Born Reciprocal (Non-inertial) Relativity, Phase Space Trajectories and Strings with variable Tension

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Abstract

Pursuing further our work on Born Reciprocal (Non-inertial) Relativity theory we find that the family of hyperbolic trajectories of particles living in a four-dim phase space of signature $(2,2)$ automatically satisfy the maximal speed of light and the maximal proper force conditions. A direct link between the maximal proper force b and the minimum value of the initial positions $(x_o)_{min}$ of these hyperbolic paths is found. Furthermore, when the maximal proper force is set to be equal to the Planck-mass squared $b = m_P^2$ then $(x_o)_{min} = 2GM$ happens to coincide with the numerical value of the horizon radius of the Schwarzschild black hole in four spacetime dimensions. We proceed with a rigorous analysis of how open strings sweeping a two-dim world-sheet region inside a Rindler wedge in a $D = 1+1$ spacetime can be seen as a continuum of point masses that are being collectively accelerated along the Rindler hyperbolic trajectories. The upshot of this analysis reveals that these strings can naturally acquire a *variable* tension which is consistent with the dynamical tension model of strings and branes proposed by Guendelman in the past years [9].

Keywords : Born Reciprocal Relativity; Non-inertial Relativity; Phase Space; Strings; variable Tension string/brane models.

1 Spacetime Trajectories

We shall begin by studying the behavior of the $D = 1+1$ spacetime trajectories of a massive particle of proper (rest) mass m subjected to a variable proper acceleration $g(\tau)$. The units are chosen to be $\hbar = c = 1$. After defining

$$u_i \equiv \frac{dx_i}{d\tau}, \quad \dot{u_i} \equiv \frac{du_i}{d\tau} = \frac{d^2 x_i}{d\tau^2}, \quad i = 0, 1 \tag{1}$$

where the derivatives are taken with respect to the proper time τ, let us find solutions to the equations

$$(u_0)^2 - (u_1)^2 = 1, \quad (\dot{u_0})^2 - (\dot{u_1})^2 = - g^2(\tau), \tag{2}$$

corresponding, respectively, to the normalization condition on the timelike velocity of a massive particle, and subjected to a variable spacelike proper acceleration $g(\tau)$, in a $D = 1 + 1$ spacetime of signature $(+, -)$. Solving u_0 in terms of u_1 yields $u_0 = \sqrt{1 + (u_1)^2}$ in the first equation. Substituting this value into the second equation leads to the differential equation

$$\frac{(du_1/d\tau)^2}{1 + (u_1)^2} = g^2(\tau) \tag{3}$$

and whose solution is given by

$$u_1 = \frac{dx}{d\tau} = \sinh\left(\int g(\tau) d\tau\right) \Rightarrow x(\tau) = \int_0^\tau \sinh\left(\int_0^{\tau'} g(\tau'') d\tau''\right) d\tau' \tag{4}$$

The solution for $t(\tau)$ turns out to be

$$t(\tau) = \int_0^\tau \cosh\left(\int_0^{\tau'} g(\tau'') d\tau''\right) d\tau' \tag{5}$$

One can verify by a mere *substitution* that the solutions in eqs-(4,5) for $t(\tau), x(\tau)$ obey

$$(\frac{d^2 t}{d\tau^2})^2 - (\frac{d^2 x}{d\tau^2})^2 = -(g(\tau))^2 \tag{6}$$

for *all* functions $g(\tau)$.

In the special case that $g(\tau) = g_o$ constant, the integrals (4,5) lead to

$$t(\tau) = \frac{1}{g_o} \sinh(g_o \tau), \quad x(\tau) = \frac{1}{g_o} \cosh(g_o \tau) \tag{7}$$

and one recovers the hyperbolic trajectories $x^2 - t^2 = (g_o)^{-2}$ associated with the motion of a massive particle subjected to a uniform proper acceleration g_o.

In the next section we shall analyze the phase space trajectories after implementing the maximal proper force condition $|mg(\tau)| \leq b$. The proper force is bounded by the physical numerical constant b of units of $(mass)^2$, and is associated with the construction of the Born Reciprocal Relativity Theory (Noninertial Relativity) in phase space involving x, t, p, E [2], [3], [4], [5], [7].

