Algebraic dynamics

Subtitle - The first stones

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This exploration focuses attention on the kernels related to the decomposition of deformed cross products. It tries to use them as commutative operator describing the evolution of the polynomials which are associated with the decomposition of these deformed products. The long range purpose is the construction of an algebraic dynamics shedding a new light on our reality.

keywords : Deformed cross products ; decomposition ; kernels ; commutative operators, polynomials, metrics, evolution.

Contents

1 Motivations

Two former discussions, (i) one trying to understand why particles do sometimes have three visages [a] and (ii) another one analysing the electromagnetic duality from a different perspective as it is usually done [b], have introduced a special set of (3-3) matrices: the so-called ordinary Perian matrices (OPMs). This document continues a systematic exploration of their mathematical particularities. The rough analogy between their generic formalism and the one of kernels resulting from the decomposition of deformed cross products, [c] and [d], attracts attention. The latter is reenforced through the existence of a coincidence between the OPMs and the representation of Euler-Rodrigues parametrisations (ERp).

Since: (i) any kernel is systematically associated with a polynomial depending on the three components of the first vector (the so-called *projectile*) involved in the deformed cross product being decomposed, $\Lambda(\mathbf{projectile})$, (ii) each kernel is not obligatorily an OPM, the document focuses attention on the Hessian matrices which can be identified with the symmetric part of OPMs; it call them *kernel-compatible matrices*.

Following the same logical vein, and since: (i) the anti-symmetric part of any kernel is related to either a singular vector (Class I) or to a pseudo-singular vector (Class II), (ii) the argument of the polynomial does only accidentally coincide with its (pseudo-)singular vector, one concludes that any OPM, hence any Erp as well, is associated with an accidental event.

Perhaps more important, one progressively becomes aware that, within this context, the deformed cross products and theirs associated polynomials are the main actors of the theory. One also understands the possibility to develop an algebraic dynamics based on these kernels because they are the mathematical representations of the polynomials.

This understanding is crucial for physics because front waves can be understood as polynomials in evolution and one knows that waves interfere. This physical fact can be described here in saying that polynomials interfere. There must exist a set of mathematical tools accounting for these interferences. In this document, one bets that the kernels resulting from the decomposition of deformed cross products are the tools associated with the front waves. This hypothesis justifies the quest for conditions allowing interpreting the kernels as commutative operators.

2 The ordinary Perian matrices

2.1 Basics

The author recommands the lecture of [c] and [d].

Definition 2.1. *The ordinary Perian matrice (OPM).*

Let consider a pair $(\mathbf{a},\, \mathbf{b})$ in $E_3 \times E_3$ and suppose that $(\alpha,\, \beta,\, \chi)$ represents $\mathbf{a}$ in K^3. These ingredients allow the construction of what will be called "an ordinary Perian matrix" (short: OPM):

$$[M(\mathbf{a}, \mathbf{b})] \,=\, \alpha \,.\, Id_3 \,+\, \beta \,.\, T_2(\otimes)(\mathbf{b}, \mathbf{b}) \,+\, \chi \,.\, _{[J]}\Phi(\mathbf{b}) \,\in\, M(3,\, K)$$

This matrix can also be understood as the image of an application denoted f such that:

$$f \,:\, (\mathbf{a}, \mathbf{b}) \,\in\, E_3^2 \rightarrow f(\mathbf{a}, \mathbf{b}) \,=\, [M(\mathbf{a}, \mathbf{b})] \,\in\, M(3,\, K)$$

Nota bene 01: The set of all OPMs will be denoted with the Greek letter Π. Per convention, the subset of Π containing all OPMs with *a given* **b** will be written Π(**b**) whilst the subset of Π containing all OPMs with *a given* **a** will be written (**a**)Π.

Nota bene 02: The addition and the multiplication on the left side by an element in K are intern operations in Π(**b**) but not in (**a**)Π because of the presence of the Pythagorean table. The product of two distinct OPMs is in general not intern (equiv.: not in Π). But, when $K = \mathbb{R}$ or $K = \mathbb{C}$, the product of two distinct elements in Π(**b**) is yet an element in Π(**b**).

