Non commutative Fourier duality on one-dimension and its applications

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Abstract

The present article aims to present an extension of the Fourier duality to non-commutative groups and algebraic structures,analyzing the non-commutative Fourier transform on one dimensional structures along with their unitary representation. This approach gives a comprehensive insight into the harmonic analysis of operator-valued functions, providing mathematical foundations for analyzing physical systems that exhibit non-commutative symmetries. Hence the article discusses applications of non-commutative harmonic analysis into emerging fields of physics such as quantum mechanics and models of quantum gravity , opening paths for exploration of the connections between non-commutative algebras, harmonic analysis, and theoretical Physics.

1 Introduction

The study of classical Fourier analysis is the first step one should take before proceeding in the study of non commutative Fourier Duality , since it is an extension of the Classical version of Fourier analysis to non-abelian groups; i.e groups in which operations do not commute. Thus , in this introductory sessions is given a brief summary on the Classical Fourier Transforms and its application in physics.

1.1 What is a Duality ?

The concepts of Duality is of major interest in many areas of Mathematics from Category theory and Functional Analysis to Algebraic and Geometric Topology , because it facilitates the study of structures, due to isomorphism[1] that establishes between them.

Definition: Given two Categories A and B , the duality between them is a contravariant equivalence F : A $\to$ B reversing arrows.Duality can be also

[1]In simple words , an isomorphism is an invertible linear map

defined as a contravariant functor that establishes isomorphism of categories with reversed morphisms[5].However , is important recognize that the Concepts of Duality is not limited only to Categories but can be applied to a broader range of mathematical structures such as spaces and groups. Thus , in a broader sense Duality can be thought of as the correspondence between two mathematical concepts , theories,theorems or objects such that one domain can be expressed into the other , preserving arrows or morphisms[11].

In a broader sense duality can be thought of as correspondence between theorems, concepts,structures,theories or problems , so that properties applied in one domain translates mutually into the other, preserving arrows or morphisms.This intuitive concepts of Duality is an extension of the formal definition stated in the first paragraph , since it it is applied only when the given structure and its dual are categories , suggesting that many dualities between structures , corresponds to pairing or bilinear functions from a given family of scalars[11]. Additionally, this analogy gives rise to an important observation , that across mathematics we can find diverse types of dualities , which are named after the kind of structures that they map , for example in Linear Algebra duality is viewed as a bilinear maps between vectors spaces to scalars, on the other hands , there are types of dualities such as the Poincaré duality that is viewed as pairings between a submanifold of a given manifold[2] , open the idea that the concepts of duality may also be applied to geometries.

Examples :

1. As a concrete example let us analyze as special type of duality that frequently appear in mathematical analysis a , they are called **"Analytic dualities"** which is widely used to solve problems by implementing a dual description of functions and operators.A Classical case of an analytic duality is the **Fourier transform** and the **Laplace transform**.

   The Fourier transform interchanges between functionals[3] on a vector space and its dual , that is :

$$\mathrm{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx$$

   And in the same way ,

$$\mathrm{f(x)} = \int_{-\infty}^{\infty} \mathrm{f}(\xi)\, \mathrm{e}^{-2\pi \mathrm{i} \mathrm{x}\xi}\, \mathrm{dx}$$

2. As mentioned earlier another classical example of duality is the Laplace transform, that interchanges between operators of multiplication and poly-

[2]"Intuitively , a manifold is a topological space that resembles the euclidean space near each point,it can also refer to a space that may be curved and have a complicated topology , but locally looks like $\mathrm{R^n}$"[3][15]

[3]Explicitly a functional is a specific kind of function , its description may vary from one field to another and depending on the author, for example in Linear Algebra the concepts of functional is quite distinct to that in Functional Analysis textbooks.

nomials with constant coefficient linear differential operators[13]. Mathematically the Laplace transform for a function f(s) is written as :

$$\mathrm{F(s)} = \int_{-\infty}^{\infty} \mathrm{f(u)\, e^{-su}\, du}$$

1.2 The Classical Fourier Transform on dual groups

In this session[4] the article discusses the properties of the classical Fourier transforms of functions on dual groups. It's a good time since we are already familiarized with the concept of duality. The first step is to understand when does a groups is the dual of the other group , to start let's now define the concepts of Character .