One can enforce the maximal proper force condition by writing

$$\frac{F(\tau)}{b} = \frac{mg(\tau)}{b} = \tanh[\xi(\tau] \Rightarrow g(\tau) = \frac{b}{m} \tanh[\xi(\tau)] \tag{8}$$

where $\xi(\tau)$ is any arbitrary function of the proper time τ and is associated with a variable force-boost rapidity. Inserting $g(\tau)$ given by eq-(8) into eqs-(4,5) yields

$$t(\tau) = \int_0^\tau \cosh\left(\int_0^{\tau'} \frac{b}{m}\tanh[\xi(\tau''] \, d\tau''\right) d\tau' \tag{9}$$

$$x(\tau) = \int_0^\tau \sinh\left(\int_0^{\tau'} \frac{b}{m}\tanh[\xi(\tau'')] \, d\tau''\right) d\tau' \tag{10}$$

One can verify that the velocity never exceeds the speed of light $c = 1$

$$v = \frac{dx}{dt} = \frac{(dx/d\tau)}{(dt/d\tau)} = \tanh\left(\int_0^\tau \frac{b}{m}\tanh[\xi(\tau'')] \, d\tau''\right) \tag{11}$$

Because the hyperbolic tangent function is bounded between 1 and -1, from eq-(11) one learns that $|v| = |(dx/dt)| \leq 1$ for all the expressions provided by $\xi(\tau')$.

We shall examine 3 cases :

Case **1** . When $\xi(\tau) = \xi_o$ is constant one arrives at the solutions described by eqs-( ) with $g_o = \frac{b}{m}\tanh(\xi_o)$.

Case **2** . When $\xi(\tau) = g_o\tau$ is linear in τ, and when $mg_o = b$ coincides with the maximal proper force, after some algebra and neglecting integration constants, one arrives at

$$t(\tau) = \frac{1}{2g_o} \left(\sinh(g_o\tau) + arctan[\sinh(g_o\tau)] \right) \tag{12}$$

$$x(\tau) = \frac{1}{2g_o} \left(\sinh(g_o\tau) - arctan[\sinh(g_o\tau)] \right) \tag{13}$$

Note the $different$ functional behavior of the solutions (12,13) compared to the hyperbolic trajectories. The solutions provided in (12,13) implement the maximal force condition $|mg(\tau)| \leq b$ by construction.

Case **3** . When $\xi(\tau)$ is any arbitrary function one ends up with

$$t(\tau) = \int_0^\tau \cosh[\Omega(\tau')] \, d\tau', \quad x(\tau) = \int_0^\tau \sinh[\Omega(\tau')] \, d\tau',$$

$$\Omega(\tau') \equiv \int_0^{\tau'} \frac{b}{m}\tanh[\xi(\tau'')] \, d\tau'' \tag{14}$$

Having explored the trajectories in a $D = 1+1$ spacetime in the next section we shall extend our analysis to the four-dim phase space of signature $(2,2)$ associated with a $D = 1+1$ spacetime.

2 Phase Space Trajectories

The principle behind the concept of "Born reciprocal relativity theory", or non-inertial relativity to be more precise[1], was advocated by [2], [3], [4] and it was based on the idea proposed long ago by [1] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. Hence, a *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [4]. The principle of maximal acceleration was advocated earlier on by [6].

The generalized velocity and force (acceleration) boosts (rotations) transformations of the $flat$ $8D$ Phase space coordinates , where $X^i, t, E, P^i; i = 1,2,3$ are **c** -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dt \wedge dE + \delta_{ij} dX^i \wedge dP^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $flat$ $8D$ phase-space

$$(d\omega)^2 \ = \ c^2 (dt)^2 \ - \ (dX)^2 \ - \ (dY)^2 \ - \ (dZ)^2 \ + $$
$$\frac{1}{b^2} \ \left((dE)^2 \ - \ c^2 (dP_x)^2 \ - \ c^2 (dP_y)^2 \ - \ c^2 (dP_z)^2 \right) \tag{1.1}$$

The maximal proper force is set to be given by b. The symplectic group is relevant because $U(1,3) = Sp(8,R) \cap O(2,6)$; $U(3,1) = Sp(8,R) \cap O(6,2)$, and $U(2,2) = Sp(8,R) \cap O(4,4)$.