$$[M(\mathbf{a},\, \mathbf{b})] \,.\, [M(\mathbf{a}',\, \mathbf{b})]$$

$$=$$

$$\{\alpha \,.\, Id_3 + \beta \,.\, T_2(\otimes)(\mathbf{b}, \mathbf{b}) + \chi \,.\, _{[J]}\Phi(\mathbf{b})\} \,.\, \{\alpha' \,.\, Id_3 + \beta' \,.\, T_2(\otimes)(\mathbf{b}, \mathbf{b}) + \chi' \,.\, _{[J]}\Phi(\mathbf{b})\}$$

$$=$$

$$\begin{gathered}(\alpha \,.\, \alpha' \,-\, \chi \,.\, \chi' \,.\, ||\mathbf{b}||^2) \,.\, Id_3 \\ + \alpha \,.\, \beta' \,.\, T_2(\otimes)(\mathbf{b}, \mathbf{b}) \,+\, \beta \,.\, \beta' \,.\, ||\mathbf{b}||^2 \,.\, T_2(\otimes)(\mathbf{b}, \mathbf{b}) \,+\, \beta \,.\, \alpha' \,.\, T_2(\otimes)(\mathbf{b}, \mathbf{b}) \\ + \beta \,.\, \chi' \,.\, T_2(\otimes)(\underbrace{\mathbf{b} \wedge \mathbf{b}}_{=\mathbf{0}}, \mathbf{b}) \,+\, \chi \,.\, \beta' \,.\, T_2(\otimes)(\mathbf{b}, \underbrace{\mathbf{b} \wedge \mathbf{b}}_{=\mathbf{0}}) \,+\, \chi \,.\, \chi' \,.\, T_2(\otimes)(\mathbf{b}, \mathbf{b}) \\ + \alpha \,.\, \chi' \,.\, _{[J]}\Phi(\mathbf{b}) \,+\, \chi \,.\, \alpha' \,.\, _{[J]}\Phi(\mathbf{b})\end{gathered}$$

$$=$$

$$(\alpha \,.\, \alpha' \,-\, \chi \,.\, \chi' \,.\, ||\mathbf{b}||^2) \,.\, Id_3$$

$$+ (\alpha . \beta' + \beta . \beta' . ||\mathbf{b}||^2 + \beta . \alpha' + \chi . \chi') . T_2(\otimes)(\mathbf{b}, \mathbf{b})$$
$$+ (\alpha . \chi' + \chi . \alpha') ._{[J]} \Phi(\mathbf{b})$$

Nota bene 03: In future works, the concept of *generalized Perian matrix* (GPM) will be introduced. A GPM is written:

$$[M(\blacktriangledown C, \mathbf{b})] = [\alpha] + [\beta] . T_2(\otimes)(\mathbf{b}, \mathbf{b}) + [\chi] ._{[J]} \Phi(\mathbf{b}) \in M(3, K)$$

... where $[\alpha]$, $[\beta]$, $[\chi]$ are elements in M(3, K). They build a cube ▼C in ⊞(3, K).

Remark 2.1. *The determinant of an OPM.*

One can prove that (see annex 4.1):

$$|M(\mathbf{a}, \mathbf{b})| = (\alpha + \beta . ||\mathbf{b}||^2) . (\alpha^2 + ||\mathbf{b}||^2 . \chi^2)$$

Remark 2.2. *The trace of OPM.*

One easily states that:

$$Tr[M(\mathbf{a}, \mathbf{b})] = 3 . \alpha + \beta . ||\mathbf{b}||^2$$

It is also the trace of its symmetric part:

$$Tr[M(\mathbf{a}, \mathbf{b})] = Tr[S] = Tr\{\alpha . Id_3 + \beta . T_2(\otimes)(\mathbf{b}, \mathbf{b})\}$$

Remark 2.3. *The sum of all entries in an OPM.*

One can easily verify that:

$$[M(\mathbf{a}, \mathbf{b})]^{\oplus} = 3 . \alpha + \beta . (\mathbf{b}^{\oplus})^2$$

The symbol ⊕ signs a sum on the components of the mathematical object preceeding it. Since the sum of the components of any axial-rotation matrix is null:

$$[M(\mathbf{a}, \mathbf{b})]^{\oplus} = [S]^{\oplus}$$

Definition 2.2. *IsoΠ(**b**).*

Per convention, the set of all elements in $\Pi(\mathbf{b})$ when **b** is an isotropic vector will be denoted $\text{Iso}\Pi(\mathbf{b})$. One easily remarks that for all elements in $\text{Iso}\Pi(\mathbf{b})$:

$$\forall \mathbf{b} : ||\mathbf{b}||^2 = 0 \Rightarrow |M(\mathbf{a}, \mathbf{b})| = \alpha^3$$

Definition 2.3. *The ordinary truncated Perian matrices.*

Per convention, a given OPM is truncated (short is a TOPM) when one of the components of its first argument (precisely: the vector **a**) is null. Therefore, three families of TOPMs exist:

1. The first family contains the OPMs such that **a** is represented by $(0, \beta, \chi)$; each element in this family has the generic deteminant:

$$|M(\mathbf{a}, \mathbf{b})| = \beta . ||\mathbf{b}||^4 . \chi^2$$

2. The second family contains the OPMs such that **a** is represented by (α, 0, χ); each element in this family has the generic deteminant:

$$|M(\mathbf{a},\mathbf{b})| \;=\; \alpha\,.\,(\alpha^2 \,+\, ||\mathbf{b}||^2\,.\,\chi^2)$$

3. The third family contains the OPMs such that **a** is represented by (α, β, 0); each element in this family has the generic deteminant:

$$|M(\mathbf{a},\mathbf{b})| \;=\; \alpha^2\,.\,(\alpha \,+\, \beta\,.\,||\mathbf{b}||^2)$$

Example 2.1. *Degenerated elements in Iso*Π*(**b**).*

All degenerated elements in IsoΠ(**b**) are in the first family of truncated OPMs. Proof: Any degenerated matrix in IsoΠ(**b**) is characterized by a vanishing component α.

Remark 2.4. *Useful rules, relations and conventions for future calculations.*

One can easily verify that:

$$\begin{array}{cc} \forall\,\mathbf{b} & \begin{array}{ccc} Id & T & \Phi \end{array} \\ \begin{array}{c} Id \\ T \\ \Phi \end{array} & \left[\begin{array}{ccc} Id & T & \Phi \\ T & ||\mathbf{b}||^2\,.\,T & [0] \\ \Phi & [0] & T - ||\mathbf{b}||^2\,.\,Id \end{array}\right] \end{array}$$

$$||\mathbf{b}||^2 \;=<\mathbf{b},\,\mathbf{b}>_{Id_3} = (b^1)^2 \,+\, (b^2)^2 \,+\, (b^3)^2$$

1.
$$T_2(\otimes)(\mathbf{M},\,\mathbf{N}) \,-\, T_2(\otimes)(\mathbf{N},\,\mathbf{M}) \;=\; {}_{[J]}\Phi(\mathbf{M}\wedge\mathbf{N})$$

2.
$$T_2(\otimes)(\mathbf{M},\,\mathbf{M})\,.\,{}_{[J]}\Phi(\mathbf{N}) \;=\; T_2(\otimes)(\mathbf{M}\wedge\mathbf{N},\,\mathbf{M})$$

3.
$${}_{[J]}\Phi(\mathbf{N})\,.\,T_2(\otimes)(\mathbf{M},\,\mathbf{M}) \;=\; T_2(\otimes)(\mathbf{M},\,-\mathbf{M}\wedge\mathbf{N})$$

4.
$$T_2(\otimes)(\mathbf{M},\,\mathbf{M})\,.\,{}_{[J]}\Phi(\mathbf{N}) \,+\, {}_{[J]}\Phi(\mathbf{N})\,.\,T_2(\otimes)(\mathbf{M},\,\mathbf{M})$$
$$=$$
$$T_2(\otimes)(\mathbf{M}\wedge\mathbf{N},\,\mathbf{M}) \,-\, T_2(\otimes)(\mathbf{M},\,\mathbf{M}\wedge\mathbf{N})$$
$$=$$
$${}_{[J]}\Phi((\mathbf{M}\wedge\mathbf{N})\wedge\mathbf{M})$$
$$=$$
$$||\mathbf{M}||^2\,.\,{}_{[J]}\Phi(\mathbf{N}) - <\mathbf{M},\,\mathbf{N}>_{Id_3}\,.\,{}_{[J]}\Phi(\mathbf{M})$$

5.
$${}_{[J]}\Phi^2(\mathbf{M}) \;=\; T_2(\otimes)(\mathbf{M},\,\mathbf{M}) \,-\, ||\mathbf{M}||^2\,.\,Id_3$$

6.
$$T_2^2(\otimes)(\mathbf{M},\,\mathbf{M}) \;=\; ||\mathbf{M}||^2\,.\,T_2(\otimes)(\mathbf{M},\,\mathbf{M})$$