Character: Let's consider a complex function K , on a locally compact abelian group[5] G , we say that this function is a **character** on G if the following equations are satisfied :

- $|\mathrm{K(x)}| = 1$, for all $\mathrm{x} \in \mathrm{G}$;
- $\mathrm{K(x+y)} = \mathrm{K(x)K(y)}$,for all $\mathrm{x, y} \in \mathrm{G}$.

Now , let Γ define the group of all continuous character of G, then Γ is called a dual group of G,if for two characters $\mathrm{K_1, K_2} \in \Gamma$ the addition operation satisfies:

$$\mathrm{(K_1 + K_2)(x)} = \mathrm{K_1(x)K_2(x)}$$

, for all $\mathrm{x} \in \mathrm{G}$. In simple words , Γ is a group homomorphism under addition , since as the equations above indicates it preserves the group structure of its domain and codomain.

Theorem: Let $\mathrm{k} \in \Gamma$ be a character , such that :

$$\mathrm{f(\hat{k})} = \int_{\mathrm{G}} \mathrm{f(x)\, k(-x)\, dx} \quad , \mathrm{f} \in \mathrm{L^1(G)}$$

then the map $\mathrm{f} \to \mathrm{\hat{f}(x)}$ is a complex homomorphism of $\mathrm{L^1(G)}$ and is not identically zero. Conversely , every nonzero complex homomorphism in this group is obtained in this way , and distinct characters induces distinct homomorphisms.

[4]The main results of this sessions , including proofs and theorems can be found in [9], however in order to facilitate the understanding , some notations were adapted.

[5]A topological space X is said to be locally compact ,if for every point $\mathrm{x} \in \mathrm{X}$ there is an open neighborhood U, such that the closure $\bar{\mathrm{U}}$ is compact(that is for every cover there is a finite number of subcover)

Proof. Assume f, g two functions on $L^1(G)$, such that $f * g$ is their convolution defined by :

$$(f * g)(x) = \int_G f(x-y)g(y)dy \quad (1)$$

By definition for a character $k \in \Gamma$, $(f * g)(k)$, is defined to be :

$$(f * g)(k) = \int_G (f * g)(k).k(-x)\, dx \quad (2)$$

Substituting (1) in (2) , result in :

$$(f * g)(k) = \int_G \int_G f(x-y)g(y)\, k(-x)\, dydx$$

$$(f * g)(k) = \int_G k(-x)dx \int_G f(x-y)g(y)dy$$

From the construction of a Haar measure on G is known that every linear functional is translation invariant on $C_c(G)$, the space of all continuous complex functions on G with compact support, thus by definition of a character we can verify that $k(-x) \in C_c(G)$, thus the Haar measure suggests that $k(-x) = k(y-k) \quad \forall y \in G$, thus:

$$k(-x) = k(-y+y-k) = k(-y)\, k(y-k)$$

$$(f * g)(k) = \int_G g(y)\, k(-y)\, dy \int_G f(x-)y\, k(y-x)\, dx$$

since $\hat{g}(k) = \int_G g(y)\, k(-y)\, dy$ and $\hat{f}(k) = \int_G f(x-y)\, k(-x+y)\, dx$, thus

$$(f * g)(k) = \hat{g}(k)\, \hat{f}(k)$$

This suggests that the linear map $f \to \hat{f}(k)$ is multiplicative on Banach Algebra and thus is a homomorphism on $L^1(G)$. To prove the converse case,let's suppose that h is a nonzero complex homomorphism on $L^1(G)$, then h is a bounded linear functional of norm 1, so that :

$$h(f) = \int_G f(x)\phi(x)\, dx \quad f \in L^1(G) \quad (3),$$

now taking g a function on $L^1(G)$,let's define another nonzero complex homomorphism $h(g)$ on $L^1(G)$, defined as,

$$h(g) = \int_G g(y)\phi(y)\, dy \quad (4),$$

hence multiplying the right hand side of (3) and (4) , we will get :

$$\int_G f(x)\phi(x)\, dx \int_G g(y)\phi(y)\, dy = h(f)h(g) = h(f * g)$$

Defining $h(f * g)$:

$$h(f*g) = \int_G (f*g)(x)\phi(y)\,dx = \int_G g(y)\,dy \int_G f(x-y)\phi(x)\,dx = \int_G g(y)h(f_y)\,dy$$