These transformations can be $simplified$ drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \ = \ c^2 (dt)^2 \ - \ (dX)^2 \ + \ \frac{(dE)^2 \ - \ c^2 (dP)^2}{b^2} =$$
$$(d\tau)^2 \ \left(1 \ + \ \frac{(dE/d\tau)^2 \ - \ c^2 (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \ \left(1 \ - \ \frac{F^2}{F^2_{max}} \right), \ F_{max} = b \tag{1.14}$$

The phase space infinitesimal interval in $c = 1$ units is

$$(d\omega)^2 \ = \ (dt)^2 \ - \ (dX)^2 \ + \ \frac{(dE)^2 \ - \ (dP)^2}{b^2} =$$
$$(d\tau)^2 \ \left(1 \ + \ \frac{(dE/d\tau)^2 \ - \ (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \ \left(1 \ - \ \frac{F^2}{F^2_{max}} \right), \ F_{max} = b \tag{15}$$

[1] We thank one of the referees of a previous article for highlighting this fact in order to clarify the point that Born did not propose a reciprocal relativity theory

where one has factored out the non-vanishing proper time infinitesimal $(d\tau)^2 = dt^2 - dX^2 \neq 0$ in (15). The numerical quantity F^2 is positive by definition. The proper force on a massive particle is given by $F = ma$, where a is the proper acceleration and m is the rest mass. We refrained from factoring out $(dt)^2$ in (15) because it is not Lorentz invariant, whereas $(d\tau)^2$ is Lorentz invariant.

It is very important to emphasize that there are *no* factors of $(1 + F^2/b^2)$ appearing in the above factorization process because in the superluminal case $(d\tau)^2 < 0$ (spacelike interval) one still has $m^2 a^2 < 0$ despite that $a^2 > 0$ (timelike proper acceleration), because $m^2 < 0$ due to the *imaginary* mass of tachyons. When $(d\tau)^2 > 0$ (timelike interval) $\Rightarrow m^2 a^2 < 0$ since $a^2 < 0$ (spacelike proper acceleration) and $m^2 > 0$. Hence we shall always have the factor $(1 - F^2/b^2)$ as expected with $m^2 a^2 = -F^2 < 0$. This is also compatible with the fact that if $(d\tau)^2 > 0$, then $(dE)^2 - (dP)^2 < 0$, and vice versa, if $(d\tau)^2 < 0$, then $(dE)^2 - (dP)^2 > 0$. An exception occurs for a free particle (or at rest) giving $(dE)^2 - (dP)^2 = 0$ because $dE = dP = 0$.

The same results occur in higher dimensions. Given $E^2 - \vec{p} \cdot \vec{p} = E^2 - |\vec{p}|^2 = m^2 > 0$ when $(d\tau)^2 > 0$ gives upon differentiation
$\frac{dE}{d|\vec{p}|} = \frac{|\vec{p}|}{E} < 1$, and resulting in $(dE)^2 - d\vec{p} \cdot d\vec{p} = E^2 - |d\vec{p}|^2 < 0$. And vice versa, in the case of $E^2 - \vec{p} \cdot \vec{p} = E^2 - |\vec{p}|^2 = m^2 < 0$ (tachyon), when $(d\tau)^2 < 0$, gives upon differentiation $\frac{dE}{d|\vec{p}|} = \frac{|\vec{p}|}{E} > 1$, and resulting in $(dE)^2 - d\vec{p} \cdot d\vec{p} = E^2 - |d\vec{p}|^2 > 0$.

Consequently, the *negative* sign appearing inside the parenthesis in the last term of eq-(15) furnishes the analog of the Lorentz relativistic factor in special relativity and it involves the ratio of the square of two *proper* forces. The result (15) in the 4-dim phase space can be generalized to the $8D$-dim phase space (and to higher dimensions)

In the case of hyperbolic trajectories associated with the motion of a massive particle subjected to a uniform proper acceleration g_o , the phase space infinitesimal interval (15) is given by $d\omega = \sqrt{1 - \frac{m^2 g_o^2}{b^2}} d\tau$. Due to the $U(1,1)$ invariance of $d\omega$, under force boosts transformations one has the relation

$$d\omega \ = \ \sqrt{1 - \frac{(m g_o)^2}{b^2}} \ d\tau \ = \ \sqrt{1 - \frac{(m' g'_o)^2}{b^2}} \ d\tau' \qquad (16)$$

which reveals that $F = m g_o \neq F' = m' g'_o$ and $\tau \neq \tau'$. Namely, the proper force $m g_o$ and the proper time τ are *not* fully $U(1,1)$ invariant, they are $(SO(1,1))$ Lorentz invariant. The explicit transformations relating $m' g'_o$ with $m g_o$, and relating τ' with τ, in terms of the force-boost rapidity parameter ξ_a can be found in [5].

