0$ [19] .

Property 6.16. $P^3 = P$.

Since we have often seen matrices where $\mathrm{tr} \neq 0$, *Property* 6.15 intrigues us in a sense, which leads us to compute

$$P^{\mathrm{T}}P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix},$$

[17] *Cf. Property* 6.11.
[18] *Cf. Property* 6.8.
[19] *Cf. Property* 6.9.

for that matter; we point out the following.

Property 6.17. $P^{\mathrm{T}}P$ is a SYM, since $(P^{\mathrm{T}}P)^{\mathrm{T}} = P^{\mathrm{T}}P$.

Property 6.18. $P^{\mathrm{T}}P \neq PP^{\mathrm{T}}(= A)$ [20] .

7 Implication of CHT?

Inspired by *Properties* 6.1, 6.2, and so on, we refer to CHT, which states that for a 3×3 matrix A, we have

$$A^3 - (\mathrm{tr}A)A^2 + \frac{1}{2}\{(\mathrm{tr}A)^2 - \mathrm{tr}A^2\}A - (\det A)I_3 = O_3. \tag{9}$$

Substituting $\det A = \mathrm{tr}A = 0$ into the LHS of (9) yields

$$A^3 - \left(\frac{\mathrm{tr}A^2}{2}\right)A = O_3,$$

and characteristic polynomial of matrix A is

$$t^3 - \left(\frac{\mathrm{tr}A^2}{2}\right)t. \tag{10}$$

If we are allowed to replace t and $\frac{\mathrm{tr}A^2}{2}$ in (10) by x and d^2, respectively, we encounter $x^3 - d^2x$, the RHS of (1), which suggests that CHT has something to do with an EC. By the way, since *SING* computation of (1) yielded $\frac{1}{\sqrt{3}}$'s in (3), we will mention them in what follows.

7.1 Encountering $\frac{1}{\sqrt{3}}$ repeatedly

Now we should like to know whether such $\frac{1}{\sqrt{3}}$'s are related to P^{T}. Let us consider the following.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P^{\mathrm{T}} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} Y - Z \\ X - Z \\ X - Y \end{pmatrix}.$$

Substituting the above relation into $S^2 : x^2 + y^2 + z^2 = 1$, one gets

$$(Y - Z)^2 + (X - Z)^2 + (X - Y)^2 = 1, \tag{11}$$

which is visualised as follows.

[20] *Cf. Property* 6.10.

```
(%i2)  load(draw)$
       draw3d(proportional_axes=xyz,enhanced3d=true,
              xlabel="X",ylabel="Y",zlabel="Z",
              implicit((Y-Z)^2+(X-Z)^2+(X-Y)^2=1,X,-2,2,Y,-2,2,Z,-2,2));

(%o2)  [gr3d(implicit)]
```

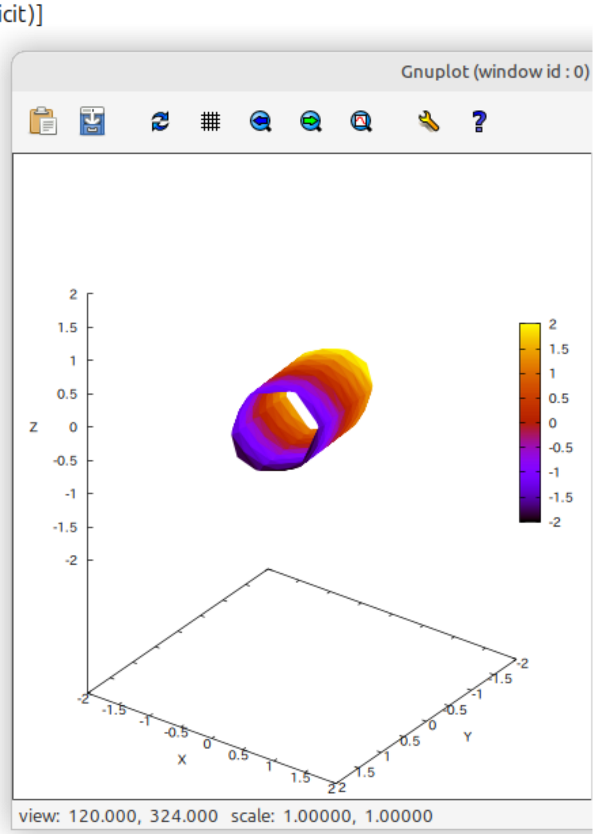


Fig. 8. Visualisation of (11)

What is visualised in the above figure looks cylindrical. To understand it more precisely, we introduce the following matrix.

$$Q = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} . \tag{12}$$

Property 7.1.1. Q is an orthogonal matrix, since $Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I_3$.