Thus, $h(f)\phi(y) = h(f_y) \quad (5)$,

and choosing $x + y$ and f_x from (5) we will then have :

$$h(f)\phi(x + y) = h(f_{x+y}) = h(f)\phi(x)\phi(y) \quad (x, y \in G)$$

Since $|\phi(x)| \leq 1$ for all x and since the equation above implies that $\phi(-x) = \phi(x)^{-1}$, it follows that $|\phi(x)| = 1$, hence $\phi \in \Gamma$, finally, if $\hat{f}(k_1) = \hat{f}(k_2)$, the definition of $\hat{f}(k)$ implies that $k_1(-x) = k_2(-x), \forall x \in G$, and since k_1, k_2 are continuous , the equality holds so that $k_1 = k_2$. As desired ! □

Now based on the theorem stated above, we are ready to define the Fourier transform of a function $f \in L^1(G)$.

1.2.1 The Fourier transform

Definition: given a function $f \in L^1(G)$,the function $\hat{f}$ on Γ defined by :

$$\hat{f}(k) = \int_G f(x)\,k(-x)\,dx \quad ,\forall k \in \Gamma$$

, is called the Fourier transform of f.

Theorem : If G is discrete then Γ is compact,conversely if G is compact then Γ is discrete.

Proof. If G is compact and its Haar measure is normalized so that $m(G) = 1$, where $m(G)$ is the Haar measure of G,then $L^1(G)$ has a unit and its maximal ideal space Γ is therefore compact. To prove the converse case, let's assume that G is compact and its Haar measure is normalized so that $m(G) = 1$,and the orthogonality relation holds:

$$\int_G k(x)\,dx = \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \neq 0 \end{cases}$$

,if $k \neq 0 => k(x_o) \neq 1, \forall x \in G$ and

$$\int_G k(x)\,dx = k(x_o)\int_G k(x - x_o)\,dx = k(x_o)\int_G k(x)\,dx$$

,if $f(x) = 1$ for all $x \in G$, this implies that $f \in L^1(G)$, since G is compact and $\hat{f}(0) = 1$, on the other hand , if $\hat{f}(k) = 0$, since $\hat{f}$ is continuous the set consisting of 0 alone is open in Γ , hence Γ is discrete. As desired ! □

Classical groups of Fourier analysis:

1. The additive group $G = \mathbf{R}$ of the real numbers,with natural topology of the real line.And for this case, the Fourier transform is expressed by :

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)\, e^{-iyx}\, dx \quad , \forall\, y \in \mathbf{R}$$

2. The circle group $G = \mathbf{T}$, that is the multiplicative group of absolute value 1.In this case the Fourier transform is expressed by :

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta$$

3. The additive group $G = \mathbf{Z}$ of the integers, and in this case the Fourier transform is expressed as :

$$\hat{f}(e^{i\alpha}) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-in\alpha} \quad , (e^{i\alpha} \in \mathbf{T})$$

2 Applications of the Classical Fourier Duality

As every mathematical tool , the duality between functions that the Fourier transform establish, has important applications in the analysis and solution of problems in many fields of engineering, medicines and most importantly in classical and modern Physics,a common example is the solution of the Classical wave function in classical wave theory.In this session , is given a detailed demonstration of the applicability of the Classical Fourier duality on the additive group of real numbers, in Quantum mechanics, techniques of spectroscopy and signal processing , we will focus on both conceptual explanations and mathematical proofs,moreover, all in one-dimensional case.