Therefore, ω is the truly $U(1,1)$-invariant evolution parameter that must be used in order to describe the phase space trajectories of a particle instead of the proper time τ which is *not* $U(1,1)$ invariant. Consequently, instead of having the expressions associated to hyperbolic trajectories

$$t(\tau) \ = \ \frac{1}{g_o} \ \sinh(g_o \tau), \quad x(\tau) \ = \ \frac{1}{g_o} \ \cosh(g_o \tau) \qquad (17)$$

$$p(\tau) = m\frac{dx}{d\tau} = m\sinh(g_o\tau), \quad E(\tau) = m\frac{dt}{d\tau} = m\cosh(g_o\tau) \tag{18}$$

one must have expressions defined in terms of ω as follows

$$Z_I(\omega) \ \equiv \ \{ \ t(\omega), \ x(\omega), \ \frac{1}{b}E(\omega), \ \frac{1}{b}p(\omega) \ \}, \quad I = 1,2,3,4 \tag{19}$$

The analog of the timelike velocity condition $V_\mu V^\mu = 1$ in special relativity is now given in phase space by

$$\dot{Z}_I \ \dot{Z}^I \ = \ (\frac{dt}{d\omega})^2 \ - \ (\frac{dx}{d\omega})^2 \ + \frac{1}{b^2} \ (\frac{dE}{d\omega})^2 \ - \frac{1}{b^2} \ (\frac{dp}{d\omega})^2 \ = \ 1 \tag{20}$$

The extension to phase space of the spacelike condition on the proper acceleration in spacetime is

$$\ddot{Z}_I \ \ddot{Z}^I \ = \ (\frac{d^2t}{d\omega^2})^2 \ - \ (\frac{d^2x}{d\omega^2})^2 \ + \frac{1}{b^2} \ (\frac{d^2E}{d\omega^2})^2 \ - \frac{1}{b^2} \ (\frac{d^2p}{d\omega^2})^2 \ = \ -\mathcal{A}^2(\omega) \tag{21}$$

where $\mathcal{A}(\omega)$ is the phase space analog of the proper spacetime acceleration and must not be confused with $g(\tau)$. Given $\dot{Z}_I\dot{Z}^I = 1$, a differentiation yields $\ddot{Z}_I\dot{Z}^I = 0$, consequently the phase space analog of the proper acceleration is orthogonal to the phase space analog of velocity, hence if the velocity is timelike, the acceleration is spacelike, and vice versa.

A thorough study of the spacelike $(d\omega)^2 < 0$, null $(d\omega)^2 = 0$, and timelike $(d\omega)^2 > 0$ intervals in phase space, and their relation to the intervals $(d\tau)^2$ in spacetime, can be found in [5] where it was shown in many examples that there are *no* crossovers in the spacetime intervals when one performs a force boost transformation for any value of the force boost rapidity ξ_a rapidity parameter. Consequently, one has that if $(d\tau)^2 > 0 \Rightarrow (d\tau')^2 > 0$. And viceversa, if $(d\tau)^2 < 0 \Rightarrow (d\tau')^2 < 0$.

Before we discuss the energy and momentum it is very important to invoke the construction of the quadratic Casimir invariants of the Quaplectic group studied by Low [2], [3]. The Quaplectic group in four spacetime dimensions (eight phase space dimensions) is the semi-direct product of $U(1,3)$ with the translations in phase space and including the unit central element associated with the Weyl-Heisenberg algebra. The relevance of the quadratic Casimir is that it correctly defines the analog of proper mass in phase space $\mathcal{M}$. Therefore, upon defining

$$\mathcal{M}\frac{dt}{d\omega} \ = \ E, \ \ \mathcal{M}\frac{dx}{d\omega} \ = \ p \tag{22}$$

where $\mathcal{M}$ is the $U(1,1)$-invariant proper mass in phase space, and which is not the same as m, the two eqs-(20,21) become a system of two simultaneous differential equations for the two functions $E(\omega), p(\omega)$ given by