2.2 The pedagogical example of the Euler-Rodrigues parametrisation

Let consider the equation of a unit-four-dimensional sphere (it has a radius equal to 1):

$$(\varsigma^0)^2 + (\varsigma^1)^2 + (\varsigma^2)^2 + (\varsigma^3)^2 = 1$$

As this has been known for a long time, this sphere can be represented in $\mathrm{M}(3, \mathbb{C})$ with (what has been called) a Euler-Rodrigues parametrisation, [03; p. 282 ([1])], [09; §3, p. 423] and this representation is an ordinary Perian matrix because it can be rewriten as:

$$[M({}^{(4)}\varsigma)]$$

$$=$$

$$\begin{bmatrix} (\varsigma^0)^2 + (\varsigma^1)^2 - (\varsigma^2)^2 - (\varsigma^3)^2 & 2\,.\,(\varsigma^1\,.\,\varsigma^2 - \varsigma^0\,.\,\varsigma^3) & 2\,.\,(\varsigma^0\,.\,\varsigma^2 + \varsigma^1\,.\,\varsigma^3) \\ 2\,.\,(\varsigma^1\,.\,\varsigma^2 + \varsigma^0\,.\,\varsigma^3) & (\varsigma^0)^2 - (\varsigma^1)^2 + (\varsigma^2)^2 - (\varsigma^3)^2 & 2\,.\,(\varsigma^2\,.\,\varsigma^3 - \varsigma^0\,.\,\varsigma^1) \\ 2\,.\,(\varsigma^1\,.\,\varsigma^3 - \varsigma^0\,.\,\varsigma^2) & 2\,.\,(\varsigma^2\,.\,\varsigma^3 + \varsigma^0\,.\,\varsigma^1) & (\varsigma^0)^2 - (\varsigma^1)^2 - (\varsigma^2)^2 + (\varsigma^3)^2 \end{bmatrix}$$

$$=$$

$$(2\,.\,(\varsigma^0)^2 - 1)\,.\,Id_3 + 2\,.\,T_2(\otimes)({}^{(3)}\varsigma,\ {}^{(3)}\varsigma) + 2\,.\,\varsigma^0\,.\,{}_{[J]}\Phi({}^{(3)}\varsigma)$$

With:

$${}^{(3)}\varsigma\ :\ (\varsigma^1,\ \varsigma^2,\ \varsigma^3)$$

$${}^{(4)}\varsigma\ :\ (\varsigma^0,\ \varsigma^1,\ \varsigma^2,\ \varsigma^3)$$

Theorem 2.1. *Ordinary Perian matrices and Euler-parametrisations of the unit four-dimensional spheres.*

Any representation $[\mathrm{M}({}^{(4)}\varsigma)]$ of the unit four-dimensional sphere in $\mathrm{M}(3, \mathbb{C})$ is an ordinary Perian matrix $[\mathrm{M}({}^{(3)}\mathbf{a},\ {}^{(3)}\mathbf{b})]$ characterized by a pair of arguments:

$${}^{(3)}\mathbf{a}\ :\ (\alpha,\ \beta,\ \chi) = (2\,.\,(\varsigma^0)^2 - 1,\ 2,\ 2\,.\,\varsigma^0)$$

$${}^{(3)}\mathbf{b} = {}^{(3)}\varsigma$$

Remark 2.5. *Some specific properties concerning the ERps.*

Let enumerate diverse characteristics.

1. The determinant; due to the equation of the unit four-dimensional sphere and to the remark 2.1:

$$|{}^{(3)}M({}^{(4)}\varsigma)|$$

$$=$$

$$(\alpha + \beta\,.\,||\mathbf{b}||^2)\,.\,(\alpha^2 + ||\mathbf{b}||^2\,.\,\chi^2)$$

$$=$$

$$\underbrace{(2\,.\,(\varsigma^0)^2 - 1 + 2\,.\,||{}^{(3)}\varsigma||^2}_{=1}\,.\,((2\,.\,(\varsigma^0)^2 - 1)^2 + ||{}^{(3)}\varsigma||^2\,.\,4\,.\,(\varsigma^0)^2)$$

$$=$$

$$4\,.\,(\varsigma^0)^4 - 4\,.\,(\varsigma^0)^2 + 1 + 4\,.\,(1 - (\varsigma^0)^2)\,.\,(\varsigma^0)^2)$$

$$=$$

$$1$$

1. with a small correction

2. The trace; due to the remark 2.2:

$$Tr[{}^{(3)}M({}^{(4)}\varsigma)] = 3\,.\,\alpha\,+\,\beta\,.\,||\mathbf{b}||^2 = 3\,.\,(2\,.\,(\varsigma^0)^2\,-\,1)\,+\,2\,.\,||{}^{(3)}\varsigma||^2 = 4\,.\,(\varsigma^0)^2\,-\,1$$

3. The sum of all entries; due to the remark 2.3:

$$[{}^{(3)}M({}^{(4)}\varsigma)]^{\oplus} = 3\,.\,\alpha\,+\,\beta\,.\,(\mathbf{b}^{\oplus})^2 = 3\,.\,(2\,.\,(\varsigma^0)^2\,-\,1)\,+\,2\,.\,({}^{(3)}\varsigma^{\oplus})^2 = 3\,.\,(\varsigma^0)^2\,-\,3\,.\,||{}^{(3)}\varsigma||^2\,+\,2\,.\,({}^{(3)}\varsigma^{\oplus})^2$$