Property 7.1.2. Moreover, since $\det Q = 1$, Q is a special orthogonal matrix .

Remark 7.1.3. Q can be decomposed into RS, the product of

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}.$$

Property 7.1.4. Since $RR^{\mathrm{T}} = R^{\mathrm{T}}R = SS^{\mathrm{T}} = S^{\mathrm{T}}S = I_3$, and $\det R = \det S = 1$, both R and S are special orthogonal matrices [21] .

Seeing a few $\frac{1}{\sqrt{3}}$'s in (12), we use Q for relating (X, Y, Z) to (x, y, z) as follows.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = Q \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Explicitly,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}x}{\sqrt{3}} + \frac{z}{\sqrt{3}} \\ -\frac{x}{\sqrt{6}} + \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{3}} \\ -\frac{x}{\sqrt{6}} - \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{3}} \end{pmatrix}, \tag{13}$$

and we are ready for making a claim.

Claim 7.1.5. The stuff visualised in Fig. 8 is an open cylinder .

Proof. We substitute (13) into the LHS of (11) and compute

$\{-\frac{x}{\sqrt{6}} + \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{3}} - (-\frac{x}{\sqrt{6}} - \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{3}})\}^2 + \{\frac{\sqrt{2}x}{\sqrt{3}} + \frac{z}{\sqrt{3}} - (-\frac{x}{\sqrt{6}} - \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{3}})\}^2 + \{\frac{\sqrt{2}x}{\sqrt{3}} + \frac{z}{\sqrt{3}} - (-\frac{x}{\sqrt{6}} + \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{3}})\}^2 = (\sqrt{2}y)^2 + (\frac{3x}{\sqrt{6}} + \frac{y}{\sqrt{2}})^2 + (\frac{3x}{\sqrt{6}} - \frac{y}{\sqrt{2}})^2 = 3x^2 + 3y^2.$

So (13) transforms (11) into $3x^2 + 3y^2 = 1$, which is the same as

$$x^2 + y^2 = \frac{1}{3}. \tag{14}$$

Clearly, this is a circle in a Cartesian coordinate system , whose radius is $\frac{1}{\sqrt{3}}$. However, if we take for granted the three-dimensional [22] , (14) can be regarded as an open cylinder as shown below.

[21] *Cf. Property* 7.1.2.
[22] *Cf.* here .

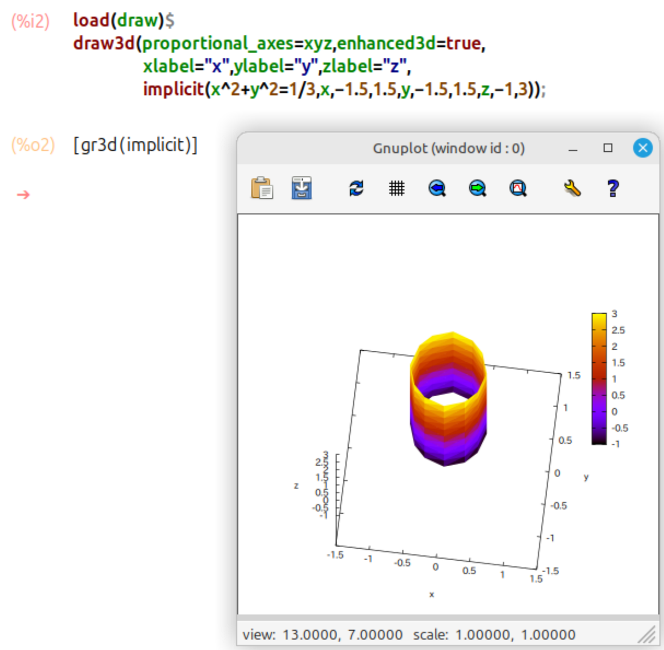


Fig. 9. Visualisation of (14)