1. **The classical Fourier transform in spectroscopy-**The classical Fourier transform on the additive group of real numbers , plays an important role in spectroscopical analysis , a common illustration of this is the "nuclear magnetic resonance "also denoted by N.M.R , in which an an exponential decay signal is "Fourier transformed" from the time domain to the frequency domain, moreover , it is also applied in "techniques of magnetic resonance imaging and spectrometry"[12].

2. **The classical Fourier transform in signal processing-** The classical Fourier transform is widely used in the processing and decomposition of signals, mainly in the decomposition of noisy input to deliver the desired speech signal.In this case , the input signal is "Fourier transformed" , in such a way that establishes a duality with the desired speech signal, and after that the inverse Fourier transform is then computed to recover the

optimal or desired speech signal, this process is often executed in computational time through algorithms , for example the fast Fourier transform (F.F.T) and the short time Fourier transform (S.T.F.T).A particular example of the short time Fourier transform used in the analysis of the frequency and phase content of local sections of signals as it changes over time [10], is given below, note that $w(t-\tau)$ denotes the window function τ is the time delay:

$$STFT[x(t)](\tau;\omega) \equiv X(\tau;\omega) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-iwt}dt$$

3. **The classical Fourier transform in Quantum Mechanics-** In Quantum Mechanics , the Fourier transform is used in the study of the state function of a particle following the principle of complementarity , which dictates the impossibility of know the spatial information and the momentum of a particle,such that the Fourier transform help us to state the duality between these two states function[12], so that if we assume a the spatial state variable and b the momentum state variable, we can relate the two states function of space and momentum by a Fourier transform in the form :

$$\phi(a) = \int \psi(b)e^{-iab/h}da; \psi(b) = \int \phi(a)e^{iab/h}db$$

This process of using Fourier methods to transform states from one frame to another, is the underlying foundation of the "Heisenberg uncertainty principle", and not just a convenient mathematical tool, showing the importance of the study of Fourier analysis in one-dimensional settings , when it comes with analysis the interplay between states functions in terms of one variable, as shown in the example above[12].Another example of the application of the classical Fourier duality, is regarding the solution of the Schrödinger's equation. In non relativistic quantum mechanics , the free particle wave equation is :

$$-i\hbar\frac{\partial\psi(x;t)}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2\psi(x;t)}{\partial x^2} \quad (1)$$

Below is given the solution of this equation by means of Fourier methods. Firstly let's determine the fourier transform of the wave function $\psi(x;t)$ and its inverse :

$$\tilde{\psi}(k;t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(x;t)e^{-ikx}dx$$

$$\psi(x;t) = \int_{-\infty}^{\infty} \tilde{\psi}(k;t)e^{ikx}dk \quad (2)$$

Now, let's Fourier transform the second order partial derivative of the wave function , note that the Fourier transform of the partial derivative is equivalent to multiplication in the Fourier space , that is :

$$\mathcal{F}\left[\frac{\partial^2\psi(x;t)}{\partial x^2}\right] = -k^2\psi(x;t)$$

Thus, from (1) we will get :

$$\frac{\partial\tilde{\psi}(k;t)}{\partial t} = -\frac{\hbar k^2}{2m}\tilde{\psi}(k;t)$$

Solving this ordinary differential equation :

$$\tilde{\psi}(k;t) = \tilde{\psi}(k;0)e^{-i\frac{\hbar k^2}{2m}t}$$

Where $\tilde{\psi}(k;0)$ is the initial wave function in momentum space, and substituting this solution in (2),we will get the solution to the free particle wave equation :

$$\psi(x;t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\psi}(k;0)e^{i(kx-w(k))t}dk \quad ,w(k) = \frac{\hbar k^2}{2m}$$

However, the equation (1) is only valid in the free particle case , so that in a more generalized sense , we have to include as well the potential energy function V,such that :

$$\frac{\partial\psi(x;t)}{\partial x} + V\psi(x;t) = -i\frac{\hbar}{2m}\frac{\partial^2\psi(x;t)}{\partial x^2}$$