Definition 2.4. *The Ko-ratio.*

For future discussions, one introduces the so-called Ko-ratio; to each element in E(3, K) of which the sum of the components does not vanish, one associates an element in K:

$$\forall\,\mathbf{w}\,\in\,E(3,\,K),\;{}^{(3)}\mathbf{w}^{\oplus}\,\neq\,0\;\xrightarrow{Ko}\;Ko({}^{(3)}\mathbf{w})\;=\;\frac{||{}^{(3)}\mathbf{w}||^2}{({}^{(3)}\mathbf{w}^{\oplus})^2}\,\in\,K$$

This ratio plays presumably a role in physics when the components of the vector are the different possible masses for a given type of particles. For example, its value is 2/3 when it concerns electrons; see [a].

2.3 The kernel-compatible matrices

In observing an OPM and a given kernel related the decomposition of a deformed cross product, a question arises: "Is this OPM an accidental representation for this kernel? More generally, is there a logical link between the kernels and the OPMs; if yes: which one?

Definition 2.5. *The kernel-compatible matrices.*

Per convention, a kernel-compatible matrix is the symmetric part of a Perian matrix (equivalently: is a truncated Perian matrix in the third family) that can be identified with the half of the classical Hessian of at least one polynomial $\Lambda(\mathbf{b})$. The set of all kernel-compatible matrices is a subset denoted KernΠ:

$$\exists\,\Lambda \in F(E_3 \to K), \forall\,\mathbf{b} \in E_3 \xrightarrow{\Lambda} \Lambda(\mathbf{b}) \in K :$$

$$[S_{\lambda\mu}] = [\alpha\,.\,\delta_{\lambda\mu} + \beta\,.\,b^{\lambda}\,.\,b^{\mu}] = \frac{1}{2}\,.\,[\frac{\partial^2\Lambda(\mathbf{b})}{\partial b^{\lambda}\partial b^{\mu}}] = \frac{1}{2}\,.\,[H_{\lambda\mu}] = \frac{1}{2}\,.\,[Hess_{(\mathbf{b},\,0)}\Lambda(\mathbf{b})]$$

$$\Downarrow$$

$$\frac{1}{2}\,.\,\{[M(\mathbf{a},\mathbf{b})] + [M(\mathbf{a},\mathbf{b})]^t\} = [S] \in Kern\Pi \subset \Pi$$

Remark 2.6. *The determinant of the symmetric part of a Perian matrix.*

Since any kernel-compatible matrix is a truncated matrix in the third family (definition 2.3), one easily proves that (remark 2.1):

$$\forall\,\mathbf{b} \in E_3 : |S| = \alpha^2\,.\,(\alpha + \beta\,.\,||\mathbf{b}||^2)$$

Recalling what has been explained in [a], one can classify the kernel-compatible matrices into two classes:

1. Class I:
$$|S| = \alpha^2\,.\,(\alpha + \beta\,.\,||\mathbf{b}||^2) \neq 0$$
2. Class II:
$$|S| = \alpha^2\,.\,(\alpha + \beta\,.\,||\mathbf{b}||^2) = 0$$

   Hence, a kernel-compatible matrix in class II is associated with a degenerated Hessian. This happens when either:

   (a) sub-class I:
   $$\alpha = 0$$

   (b) or, sub-class II:
   $$\alpha + \beta\,.\,||\mathbf{b}||^2 = 0$$

Remark 2.7. *The eigenvalues of the symmetric part of a Perian matrix.*

When K = ℝ or K = ℂ, then (see Annex 4.2):

$$W = \alpha\,.\,(3\,.\,\alpha + 2\,.\,\beta\,.\,||\mathbf{b}||^2)$$

Within this context, the chracteristic polynomial of the symmetric part [S] is:

$$P(\lambda)$$

$$=$$

$$\begin{vmatrix} S_{11} - \lambda & S_{12} & S_{13} \\ S_{12} & S_{22} - \lambda & S_{23} \\ S_{13} & S_{23} & S_{33} - \lambda \end{vmatrix}$$

$$=$$

$$|S| - W \, . \, \lambda + Tr[S] \, . \, (\lambda)^2 - (\lambda)^3$$

$$=$$

$$\alpha^2 \, . \, (\alpha + \beta \, . \, ||\mathbf{b}||^2) - \alpha \, . \, (3 \, . \, \alpha + 2 \, . \, \beta \, . \, ||\mathbf{b}||^2) \, . \, \lambda + (3 \, . \, \alpha + \beta \, . \, ||\mathbf{b}||^2) \, . \, \lambda^2 - \lambda^3$$