Schematically,

$$\text{stuff in Fig. } 8 \xrightarrow{\text{some transformation}} \text{open cylinder in Fig. } 9.$$

Since the above transformation is dictated by a special orthogonal matrix [23] , and every rotation can be represented uniquely by an orthogonal matrix with unit determinant , the transformation is essentially a sheer rotation, which doesn't affect the so-called shape. So one can say the stuff in Fig. 8 has been transformed into another stuff without change of shape. Hence, the stuff in Fig. 8 is also an open cylinder .

$\square$

After all, we have encountered the values $\frac{1}{\sqrt{3}}$'s again and again, which is typically seen in this proof.

8 Discussion

As suggested in [7, **Discussion**], we have tried to get a deeper understanding of the BSD conjecture.

[23] *Cf. Properties* 7.1.2 and 7.1.4.

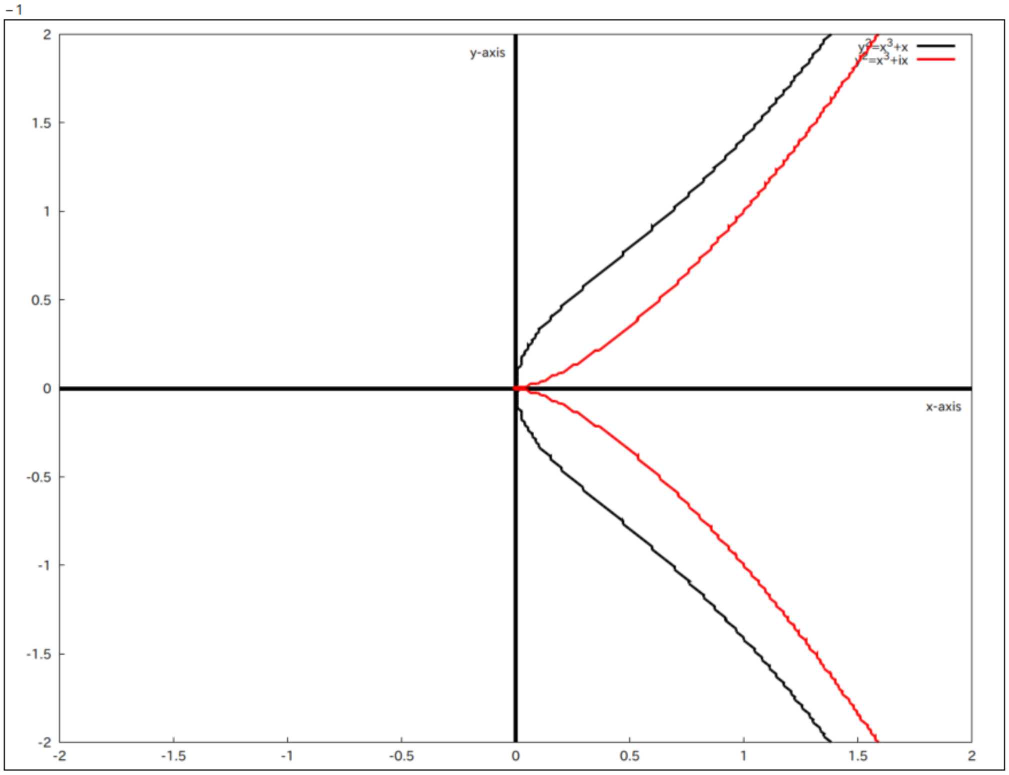
First, we should like to discuss the RHS of (1). We have implicitly assumed that $d \in \mathbb{N}$ [24] as in *Example* 3.2, where $d = 5$. But what if we go into $\mathbb{C}$? Here we recall the point $(-1, 0)$ in Fig. 3 to imagine it is a Gaussian integer . Specifically, the point is read as $-1 + 0 \cdot \imath$, which leads us to dare to replace some coefficients in the RHS's of EC's by pure imaginary number s. Here are two examples:

In the following figure, the RHS of $y^2 = x^3 + x$ (black curve) is read as $x^3 + 1 \cdot x$. Then, the coefficient 1 is replaced by $\imath$ and rewritten as $y^2 = x^3 + \imath x$ (red curve).

```
(%i2)   %i^2; /* This is confirmation.*/
        wxdraw2d(color=black,key="y^2=x^3+x",line_width=3,implicit(y^2=x^3+x,x,–2,2,y,–2,2),
                  color=red,key="y^2=x^3+ix", /* 'i' denotes the imaginary unit. See Glossary.*/
                  line_width=3,implicit(y^2=x^3+%i·x,x,–2,2,y,–2,2),
                  xaxis=true, yaxis=true,xaxis_type=solid,yaxis_type=solid,
                  xaxis_width=5, yaxis_width=5,
                  label_alignment=left,color=black,label(["x-axis",1.8,–0.1]),label(["y-axis",–0.2,1.9]));

(%o1)   –1
```

(%t2)



(%o2)

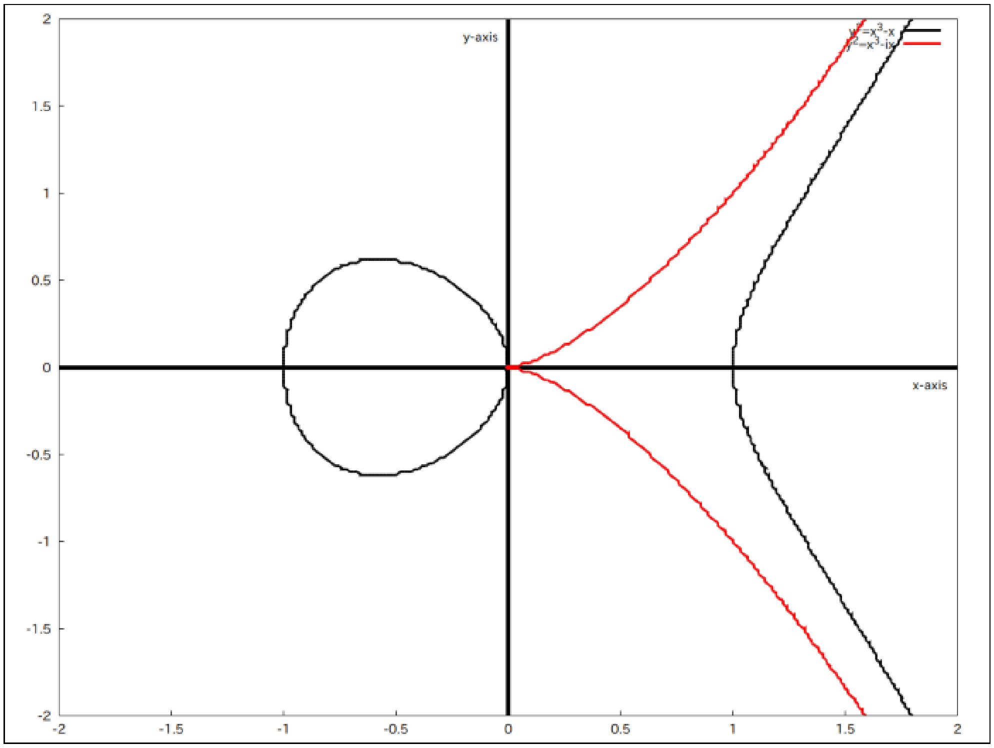
Fig. 10. Outcome of replacement of ‘$+x$’ by ‘$+\imath x$’

Likewise, $y^2 = x^3 - x$ (black curve) is rewritten as $y^2 = x^3 - \imath x$ (red curve) and visualised.