$$(\frac{E}{\mathcal{M}})^2 \ - \ (\frac{p}{\mathcal{M}})^2 \ + \frac{1}{b^2} \ (\frac{dE}{d\omega})^2 \ - \frac{1}{b^2} \ (\frac{dp}{d\omega})^2 \ = \ 1 \tag{23}$$

$$(\frac{1}{\mathcal{M}})^2\ (\frac{dE}{d\omega})^2\ -\ (\frac{1}{\mathcal{M}})^2\ (\frac{dp}{d\omega})^2\ +\frac{1}{b^2}\ (\frac{d^2E}{d\omega^2})^2\ -\frac{1}{b^2}\ (\frac{d^2p}{d\omega^2})^2\ =\ -\mathcal{A}^2(\omega) \quad (24)$$

In the $b \to \infty$ limit, eq-(23) becomes $E^2 - p^2 = \mathcal{M}^2$, and one recovers the on-shell condition since $\mathcal{M} \to m$ in the $b \to \infty$ limit. Whereas eq-(24) becomes the spacelike proper force squared as it occurs in special relativity since $\omega \to \tau$ and $\mathcal{A}(\omega) \to g(\tau)$

$$(\frac{dE}{d\tau})^2\ -\ (\frac{dp}{d\tau})^2\ =\ -(mg(\tau))^2 \quad (25)$$

Let us find a one-parameter family of solutions to eqs-(23,24) parametrized by the parameter κ after setting $\mathcal{A} = \mathcal{A}(\kappa)$ to be a one-parameter family of accelerations *independent* of the phase space evolution parameter ω. A careful inspection reveals that a one-parameter family of solutions to eqs-(23,24) is given by

$$t(\omega;\kappa)\ =\ \frac{\kappa}{\mathcal{A}(\kappa)}\ \sinh[\mathcal{A}(\kappa)\ \omega],\quad x(\omega;\kappa)\ =\ \frac{\kappa}{\mathcal{A}(\kappa)}\ \cosh[\mathcal{A}(\kappa)\ \omega] \quad (26)$$

$$E(\omega;\kappa)\ =\ \kappa\ \mathcal{M}\ \cosh[\mathcal{A}(\kappa)\ \omega],\quad p(\omega;\kappa)\ =\ \kappa\ \mathcal{M}\ \sinh[\mathcal{A}(\kappa)\ \omega] \quad (27)$$

where κ is a numerical parameter.

From eq-(26) one infers that as $\omega \to \infty$ the particle reaches the speed of light $\frac{dx}{dt} = \frac{(dx/d\omega)}{(dt/d\omega)} = \tanh[\mathcal{A}(\kappa)\omega] \to 1$ $(c = 1)$.

Inserting the solutions given by eq-(27) into eqs-(23,24) lead to the relations

$$\frac{M^2\mathcal{A}^2(\kappa)}{b^2}\ =\ 1 - \frac{1}{\kappa^2},\quad \mathcal{M}\ \mathcal{A}(\kappa)\ =\ b\ \sqrt{1 - \frac{1}{\kappa^2}} \leq b,\quad \kappa \geq 1 \quad (28)$$

In order to avoid complex values for $\mathcal{M}\mathcal{A}(\kappa)$, one must choose $\kappa \geq 1$. A value of $\kappa = 1$ yields $\mathcal{M}\mathcal{A}(\kappa = 1) = 0$. The condition $\mathcal{M}\mathcal{A}(\kappa) \leq b$ in eq-(29) is also a sign of consistency such that the maximal upper bound of b is not exceeded. In the limit $\kappa \to \infty$ one has $\mathcal{M}\mathcal{A}(\kappa) \to b$ and the upper bound b is saturated.

The solutions (26,27) describe a one-parameter family of elliptic-hyperboloids in four dimensions defined by the algebraic equation

$$x^2\ +\ \frac{p^2}{b^2}\ -\ t^2\ -\ \frac{E^2}{b^2}\ =\ \frac{1}{\mathcal{A}(\kappa)^2}\ =\ \frac{\mathcal{M}^2}{b^2(1 - 1/\kappa^2)} \quad (29)$$

The above algebraic equation for the elliptic-hyperboloid is $U(1,1)$-invariant.

