A striking feature in the projection operator and the identity operator utilized by the Kochen–Specker theorem

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Abstract

We explicitly investigate the significant striking aspect of the most basic two quantum observables: (1) an observable that is obtained through a single projection operator and (2) an observable that is obtained through a single identity operator. Both of quantum operators are well studied by Kochen and Specker in terms of the inconsistency with hidden variable theories. They show that some value assignment to elements of a set of projection operators and the identity operator should fail. Now, we greatly simplify the Kochen–Specker theorem. Our simplified version says that we cannot assign the predetermined hidden result as +1 to both of quantum operators (the projection operator and the identity operator) even though the assignment is independent of each other. Both of quantum operators do not have a classical counterpart.

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I. INTRODUCTION

Quantum mechanics (cf. [1–8]) gives explanations for the microscopic behaviors of the nature. The incompleteness argument for quantum mechanics itself is discussed by Einstein, Podolsky, and Rosen [9]. A hidden variable interpretation of quantum mechanics is an interesting topic of research [3, 4] and the inconsistency with hidden variable theories is discussed by Bell, Kochen, and Specker [10, 11].

Multipartite inequality as tests for the Kochen–Specker theorem is proposed by Nagata [12]. He studies the Kochen–Specker theorem as a precondition for secure quantum key distribution [13]. And, Nagata and Nakamura discover the Kochen–Specker theorem as a precondition for quantum computing [14]. Nagata, Heidari, and Nakamura investigate a violation of the multipartite inequality on the basis on binary logic [15]. It turns out that the Kochen–Specker theorem becomes a quite strong theorem when the dimension of the multipartite state highly increases, regardless of entanglement properties. A strengthened Kochen–Specker theorem, i.e., the free will theorem is discussed by Conway and Kochen [16]. An inconsistency with hidden variable theories is discussed, using two symmetric measurements, (they are independent of the order of measurements themselves.) by Nagata, Diep, and Nakamura [17].

In this paper, we explicitly investigate the significant striking aspect of the most basic two quantum observables: (1) an observable that is obtained through a single projection operator and (2) an observable that is obtained through a single identity operator. Both of quantum operators are well studied by Kochen and Specker in terms of the inconsistency with hidden variable theories. They show that some value assignment to elements of a set of projection operators and the identity operator should fail.

Now, we greatly simplify the Kochen–Specker theorem. We here formulate the most basic two quantum operators on the two-dimensional space as: $H(x) \equiv \begin{pmatrix} +1 & 0 \\ 0 & +x \end{pmatrix}, (x = 0, +1)$. x is a fixed constant.

We are in the inconsistency with hidden variable theories when the following quantum measurement process happens twice: The predetermined hidden result is +1 by measuring the quantum operator $H(x)$ in the quantum state $|\uparrow\rangle$. The two measurements of $H(x)$ are corresponding to the same x. It turns out that we cannot assign the predetermined hidden result for quantum measurement outcome as +1 when measuring the quantum operator $H(x)$ in the quantum state $|\uparrow\rangle$.

In the case that $x = 0$, Projection operator does not have a classical counterpart. In the case that $x = +1$, Identity operator does not have a classical counterpart. Our simplified version says that we cannot assign a predetermined hidden result as +1 to both of quantum operators (the projection operator and the identity operator) even though the assignment is independent of each other. Both of quantum operators do not have a classical counterpart.

The essence of our deep desire to describe our honest view is a new insight into hidden variable theories for future studies over the academic world because both of quantum operators (the projection operator and the identity operator) do not have a classical counterpart.

II. THE FUNCTIONAL RULE AS A HIDDEN VARIABLE ASSUMPTION

First, we define the functional rule as:

$$f(g(O)) = g(f(O)), \tag{1}$$

where O is a Hermitian operator and f, g are appropriate functions to be used later. We introduce the above assumption (the functional rule) for hidden variable theories. We want to investigate if the algebraic structure of quantum mechanical operators could be mirrored in the algebraic structure of the predetermined hidden result of the operators. Second, the sum rule is defined as:

$$f(A + B) = f(A) + f(B), \tag{2}$$

where A, B are two commuting Hermitian operators. Thirdly, the product rule is defined as:

$$f(A \cdot B) = f(A) \cdot f(B). \tag{3}$$

We have [4, 12, 17] the following relations between the rules for commuting operators:

$$\begin{aligned} &\text{The functional rule} \Rightarrow \text{The sum rule} \\ &\text{The functional rule} \Rightarrow \text{The product rule} \end{aligned} \tag{4}$$

For example, let us derive the sum rule and the product rule from the functional rule. Suppose now that A and B are two commuting Hermitian operators. In [4, 12, 17], we discuss that we can define functions j and k and there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. Thus we have

$$\begin{aligned} &f(A \cdot B) = f(h(O)) = h(f(O)) = j(f(O)) \cdot k(f(O)) \\ &f(j(O)) \cdot f(k(O)) = f(A) \cdot f(B), \end{aligned} \tag{5}$$

where we use the functional rule. We can introduce also a function l such that $A + B = l(O)$ where $l := j + k$. Thus we have

$$\begin{aligned} &f(A + B) = f(l(O)) = l(f(O)) = j(f(O)) + k(f(O)) \\ &f(j(O)) + f(k(O)) = f(A) + f(B), \end{aligned} \tag{6}$$

where we use the functional rule.