[24] *Cf.* [3, **Figure 1.2**].

```
(%i1) wxdraw2d(color=black,key="y^2=x^3-x",line_width=3,implicit(y^2=x^3-x,x,-2,2,y,-2,2),
          color=red,key="y^2=x^3-ix",line_width=3,implicit(y^2=x^3-%i·x,x,-2,2,y,-2,2),
          xaxis=true,yaxis=true,xaxis_type=solid,yaxis_type=solid,
          xaxis_width=5,yaxis_width=5,
          label_alignment=left,color=black,label(["x-axis",1.8,-0.1]),label(["y-axis",-0.2,1.9]));
```

(%t1)



(%o1)

Fig. 11. Outcome of replacement of '$-x$' by '$-\imath x$'

It might be noteworthy that in Fig.'s 10 and 11, (rather) simple replacement in the RHS's gave rise to cusp -looking points on O. Actually, such points came from the arguments about an acnode , which is shown as the point $(-1, 0)$ in Fig. 3, and we wonder whether the so-called acnode (and perhaps crunode as well) needs treating like statistical outlier (s) in discussing the BSD conjecture.

As for the EC in Fig. 1, *Example* 3.2 has been classified into the categories **IN** and **OUT** simultaneously; we recall we have once seen this kind of simultaneity [7, **Discussion**]. On the other hand, we now see **ON** and **OUT** in Fig. 4, **ON** and **IN** in Fig. 6. So the difference between EC and 'non-EC' seems to lie in the absence/presence of the category **ON** in terms of *SING*.

Talking of identity-based decomposition of GCM, GCM A has been decomposed into the product of B and C, where $B \neq C$. So A is the product of two different matrices, neither of which is a SYM. Incidentally, it is known that a GCM can be written as the product of a diagonal matrix and SYM . So the decomposition we have described might be of some additional interest. Moreover, *Property* 6.11 means that A can be expressed as the sum of B and C. Thus, A can be written not only as the product of two matrices, but also as the sum of the same matrices. Taken together, we have

$$A = BC = B + C. \tag{15}$$

Since a matrix decomposition or matrix factorisation is a factorisation of a matrix into a product of matrices , (15) too might be of some additional interest.

And it follows from *Properties* 6.7 and 6.10 that

$$BB^{\mathrm{T}} = B^{\mathrm{T}}B = A.$$

Since A is a GCM, calling matrices such as B and B^{T} pre-GCM's, we classify them into two categories in terms of commutative property of multiplication . That is,

Category 1: commutative pre-GCM.

Examples. B and C [25] .

Category 2: non-commutative pre-GCM.

Examples. P and P^{T} [26] .

As for *Property* 6.16, we discuss it in terms of whether

$$P^2 = P \tag{16}$$

holds.

Case 1. $P^2 = P$. Multiplying both sides of (16) by P, one gets $P^3 = P^2$. Since $P^2 = P$ in this case, $P^3 = P$ is almost trivial.

Example 8.1. $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, $T^3 = T^2 = T$.

Case 2. $P^2 \neq P$. In this case, $P^3 = P$ is not so trivial as *Case* 1.

Example 8.2. $U = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & -2 \end{pmatrix}$. Then, $U^2 = \begin{pmatrix} -1 & -4 & -1 \\ 0 & 1 & 0 \\ 2 & 4 & 2 \end{pmatrix} \neq U$, and $U^3 = U$ [27] .

By the way, a square matrix P is called a projection matrix, if $P^2 = P$ [28] , and it has been shown that *Property* 6.16 comprises both $P^2 = P$ and $P^2 \neq P$. So one might be able to say *Property* 6.16 is an 'extension' of the condition $P^2 = P$.

With regard to possible implication of CHT, we recall $\det A = 0$ [29] , $\mathrm{tr}A = 6$ [30] and compute

[25] See *Properties* 6.10 and 8.10.
[26] See *Property* 6.18.
[27] U can be slightly generalised. See **9.2**.
[28] *Cf.* [8].
[29] See *Property* 6.1.
[30] See *Property* 6.2.