Note the similarity of the family of solutions (26) with the family of Rindler hyperbolic trajectories if one replaces g, τ with $\mathcal{A}, \omega$, respectively. The parameter $\kappa \geq 1$ is arbitrary and depicts the infinite family of trajectories. The similarity with the family of Rindler trajectories which span the Rindler coordinate-grid is more clear when κ is recast as $\kappa = \exp(\mathcal{A}_o\sigma)$, with $\mathcal{A}_o$ constant, with $0 \leq \sigma \leq \infty$, so when $\sigma = 0 \Rightarrow \kappa = 1$, and $\sigma = \infty \Rightarrow \kappa = \infty$. The actual Rindler grid extends all the way to $\sigma = -\infty$ which corresponds to the origin $x = 0, t = 0$.

The initial positions of the trajectories in eq-(26) is described by the functions

$$x(\omega = 0; \kappa) \;=\; x_o(\kappa) \;=\; \frac{\kappa}{\mathcal{A}(\kappa)} \;=\; \frac{\kappa\mathcal{M}}{\mathcal{M}\mathcal{A}(\kappa)} \;=\; \frac{\kappa\mathcal{M}}{b\sqrt{1 - (1/\kappa^2)}} \tag{30}$$

$x_o(\kappa)$ is proportional to the throat-sizes of the elliptic-hyperboloids.

When $\kappa = 1$ and $\infty \Rightarrow x_o = \infty$. The minimum value of x_o is obtained by solving $\frac{dx_o}{d\kappa} = 0$. After some algebra one arrives at a quadratic equation for κ : $1 - \frac{2}{\kappa^2} = 0 \Rightarrow \kappa = \sqrt{2} > 1$. Inserting this value of κ into (30) yields

$$(x_o)_{min} \;=\; \frac{2\mathcal{M}}{b} \;=\; \frac{2G\mathcal{M}}{Gb} \tag{31}$$

If one sets the maximal proper force b to be equal to the Planck mass-squared $b = m_P^2$, in units of $\hbar = c = 1$, then $Gb = Gm_P^2 = L_P^2 m_P^2 = 1$, with L_P being the Planck scale in $4D$, such that the minimal initial position turns out to be $2G\mathcal{M}$ which coincides precisely with the numerical value of the horizon radius of the Schwarzschild black hole in four spacetime dimensions (which must not be confused with the four dimensions of the phase space we have been working with signature $(2, 2)$).

To sum up, the salient features of this work are (i) : the hyperbolic family of trajectories in phase space automatically satisfy both the maximal speed of light $|dx/dt| \leq 1$, and the maximal proper force $\mathcal{M}\mathcal{A}(\kappa) \leq b$ condition. $\mathcal{M}\mathcal{A}(\kappa)$ is $U(1, 1)$ invariant, which in turn implies $SO(1, 1)$-invariance; the converse is not true, as we saw with mg_o and τ *not* being invariant under force-boost transformations [4], [5]. (ii) one arrives at $(x_o)_{min} = \frac{2\mathcal{M}}{b}$ which establishes a direct link between the maximal proper force b and the minimum value of the initial position, a minimal distance from the origin given by eq-(31). Furthermore, (iii) : if the maximal proper force is $b = m_P^2$ then $(x_o)_{min} = 2G\mathcal{M}$ happens to coincide with the numerical value of the horizon radius of the Schwarzschild black hole in four spacetime dimensions.

The minimal value of the initial positions is a result which is in sharp contrast with what occurs in special relativity (inertial relativity). The initial positions of the family of Rindler hyperbolic trajectories in $D = 1 + 1$ can range between 0 and ∞ depending on the range of values of the uniform proper spacetime acceleration g_o given by $g_o = \infty$ and $g_o = 0$, respectively. The asymptotes to the Rindler hyperbolas are the null lines going through the origin $x_o = 0$ (zero minimal distance) corresponding to the massless photons trajectories which can be viewed as having an infinite proper acceleration $g_o = \infty$ and zero mass $m = 0$, and such that the double scaling limit is $mg_o = b$ [5], [4].

Hence we have shown how the behavior in phase space is very different and one has found a non-zero minimal value for the initial positions of the family of hyperbolic trajectories and which is determined in terms of the maximal proper force b as shown by eq-(31). It remains to explore the solutions of eqs-(23,24) for a *variable* proper phase space acceleration $\mathcal{A}(\omega;\kappa)$. However, to find solutions to eqs-(23,24) in this case when there is an explicit ω-dependence of $\mathcal{A}$ is far more difficult than finding the solutions to eqs-(1,2) in spacetime and which were displayed by eqs-(4,5).