Moreover , in relativistic Quantum Mechanics the Fourier Methods is widely used to solve the free one-dimensional wave equation also called Klein-Gordon-Schrodinger equation, a relevant equation in Quantum Field Theory in which each Fourier component wave can be thought of as separate harmonic oscillator and then quantized , in a process named "second quantization" [1]. This equation can be written as :

$$\left(\frac{1}{c^2}\right)\frac{\partial^2\psi(x;t)}{\partial t^2} - \frac{\partial^2\psi(x;t)}{\partial x^2} + \frac{m^2c^2}{\hbar^2}\psi(x;t) = 0$$

3 Transition to Non Commutative generalizations

With the realization that systems exhibiting non commutative symmetries fall outside the scope of the Abelian Fourier analysis, mathematicians and Physicists found necessary extends it to non commutative settings , motivated by the need to study quantum systems , operator algebras and more complex symmetries.Here we will explore the Non Abelian framework of the Fourier theory , with references to relevant results from representation theory,and its behavior on one-dimensional classical and quantized spaces, finally we will delve into the realm of applications in modern Physics.

Unlike in the classical Fourier duality , in which the duality is established between a space and its group of characters or one-dimensional irreducible representations , the non Abelian version presents the decomposition of a non Abelian group into a higher-dimensional irreducible representations , in such a way that the dual group is no longer a character , but instead a a space or category of representations[6]. Additionally, is important to notice that non commutative Fourier duality in one dimension does not exists , that's why the title of this article states clearly that even though there is not a one-dimensional version of the non Abelian Fourier duality, it can acts on one-dimensional classical or quantized spaces, through mechanisms from "Non commutative Algebras" and "Operator Theory" and "group actions", so that we are dealing with the "Non commutative Fourier Duality on one-dimension" and not the "Non commutative Fourier Duality in one-dimension",since it does not exists.

3.1 The unitary dual and Lie Algebras

Lie groups and their associated Lie Algebras along with the study of their unitary representations are the scaffolding for symmetry analysis in Mathematical Physics, facilitating not only the classification of systems but also the study of duality principles, particularly , the set of equivalence classes of irreducible representations(unitary dual),plays the role of analytic duality of a group allowing the study of generalizations such as the Non commutative Fourier Duality[6].That's why, is necessary a brief analysis of the Unitary Dual and the associated Lie Algebra.

1. **Lie Algebras:** is a vector space $\mathfrak{g}$ over a field $\mathbf{F}$ equipped with a binary operation called the Lie bracket[4][14] , and is denoted by : $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and satisfies the following axioms :

   - **Bilinearity:** $[ax + by, z] = a[x, z] + b[y, z], [z, ax + by] = a[z, x] + b[z, y] \quad \forall a, b \in F; x, y, z \in \mathfrak{g}$.
   - **Anti symmetry:** $[x, y] = -[y, x] \quad \forall x, y \in \mathfrak{g}$
   - **Jacobi identity:** $[x, [y, x]] + [z, [x, y]] + [y, [z, x]] = 0 \quad \forall x, y, z \in \mathfrak{g}$

   Intuitively, the Lie brackets measure non commutativity of their elements ,i.e how they fail to commute if we try to combine them in different orders, on the other hand , the Jacobi identity suggests that Lie Algebras are non associative , since it capture the idea of "Twisted associativity" , meaning that even in different orders the combinations of three elements still gives a consistent result.Another interesting result is the relation between a Lie group, that is a group that is also a differentiable manifold, and the respective Lie Algebra , such that a Lie Algebra can be considered as the infinitesimal version of a Lie group , meaning that given a smooth group of manifolds, the lie Algebra is the tangent space at the identity , capturing the possible small directions we can move near the identity[14].