The eigenvalues of [S] are the roots of P(λ). By the way and for future calculations, one may remark that:

$$P(\lambda) = (\alpha - \lambda)^3 + \beta \, . \, ||\mathbf{b}||^2 \, . \, (\alpha - \lambda)^2 = (\alpha - \lambda)^2 \, . \, (\alpha - \lambda + \beta \, . \, ||\mathbf{b}||^2)$$

This simplification rapidly yields the roots:

$$\lambda_1 = \lambda_2 = \alpha$$

$$\lambda_3 = \alpha + \beta \, . \, ||\mathbf{b}||^2$$

This result creates a classification which is overlaping with that of remark 2.6. With different words: the roots of P(λ) are associated with degenerated kernel-matrices (equiv.: they are in the class II).

Example 2.2. *The characteristic polynomial for the symmetric part of a Perian matrix in Iso*$\Pi(\boldsymbol{b})$*.*

Here, due to the definition 2.2, $||\mathbf{b}||^2 = 0$ and therefore:

$$P(\lambda) = \alpha^3 - 3 \, . \, \alpha^2 \, . \, \lambda + 3 \, . \, \alpha \, . \, \lambda^2 - \lambda^3 = (\alpha - \lambda)^3$$

Each symmetric part of a Perian matrix in Iso$\Pi(\mathbf{b})$ has a triple eigenvalue coinciding with the first component of its first argument (i.e.: the vector **a**):

$$\forall \, [S] \in Iso\Pi(\mathbf{b}) : \, P(\lambda) = 0 \Rightarrow \lambda = \alpha$$

This result is true whatever **b** is.

Remark 2.8. *The Laplacian of the polynomial associated with a kernel-compatible matrix.*

It is obvious that the trace of a classical Hessian matrix is the classical Laplacian of the polynomial $\Lambda(\mathbf{b})$ of which the Hessian contains the second order partial derivates:

$$Tr[Hess_{(\mathbf{b},0)} \Lambda(\mathbf{b})] = \Delta\Lambda(\mathbf{b})$$

With (recall):

$$\Delta\Lambda(\mathbf{b}) = \frac{\partial^2 \Lambda(\mathbf{b})}{\partial^2 b^1} + \frac{\partial^2 \Lambda(\mathbf{b})}{\partial^2 b^2} + \frac{\partial^2 \Lambda(\mathbf{b})}{\partial^2 b^3}$$

Hence, the Laplacian of the polynomial associated with a kernel-compatible matrix is:

$$\Delta\Lambda(\mathbf{b}) = H_{11} + H_{22} + H_{33} = 2 \, . \, (3 \, . \, \alpha + \beta \, . \, ||\mathbf{b}||^2)$$

And, for degenerated polynomials in class II:

1. Subclass I:
$$\Delta\Lambda(\mathbf{b}) = 2\,.\,\beta\,.\,||\mathbf{b}||^2$$

2. Subclass II
$$\Delta\Lambda(\mathbf{b}) = 4\,.\,\alpha = -4\,.\,\beta\,.\,||\mathbf{b}||^2$$

Example 2.3. *The Laplacian of the polynomial associated with any kernel-compatible matrix in Iso*$\Pi(\boldsymbol{b})$*.*

It is obvious that:

$$\forall\,[M(\mathbf{a},\mathbf{b})] \in Kern\Pi(\mathbf{b}) \cap Iso\Pi(\mathbf{b}) \Rightarrow \Delta\Lambda(\mathbf{b}) = 6\,.\,\alpha$$

This result is true whatever **b** is.

2.4 The kernels as commutative operators? An essay.

Remark 2.9. *The decomposition of deformed cross products.*

Let recall some results which have been obtained in [c] and [d]. For example, each decomposition:

$$|[\mathbf{projectile},\,\mathbf{target}]_{[A]} >= [P]\,.\,|\mathbf{target} > +|\mathbf{residual\,part} >$$

... is related to a polynomial $\Lambda(\mathbf{projectile})$ and its kernel has the generic formalism:

$$^{(3)}[N_\Lambda^{|A|}] = \frac{1}{2}\,.\,[H_\Lambda] - \frac{1}{|A|}\,.\,_{[J]}\Phi(_\Lambda\mathbf{s}),\ |A| = \pm 1$$

This fact is evident for non-degenerated polynomials (class I) and a little bit more subtile for the degenerated ones (class II). To avoid misunderstandings, one introduces a specific semantic for each class:

1. For the Class I:
   — The matrix $[\mathrm{H}_\Lambda]$ is a Hessian matrix related to the non-degenerated polynomial $\Lambda(\mathbf{projectile})$ resulting from the decomposition;

   — The vector $_\Lambda\mathbf{s}$ is the singular vector of the non-degenerated polynomial $\Lambda(\mathbf{projectile})$ resulting from the decomposition.