$\mathrm{tr}A^2 =$ [31] $\mathrm{tr}\,3A = 18$. So using (9), we obtain $A^3 - 6A^2 + 9A = O_3$, which we view as analogous to

$$x^3 - 6x^2 + 9x = 0.$$

LHS of this equation can be factored as $x(x-3)^2$. Furthermore, since $A^2 = 3A$ [32], we have $A^3 - 3A^2 = O_3$ and thus $A^3 - 3 \cdot 3A = A^3 - 9A = O_3$. That is, we can also consider $x^3 - 3x^2 = 0$ and $x^3 - 9x = 0$, whose LHS's are factored as $x^2(x-3)$ and $(x-3)x(x+3)$, respectively. So using CHT, one can derive the RHS's of the equations

1. $y^2 = x(x-3)^2$,
2. $y^2 = x^2(x-3)$,
3. $y^2 = (x-3)x(x+3)$ [33]

from a GCM. 1 and 2 are 'non-EC's', whereas 3 is an EC. To sum up, arguments about GCM and CHT we have so far made [34] don't seem pointless from a viewpoint of the BSD conjecture, to which an EC is relevant (hopefully).

By the way, what if we 'extend' (7) to consider

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = r^2, \tag{17}$$

where $r \in \mathbb{R}$? We make the following definition.

Definition 8.3. $Cyl(r) := (x-y)^2 + (y-z)^2 + (z-x)^2 - r^2$.

Remark 8.4. Setting $Cyl(1) = 0$ yields $(x-y)^2 + (y-z)^2 + (z-x)^2 = 1$, which is essentially the same as (11).

Now let r tend to 0. In other words, we consider $\lim_{r \to 0} Cyl(r)$. Then, we get $Cyl(0) = (x-y)^2 + (y-z)^2 + (z-x)^2$, the LHS of (17). Equating this with 0, we get (7) and thus the line $x = y = z$ [4, **Discussion**]. Hence, $Cyl(r)$ was made to 'shrink' to become a line, which leads us to raise the following questions.

Question 8.5. Does this kind of reasoning have something to do with non-contractibility ?

Question 8.6. Can deriving a line from such 'shrinking' offer something significant to the very concept of cylinder and/or line?

Unfortunately, we find difficulty in answering these questions at the time of writing, but it might be worth trying to answer them.

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[31] See *Property* 6.3.
[32] Ditto.
[33] *Cf.* [9, FIGURE 4.3].
[34] See **6** and **7**.

We compute $\frac{B^3}{3} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = V$, for that matter, and point out the following.

Property 8.7. $\det V = 0$.

Property 8.8. $\mathrm{tr} V = 0$.

These properties remind us of *Properties* 6.14 and 6.15; matrices whose $\det$ and tr are 0 are not unusual. For example, see [10, *Remarks* 3.4 and 3.5]. Besides, we refer to the Sobel filter [11] that uses the matrices

$$W = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad X = \begin{pmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix},$$

where $\det W = \mathrm{tr} W = \det X = \mathrm{tr} X = 0$. We further point out the following.

Property 8.9. $V^{\mathrm{T}} = -V$. So V is skew-symmetric .

Property 8.10. $VV^{\mathrm{T}} = V^{\mathrm{T}}V = A$. So V and V^{T} exemplify *Category* 1, commutative pre-GCM.

Now we try applying 'trinions' (t_r's) [12] to (1). Let x, y be

$$x = x_1 + x_2 i + x_3 j, \quad y = y_1 + y_2 i + y_3 j, \quad x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \ \ [7, (12)]. \tag{18}$$

For convenience, below is MT of t_r's [12, **Table 1**].

$\times$	1	i	j
1	1	i	j
i	i	0	0
j	j	0	0

Using (18) and this MT, one gets

$$\begin{cases} x^3 = x_1^3 + 3x_1^2(x_2 i + x_3 j), & [7,\ (13)] \\ y^2 = y_1^2 + 2y_1(y_2 i + y_3 j). \end{cases}$$

Substituting these into (1), one gets

$$y_1^2 + 2y_1(y_2 i + y_3 j) = x_1^3 + 3x_1^2(x_2 i + x_3 j) - d^2(x_1 + x_2 i + x_3 j).$$

After some computation, this becomes

$$y_1^2 - x_1^3 + d^2 x_1 + (2y_1 y_2 - 3x_1^2 x_2 + d^2 x_2)i + (2y_1 y_3 - 3x_1^2 x_3 + d^2 x_3)j = 0.$$

So we have

$$\begin{cases} y_1^2 = x_1^3 - d^2 x_1, & (19) \\ 2y_1 y_2 = 3x_1^2 x_2 - d^2 x_2, & (20) \\ 2y_1 y_3 = 3x_1^2 x_3 - d^2 x_3. & (21) \end{cases}$$