Homomorphism of Lie algebras: is a linear map between two Lie Algebras , that preserves the internal structure of their Lie brackets,formally , let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie Algebras, the Lie algebras homomorphism is a linear map such that :

$$\varphi([a,b]) = [\varphi(a), \varphi(b)]$$

Lie algebras Representation: the representation of a Lie algebras $\mathfrak{g}$ is a vector space V equipped with a homomorphism of the Lie algebras $X = \mathfrak{g} \to EndV$, the latter is the collection of all linear maps from V to itself,and stands for endomorphism of V.Another relevant results in representation theory is the Universal enveloping Algebra of a Lie Algebra $\mathfrak{g}$, that is an associative algebra that encodes the representation theory of $\mathfrak{g}$ by enveloping it into an associative framework ,and is denoted by $U(\mathfrak{g})$. This exhibit a universal property , that suggests that any Lie Algebra homomorphism from $\mathfrak{g}$ to an associative algebra A equipped with a Lie bracket , extends uniquely an associative Algebra homomorphism from $U(\mathfrak{g})$ to A[4].

2. **The unitary dual:** given a topological space G over the Hilbert space H,the unitary dual of G is the space of equivalence classes of irreducible representation of G , and is denoted by $\hat{G}$, on the other hand,$\hat{G}$ is usually equipped with a topology and a Borel structure, that is a way to measure sets of representations , this allow harmonic analysis and integration over the dual. Additionally, the unitary dual can be considered a space where we can move around between different representations and study how they are related[2][4].

   Continuous unitary representation: a continuous unitary representation of a topological group G in the Hilbert space H, is a group homomorphism $\pi : G \to U(H)$, the group of unitary operator in H. Intuitively, a topological group is continuous if smooth changes in the elements of G induces smooth changes in their unitary operators on H , such that small changes in G leads to small changes on how vectors on G are transformed[4]. Unitary representations , plays a crucial role in Harmonic analysis and Quantum mechanics, since it allows the study on how functions on G can be decomposed into waves and in the construction of quantum systems with symmetry in G, in other words, how a function can be decomposed into contributions from each kinds of irreducible representations.

3.2 The non commutative Fourier transform

Let's define a Lie algebra $\mathfrak{g}$ and its associated Lie group G, and let's consider the quantization of the classical Poisson algebra $C^\infty(G \times \mathfrak{g}*)$, then in this setting there are a group representation π_G on $L^2(G)$, with elements of the quantized Poisson algebras that acts as multiplication and invariant differential operators , and a non commutative function space $L^2_\star(\mathfrak{g}*)$ with an algebra representation

, induced by a star product correspondent to the chosen quantization of the Poisson algebra.In this non commutative settings equipped with a star product we can then define the Fourier duality[6][8] .

Fourier transform : given the representations π_G on $L^2(G)$ and $\pi_{\mathfrak{g}*}$ on $L^2_\star(G)$, the unitary map $\mathcal{F} : \pi_G \to \pi_{\mathfrak{g}*}$ is the Fourier transform in a non commutative sense, and is defined by integration against a non commutative plane wave $E(X)$, and is written :

$$\tilde{\psi}(X) = \int_G E(X)\psi(g)\, dg$$

The duality established between the non commutative plane wave and the functions on the quantized Poisson algebras , is then the non commutative Fourier duality in this settings, and unlike in the classical Fourier duality, in the non commutative case both multiplication and differentiation are deformed by the non commutative structure encoded by means of star product , such that the duality is now between the Lie group and the non commutative dual algebra through a generalized exponential wave , aware of quantization and symmetries. Let's now , examine the action of this duality on one-dimensional classical space through the idea of phase space and on one-dimensional quantized spaces[8].

Phase space: is a n-dimensional space in which each point of coordinate represents a state of the system at a given time[7].In some sense, unlike the physical space where a given state can be just described by positions or a unique system of coordinates components , the phase space include all systems of coordinates, such that in the case of a given physical system the phase space encodes both positions and momenta in association with those positions.The phase space has plays a crucial role in the system dynamics , since by knowing both positions and momenta of the system ,specifies its states and allow us to predict the system's future evolution. Formally , given the space M of all possible positions , the configuration space,the phase space is represent by the cotangent bundle T^*M, and is defined as : $T^*M = \{(a,b)|a \in M, b \in T^*_a M\}$, where a, b represent positions and momentum respectively. Additionally, the phase space has a geometrical arrangement that dictates how positions and momenta interact in the system dynamics, this geometric structure is called "simplectic geometry".