   Example 2.4. *The kernel-compatible matrices.*

   The coincidence between the kernel of a decomposition and a kernel-compatible Perian matrix in $\Pi(\mathbf{b})$ (definition 2.5) means:

$$[S] = \alpha\,.\,Id_3 + \beta\,.\,T_2(\otimes)(\mathbf{b},\mathbf{b}) = \frac{1}{2}\,.\,[H_\Lambda]$$

$$\chi\,.\,\mathbf{b} = -\frac{1}{|A|}\,.\,_\Lambda\mathbf{s}$$

   Please, keep in mind that this coincidence only is a particular example and not a systematic reality!

2. For the Class II:
   — The matrix $[H_\Lambda]$ is a pseudo-Hessian matrix related to a pair of vectors $(\mathbf{K}, \mathbf{O})$ in $E^2{}_3$:
   $$[H_\Lambda] = T_2(\otimes)(\mathbf{K},\, \mathbf{O}) + T_2(\otimes)(\mathbf{O},\, \mathbf{K})$$
   — The vector ${}_\Lambda\mathbf{s}$ is a pseudo-singular vector related to the pair of vectors $(\mathbf{K}, \mathbf{O})$ in $E^2{}_3$:
   $$-\frac{1}{|A|} \cdot {}_\Lambda\mathbf{s} = \frac{1}{2} \cdot (\mathbf{K} \wedge \mathbf{O})$$

Recall that:
- — The kernel-matrices related to a decomposition are associated with polynomials.
- — There exist planes, front and volumic waves.
- — Waves and particles represents two visages of the physical reality.
- — Waves interfere
- — The results which have been obtained in [c] and [d] hold true when $K = \mathbb{R}$ or $K = \mathbb{C}$.

Question: These facts are suggesting a question: "Can the kernel-matrices related to a decomposition represent commutative operators reporting on the behavior of waves?"

The reading of [03] and [04; pp. 62-63] is helpful when one is looking for answers to this question. Within the quantum treatment of the reality, one manages pairs of conjugated variables, for example $(\phi^\lambda,\, \pi_\mu)$.

Hypothesis: Up to now, this exploration introduces a specific set of matrices, the formalism of which is per convention the sum (i) of (up to a minus sign) the half of the Hessian of a polynomial Λ and (ii) of a matrix representing an axial rotation around some vector $\mathbf{s}$:

$$\forall\, \Lambda \in F(E_3;\, K),\, \forall\, |A| = \pm 1 \,:\, [N_\Lambda^{|A|}] = -\frac{|A|}{2} \cdot [H_\Lambda] + {}_{[J]}\Phi(\mathbf{s})$$

There is in general no link between this specific set of matrices and the set of all Perian matrices. Sometimes, the half of the Hessian matrix may eventualy be a kernel-compatible matrix (definition 2.5). Nevertheless, even if it happens, there is no reason justifying a systematic coincidence between the argument of the polynomial Λ, e.g.: the vector $\mathbf{b}$, and the vector $\mathbf{s}$. The semantic *kernel-compatible* matrix will be explained a little bit later when this discussion will introduce the matrices representing the kernels resulting from the decomposition of deformed cross products.

Each element in the specific set of matrices one will work with has a conjugated representation which is per convention *minus its transposed*. Concretely, the discussion will try to manage pairs of kernels like this one:

$$([{}^{(3)}N_{\Lambda_\lambda}^{|A|}],\, [{}^{(3)}N_{\Lambda_\mu}^{|A|}]^* = -[{}^{(3)}N_{\Lambda_\mu}^{|A|}]^t)$$

To success, one must translate [04; p. 63, Equ.(188) and Equ.(189)] into the matrix language of this theory. Let start in proposing two formal relations:

$$\forall\,\Lambda_\lambda,\ \Lambda_\mu$$

1. The translation of [04; p. 63, Equ.(188)]:

$$\forall\,|A|\;=\;\pm 1:\;[N^{|A|}_{\Lambda_\lambda}]\,.\,\{-[N^{|A|}_{\Lambda_\mu}]^t\}\;-\;\{-[N^{|A|}_{\Lambda_\mu}]^t\}\,.\,[N^{|A|}_{\Lambda_\lambda}]\;=\;[?]$$