(19) is essentially the same as (1). Considering the coordinate (x_1, x_2, y_1, y_2), one can regard (20) as something four-dimensional. (21) can also be regarded as something four-dimensional by considering the coordinate (x_1, x_3, y_1, y_3). Moreover, if we go so far as to 'merge' those coordinates to consider $(x_1, x_2, x_3, y_1, y_2, y_3)$, we will be faced with something six-dimensional. Given that we didn't present a single counterexample to the BSD conjecture, let alone its proof, it seems sensible to refrain from going too far for the moment, though one might be able to imagine four-dimensional version of the conjecture in case it should be wrong.

Finally, we should like to touch on TPC. Letting p, $g \in \mathbb{N}$ be a prime and prime gap , respectively, we state our current view on twin prime pairs :

$$\begin{cases} (p, p+2): \text{ intractable in terms of proof}, \\ (p, p+3): \text{ watershed of tractability}, \\ (p, p+g), \text{ where } g \geq 4: \text{tractable in terms of proof}. \end{cases}$$

Concretely, the above 'watershed' is $(2, 5(= 2+3))$, where 2 is a prime . Since 5 is the midpoint of 2 and 8 in $\mathbb{R}^1$, we recall (3) and *Remark* 3.1 to regard 2 and 8 as '*SING*'s' in $\mathbb{N}$ in a similar manner. Thus, we have drawn a parallel between the BSD conjecture and TPC. Of course, our arguments leading to such a parallel have been at best speculative. Nonetheless, we wonder if they can be regarded as a deeper understanding of the BSD conjecture suggested in [7, **Discussion**].

Acknowledgment. We should like to thank the developers of GIMP, PostScript , and wxMaxima for their indirect help, which enabled us to prepare figures for submission.

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9 Appendix

9.1 ‘Quasi-EC’s?

In what follows, we show a few things we have recently run across; they look relevant to our current interest for some reason.

Firstly,

$$\frac{1}{\arcsin(\frac{1}{z})} = 1, \quad \text{where } z = -x + \imath y, \tag{22}$$

which is visualised as follows.

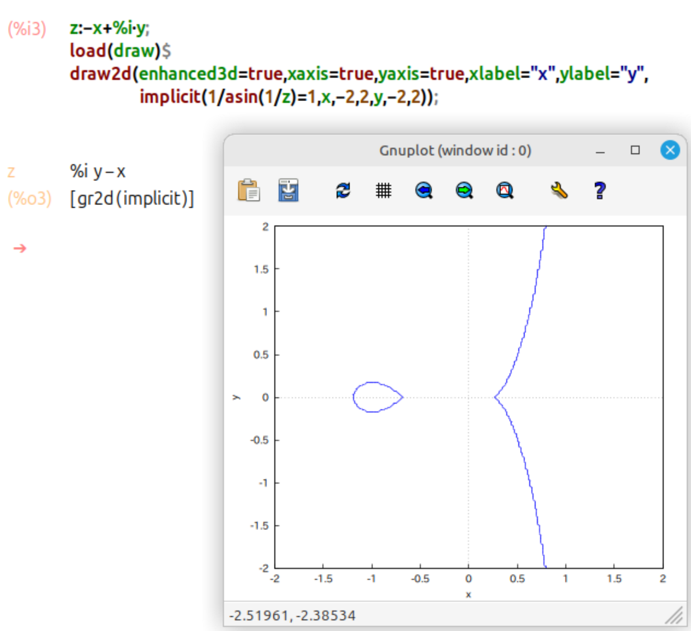

Fig. 12. Something that seems a ‘quasi-EC’

Some might feel like referring to teardrop curve , while others might get reminded of Fig. 1.