A space is said to be quantized , when the the classical idea of space as a continuum is replaced by the structures in which variables or systems of coordinates do not commute in the usual sense and they have a discrete or quantized aspect.

A classical example of a configuration space is the real line , in which the phase space is the cotangent bundle $T^*R = R \times R$. In this space the functions usually plays the role of position and momentum of the physical system they represent, moreover, both the functions on positions and functions on momentum , are obtained by means of Group representations as stated earlier, the following result gives the group representation of the function on momentum $\tilde{\psi}(p)$ and the function on position $\psi(x)$ for a given physical system in the phase space[7].

$$\begin{cases}(\pi_p(p)\tilde{\psi})(p) = p\tilde{\psi}(p) \\ (\pi_p(x)\tilde{\psi})(p) = i\partial_p\tilde{\psi}(p)\end{cases} \quad , \quad \begin{cases}(\pi_x(x)\psi)(x) = x\psi(x) \\ (\pi_x(p)\psi)(x) = -i\partial_x\psi(x)\end{cases}$$

In this one-dimensional phase space, the Fourier duality , is expressed as a self dual with commutativity throughout , and the plane wave function is simply a character of the abelian translation group, this duality is defined as :

$$\tilde{\psi}(p) = \int_R \psi(x)\, e^{-ipx}\, dx$$

Now, let's analyze the case of a one-dimensional quantized space with configuration space the circle $U(1)$ and group elements parametrized by an angle θ , such that wave functions are usually represented by $\psi(\theta)$, on the other hand, operators in the quantized space act via multiplications and differentiation[6]. In this setting the Fourier duality uses deformation quantization , and is expressed as :

$$\tilde{\psi}(x) = \int_{U(1)} E_{e^{i\theta}}(x)\psi(x)\, d\theta$$

3.3 The Fourier duality in Theoretical Physics

The non Abelian version of the Fourier duality , briefly analyzed in the previous session,is widely used in Theoretical Physics to study,which is the subject of this session, here we will see how the Fourier transform appear in many areas of modern physics from Quantum mechanics to Quantum field theory in non commutative spaces.

1. **Quantum Mechanics on Lie groups:** in Quantum mechanics , the non commutative Fourier duality allow generalization to systems where configuration spaces are non Abelian Lie groups ,as a results a in this system the quantum states can be represented explicitly not only in group variables but also in duals non commutative algebras variables. An example of this , is the case of path integrals for group valued quantum systems , where the system is represented by non commutative momentum variables ,allowing stationary phase analysis even in the case of discrete spectra. Additionally, the the introduction of non commutative algebras in quantum mechanics, allow a more detailed analysis and interpretations of quantum corrections, that are explored in path integrals and semiclassical limits[8].

2. **Quantum gravity:** in loop quantum gravity and , non commutative dualities play a crucial role in the study of the geometric content, since it resides in the Lie algebras,for example in metric representations where the variable corresponds directly to geometric structures rather than abstract group representations.Additionally , the non commutative generalization appear widely in classical limits and geometricity , particularly in the study of semiclassical limits and classical models of space time geometry , through direct analysis of the path integral over metric data , enhancing the understanding of classical gravity from quantum models[8].

3. **Quantum field theory on non commutative spaces:**In Quantum field theory, the non commutative Fourier duality along with the associated star product structure , allows the analysis of field behavior that can not be studied by means of classical model of geometry and algebras[8].

4 Conclusion

From the framework built in this article, is evident that the non commutative Fourier duality acts as a generalization of the classical Fourier duality, grounded in Lie groups and their associated Lie algebras , deformation quantization and harmonic analysis on non commutative settings, providing more sophisticated mathematical tools for models in modern physics , from the exact definition of quantum systems and models of loop quantum gravity , to quantum field theory on non commutative settings, the duality also provides essential backgrounds in the analysis of semiclassical limits and the spacetime geometry , thus establishing connections between classical theories and quantum mechanics.

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