III. THE INCONSISTENCY WITH HIDDEN VARIABLE THEORIES

Let $H(x) \equiv \begin{pmatrix} +1 & 0 \\ 0 & +x \end{pmatrix}$ be a quantum operator described by a fixed constant x. We consider the two patterns of the operator as:

1. $H(0) = \begin{pmatrix} +1 & 0 \\ 0 & 0 \end{pmatrix}$ (Projection operator)

2. $H(+1) = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}$ (Identity operator)

Let $|\uparrow\rangle$ be an eigenstate of $H(x)$ such that $H(x)|\uparrow\rangle = +1|\uparrow\rangle$. The predetermined hidden result of quantum measurement outcome is $+1$ in the ideal case.

We are in the inconsistency with hidden variable theories when the following quantum measurement process happens twice: The predetermined hidden result is $+1$ by measuring the quantum operator $H(x)$ in the quantum state $|\uparrow\rangle$. The two measurements of $H(x)$ are corresponding to the same x. It turns out that we cannot assign the predetermined hidden result for quantum measurement outcome as $+1$ when measuring the quantum operator $H(x)$ in the quantum state $|\uparrow\rangle$.

We consider a value V which is the sum of the two predetermined hidden results in a thought experiment. The predetermined hidden result of outcome is $+1$. If the number of outcomes is two, we can depict the predetermined hidden results using v_1, v_2 as: $v_1 = +1$ and $v_2 = +1$. Let us write V as follows:

$$V = v_1 + v_2 = +2. \tag{7}$$

In the following, we introduce the product rule, then we derive an inconsistency with hidden variable theories. Hence, we have

$$\begin{aligned} &\exp(+2) = \exp(V) = \exp(v_1 + v_2) \\ &= \exp((v_1) + (v_2)) \\ &= \exp(v_1) \times \exp(v_2) \\ &= \exp(v_1 \times v_2) \\ &= \exp(+1), \end{aligned} \tag{8}$$

where the product rule $f(A) \times f(B) = f(A \times B)$ is used. Here $f(\cdot) = \exp(\mathrm{tr}[\rho(\cdot)])$ and the quantum state ρ is $|\uparrow\rangle\langle\uparrow|$.

We cannot assign the truth value "1" for the proposition (8). We derive the inconsistency with hidden variable theories for quantum measurement outcomes on $H(x)$. In the case that $x = 0$, Projection operator does not have a classical counterpart. In the case that $x = +1$, Identity operator does not have a classical counterpart. Our simplified version says that we cannot assign a predetermined hidden result as $+1$ to both of quantum operators (the projection operator and the identity operator) even though the assignment is independent of each other. Both of quantum operators do not have a classical counterpart.

IV. THE EXTENSION INTO N-DIMENSIONAL UNITARY SPACE

We can extend our result into the N-dimensional unitary space. Let a quantum operator $H_N(x)$ be as:

$$H_N(x) \equiv |1\rangle\langle 1| + x|2\rangle\langle 2| + ... + x|N\rangle\langle N|. \tag{9}$$

Then we have

1. $H_N(0) = |1\rangle\langle 1|$ (Projection operator)

2. $H_N(+1) = |1\rangle\langle 1| + |2\rangle\langle 2| + ... + |N\rangle\langle N|$ (N-dimensional identity operator)

We are in the inconsistency with hidden variable theories when the following quantum measurement process happens twice: The predetermined hidden result is $+1$ by measuring the quantum operator $H_N(x)$ in the quantum state $|1\rangle$. The two measurements of $H_N(x)$ are corresponding to the same x. It turns out that we cannot assign the predetermined hidden result for quantum measurement outcome as $+1$ when measuring the quantum operator $H_N(x)$ in the quantum state $|1\rangle$. In the case that $x = 0$, Projection operator does not have a classical counterpart. In the case that $x = +1$, N-dimensional identity operator does not have a classical counterpart.

V. CONCLUSIONS AND DISCUSSIONS

In conclusions, we have explicitly investigated the significant striking aspect of the most basic two quantum observables: (1) an observable that is obtained through a single projection operator and (2) an observable that is obtained through a single identity operator. Both of quantum operators have been well studied by Kochen and Specker in terms of the inconsistency with hidden variable theories. They have shown that some value assignment to elements of a set of projection operators and the identity operator should fail.

Now, we have greatly simplified the Kochen–Specker theorem. We here have formulated the basic two quantum operators as: $H(x) \equiv \begin{pmatrix} +1 & 0 \\ 0 & +x \end{pmatrix}, (x = 0, +1)$. We have been in the inconsistency with hidden variable theories when the following quantum measurement process happens twice: The predetermined hidden result has been $+1$ by measuring the quantum operator $H(x)$ in the quantum state $|\uparrow\rangle$. The two measurements of $H(x)$ have been corresponding to the same x. It has turned out that we cannot assign the predetermined hidden result for quantum measurement outcome as $+1$ when measuring the quantum operator $H(x)$ in the quantum state $|\uparrow\rangle$.

In the case that $x = 0$, Projection operator has not had a classical counterpart. In the case that $x = +1$, Identity operator has not had a classical counterpart. Our simplified version has said that we cannot assign the predetermined hidden result as $+1$ to both of quantum operators (the projection operator and the identity operator) even though the assignment is independent of each other. Both of quantum operators have not had a classical counterpart.

The essence of our deep desire to describe our honest view has been a new insight into hidden variable theories for future studies over the academic world because both of quantum operators (the projection operator and the identity operator) have not had a classical counterpart.

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