Secondly,

$$z = \arcsin(z^2), \quad \text{where } z = -x + \imath y, \tag{23}$$

which is visualised as follows.

```
(%i3)  z:-x+%i·y;
       load(draw)$
       draw2d(enhanced3d=true,xaxis=true,yaxis=true,xlabel="x",ylabel="y",
              implicit(z=asin(z^2),x,-2,2,y,-2,2));
z      %i y-x
(%o3)  [gr2d(implicit)]
```

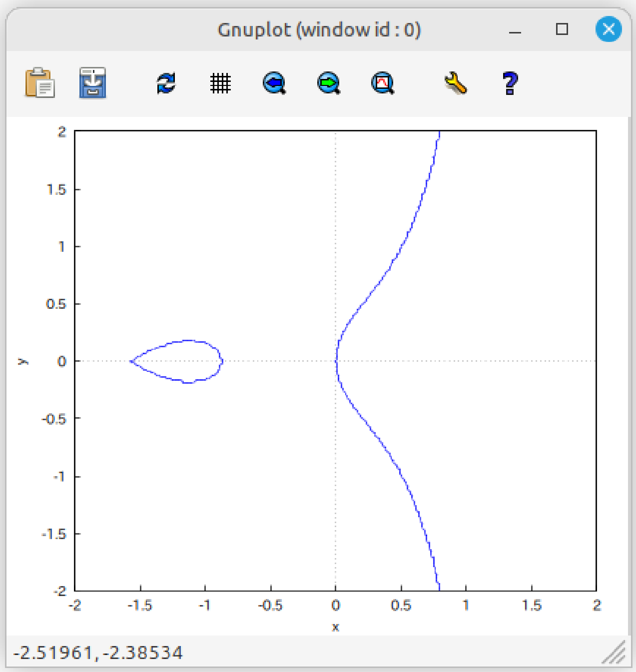

Fig. 13. Another thing that also seems a 'quasi-EC'

Some might point out a cusp -looking point around $(-1.5, 0)$, whereas others might get reminded of Fig. 1.

Thirdly,

$$\arcsin(\frac{1}{z}) = 1, \quad \text{where } z = -x + \imath y, \tag{24}$$

which is also visualised:

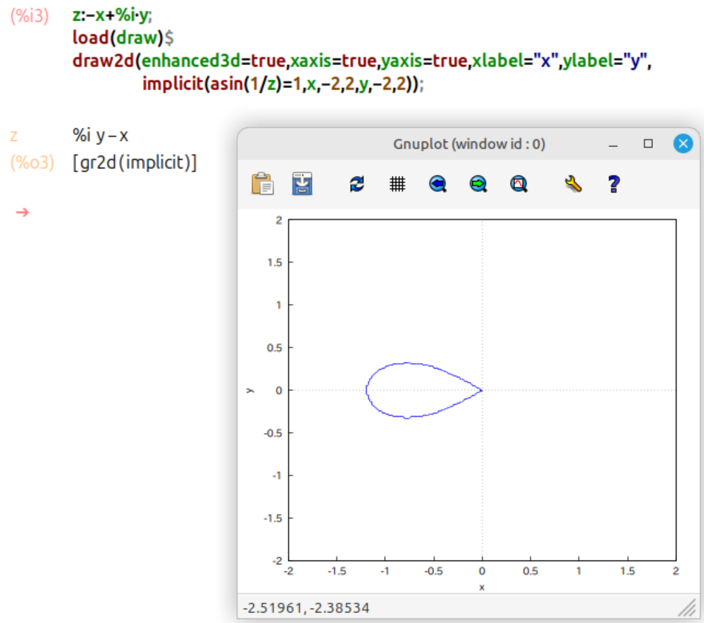

Fig. 14. Something that seems a part of 'quasi-EC'

Some might recall cubic egg , whereas others a 'fragment' of cycloid of Ceva . That said, we are not sure whether we can locate the *SING*'s of $(22) - (24)$ in Fig.'s $12 - 14$ at the time of writing.

9.2 Slight generalisation of *Example* 8.2

Let $Y = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ -2 & a & -2 \end{pmatrix}$, where $a \in \mathbb{R}$. Then, $Y^2 = \begin{pmatrix} -1 & a-4 & -1 \\ 0 & 1 & 0 \\ 2 & 4-a & 2 \end{pmatrix} \neq Y$, and Y^3

$= \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ -2 & a & -2 \end{pmatrix} = Y$. Thus, *Example* 8.2 has been slightly generalised [35] .

[35]In other words, U is a special case of Y.