So far we have studied the *flat* Born geometry. A curved geometry of the phase space (cotangent space) requires the tools of Hamilton-Cartan geometry (Lagrange-Finsler geometry in the case of tangent space). The Born interval in an 8-dim curved phase space (cotangent space) is given by

$$(d\omega)^2 \;=\; g_{\mu\nu}(x,p)\,dx^\mu\,dx^\nu + h_{ab}(x,p)\,(dp^a + A^a_\mu(x,p)\,dx^\mu)\,(dp^b + A^b_\nu(x,p)\,dx^\nu) \tag{32}$$

$g_{\mu\nu}(x,p)$ is the horizontal base spacetime metric; $\mu,\nu = 0,1,2,3$. $h_{ab}(x,p)$ is the vertical space (fiber) metric; $a,b = 0,1,2,3$. $A^a_\mu(x,p)$ is the *nonlinear* connection. The flat space limit occurs when $g_{\mu\nu} = \eta_{\mu\nu}; h_{ab} = \frac{1}{b^2}\eta_{ab}$; $A^a_\mu = 0$. See [7] and the many references therein for more details. A curved geometry of the phase space has many other applications like in resolving the problem of UV divergences in QFT. Recently, an intrinsic regularization via curved momentum space as a geometric solution to divergences in Quantum Field Theory has been proposed by [8].

To finalize let us point out the connection [10] between open strings and the one-parameter family of Rindler hyperbolic trajectories in $D = 1+1$ given by

$$t(\eta,\sigma) \;=\; \frac{\exp(g_o\sigma)}{g_o}\,\sinh(g_o\eta), \quad x(\eta,\sigma) \;=\; \frac{\exp(g_o\sigma)}{g_o}\,\cosh(g_o\eta) \tag{33}$$

$$(d\tau)^2 \;=\; (dt)^2 \;-\; (dx)^2 \;=\; \exp(2g_o\sigma)\,\left((d\eta)^2 - (d\sigma)^2\right) \tag{34}$$

Each Rindler hyperbolic trajectory corresponds to setting σ = constant in eq-(34) such that the proper time along each one of these trajectories becomes $(d\tau) = \exp(g_o\sigma)d\eta \Rightarrow \tau = \exp(g_o\sigma)\eta$, and one can write

$$g_o\,\eta \;=\; [\exp(-g_o\sigma)\,g_o]\,[\exp(g_o\sigma)\,\eta] \;=\; g(\sigma)\,\tau; \quad g(\sigma) = \exp(-g_o\sigma)\,g_o \tag{35}$$

such that the expressions in eq-(33) can be rewritten as

$$t(\tau,\sigma) \;=\; \frac{1}{g(\sigma)}\,\sinh[g(\sigma)\tau], \quad x(\tau,\sigma) \;=\; \frac{1}{g(\sigma)}\,\cosh[g(\sigma)\tau] \tag{36}$$

Note that the functional form of eq-(36) is *almost* similar to eq-(26) with the key difference that there are extra factors of $\kappa = \exp(\mathcal{A}_o\sigma)$ in eq-(26). The solutions (33) obey the string equations of motion

$$\frac{\partial^2 t}{\partial\eta^2} - \frac{\partial^2 t}{\partial\sigma^2} \;=\; 0, \quad \frac{\partial^2 x}{\partial\eta^2} - \frac{\partial^2 x}{\partial\sigma^2} \;=\; 0 \tag{37}$$

whereas the solutions in eq-(26) do *not*

$$\frac{\partial^2 t}{\partial \omega^2} - \frac{\partial^2 t}{\partial \sigma^2} \neq 0, \quad \frac{\partial^2 x}{\partial \omega^2} - \frac{\partial^2 x}{\partial \sigma^2} \neq 0, \quad \kappa = \exp(\mathcal{A}_o \sigma) \tag{38}$$

Imagine an infinite open string stretching from x_o all the way to $x = \infty$ and comprised of a continuum of point masses m where each one of these point masses is subjected to a σ-dependent proper force given by $F(\sigma) = mg(\sigma) = mg_o \exp(-g_o\sigma)$ endowing each one of these point masses with a hyperbolic trajectory such that the σ-dependent proper acceleration $g(\sigma)$ is $g_o \exp(-g_o\sigma)$. One could rewrite the expression for $F(\sigma)$ as $m(\sigma)g_o = [m_o \exp(-g_o\sigma)]g_o$, and in turn, interpret the proper force at each value of σ as that experienced by a σ-dependent mass distribution of points given by $m(\sigma)$ where each one of these mass points is experiencing the same proper acceleration g_o.

Because the physical units of a proper force F are the same as a tension $[mc^2/L] = [Tc^2]$ (the tension T is just mass per unit length m/L) one could view this continuum arrangement of accelerated point masses as belonging collectively to an infinite open string with a variable tension $T(\sigma)$ which sweeps a two-dim world-sheet lying inside a region of the Rindler wedge [10]. Equating the tension T with the mass density $\frac{dm(\sigma)}{d\sigma}$ gives a variable negative tension

$$T(\sigma) = \frac{dm(\sigma)}{d\sigma} = -m_o g_o \exp(-g_o\sigma) \tag{39}$$

One can avoid this negative tension problem by recurring instead to the phase space analog of the proper force given by eq-(28)

$$\mathcal{F}(\sigma) \equiv \mathcal{M}\ \mathcal{A}(\sigma) = b\ \sqrt{1-\exp(-2\mathcal{A}_o\sigma)}, \quad \kappa = \exp(\mathcal{A}_o\sigma) \tag{40}$$

After repeating the previous argument to select a σ-dependent mass one finds that

$$\mathcal{M}(\sigma) = \frac{b}{\mathcal{A}_o}\ \sqrt{1-\exp(-2\mathcal{A}_o\sigma)} \tag{40}$$

leading now to a *positive* variable tension

$$T(\sigma) = \frac{d\mathcal{M}(\sigma)}{d\sigma} = b\ \frac{\exp(-2\mathcal{A}_o\sigma)}{\sqrt{1-\exp(-2\mathcal{A}_o\sigma)}} \tag{41}$$

given in terms of the constant parameter $\mathcal{A}_o$ and the σ variable. Because the above tension *blows* up at $\sigma = 0$ one needs to introduce a cut-off σ_{min} that can be chosen to coincide with the value of $\kappa = \sqrt{2}$ associated with the minimal initial position $(x_o)_{min} = 2\mathcal{M}/b$ found in eq-(31). In doing so one arrives at $\sqrt{2} = \exp(\mathcal{A}_o\sigma_{min}) \Rightarrow \sigma_{min} = ln(\sqrt{2})/\mathcal{A}_o$, and the tension at σ_{min} becomes $T(\sigma_{min}) = \frac{b}{\sqrt{2}} < b$. One could choose the cutoff value for σ to be that point $0 < \sigma_* < \sigma_{min}$ such that $T(\sigma_*) = b$ is precisely the maximal proper force. The tension at $\sigma = \infty$ is zero.

Strings with a dynamical (variable) tension (both in space and time) $T(\tau, \sigma)$ have been extensively studied by Guendelman [9] over the years. In the modified measure formulation of strings/ branes, the tension appear as an additional dynamical degree of freedom. There are many important physical consequences of these variable tension models of strings and branes. Recently, Guendelman has reviewed how the model avoids the Swampland constraints making treatments for Dark energy and inflation more realistic and how strings with a different tension appear as Dark Matter to us. We refer to [9] and the many references therein for specific details.

We remarked in eq-(38) that the solutions (26) do *not* obey the string equations of motion. We must emphasize that there is *no* contradiction in assigning a string picture to the collective motion of the continuum of point masses along hyperbolic paths despite that the solutions (26) do not obey the string equations of motion (38) because these equations are associated to strings with *constant* tensions. And we have shown above how a variable string tension $T(\sigma)$ mechanism emerges. Consequently, one needs to examine the more general equations of motion for strings with a variable tension.

To finalize, we have found above how to introduce a variable tension of the form $T(\sigma)$. The next question is how to introduce an additional temporal dependence. Because the string was comprised of a continuum of point masses that are being accelerated, if they emit gravitational radiation, there will be a loss of mass, and in this fashion, one could attain the sought after temporal dependence leading to dynamical tensions as well $T(\tau, \sigma)$. This warrants further investigation.

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