   ... where the unknown matrix [?] on the right side must represent the term on the right side of [04; p. 63, Equ.(188)]; precisely:

$$\delta_{\alpha\mu}\,.\,i\,.\,\hbar\,.\,\delta^3(\mathbf{r}\;-\;\mathbf{r}')$$

2. The translation of [04; p. 63, Equ.(189)]:

$$\forall\,|A|\;=\;\pm 1:\;[N^{|A|}_{\Lambda_\lambda}]\,.\,[N^{|A|}_{\Lambda_\mu}]\;-\;[N^{|A|}_{\Lambda_\mu}]\,.\,[N^{|A|}_{\Lambda_\lambda}]\;=\;[0]$$

Remark 2.10. *Concerning the first relation.*

One must state that:

$$\forall\,\Lambda_1,\ \Lambda_2,\ \forall\,|A|\;=\;\pm 1$$
$$[[N^{|A|}_{\Lambda_1}],\,[N^{|A|}_{\Lambda_2}]^*]$$
$$=$$
$$[N^{|A|}_{\Lambda_1}]\,.\,[N^{|A|}_{\Lambda_2}]^*\;-\;[N^{|A|}_{\Lambda_2}]^*\,.\,[N^{|A|}_{\Lambda_1}]$$
$$=$$
$$\{-\frac{|A|}{2}\,.\,[H_{\Lambda_1}]\;+\;{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}\,.\,\{\frac{|A|}{2}\,.\,[H_{\Lambda_2}]\;+\;{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\}$$
$$-\,\{\frac{|A|}{2}\,.\,[H_{\Lambda_2}]\;+\;{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\}\,.\,\{-\frac{|A|}{2}\,.\,[H_{\Lambda_1}]\;+\;{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}$$
$$=$$
$$\frac{1}{4}\,.\,\{[H_{\Lambda_2}]\,.\,[H_{\Lambda_1}]-[H_{\Lambda_1}]\,.\,[H_{\Lambda_2}]\}+\{{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\,.\,{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})-{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\,.\,{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}$$
$$+\,\frac{|A|}{2}\,.\,\{-[H_{\Lambda_1}]\,.\,{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})+{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\,.\,[H_{\Lambda_1}]+{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\,.\,[H_{\Lambda_2}]-[H_{\Lambda_2}]\,.\,{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}$$

When:

$$\Lambda_1\;=\;\Lambda_2\;=\;\Lambda$$

The previous relation is drastically simplified and it can be rewritten as (the brackets have been re-introduced):

$$[[N^{|A|}_{\Lambda}],\,[N^{|A|}_{\Lambda}]^*]\;=\;-|A|\,.\,[[H_\Lambda],\,{}_{[J]}\Phi({}_\Lambda\mathbf{s})]$$

This relation is the translation of [04; p. 63, Equ.(188)] when $\lambda = \mu$. Therefore, here, the bracket on the right side plays a role which should be equivalent to:

$$i\,.\,\hbar\,.\,\delta^3(\mathbf{r}\;-\;\mathbf{r}')$$

Luckily, this bracket has a formalism mimicking the one of [04; p. 63, Equ.(190)] representing the Heisenberg's equation of motion for operators when one writes:

$$[[H_\Lambda(t)],\, {}_{[J]}\Phi({}_\Lambda\mathbf{s}(t))] \,=\, -\,i\,.\,\hbar\,.\,\frac{d_{[J]}\Phi({}_\Lambda\mathbf{s}(t))}{dt}$$

This statement suggests that one should interpret the Hessian matrix as a representation of the total Hamiltonian associated with the polynomial Λ at instant t.

The translation of [04; p. 63, Equ.(188)] when $\lambda \neq \mu$ into the matrix language of this document is intuitively:

$$[[N_\Lambda^{|A|}],\, [N_\Lambda^{|A|}]^*] \,=\, [0]$$

The original version of this relation is realized when **r'** = **r**. This fact suggests that the bracket should contain an information about a pertinent difference concerning any polynomial. The difference between the argument **b** of a given polynomial and its (pseudo-)singular vector gives the feeling of being able to satisfy this requirement. Let consider the symmetric part of any Perian matrix when is argument is equal to its (pseudo-)singular vector:

$$[H_\Lambda] \,=\, 2\,.\,\{Id_3 \,+\, T_2(\otimes)({}_\Lambda\mathbf{s},\, {}_\Lambda\mathbf{s})\}$$

It is evident that, in this case (recall the remark 2.4):

$$[[N_\Lambda^{|A|}],\, [N_\Lambda^{|A|}]^*]$$
$$=$$
$$-|A|\,.\,[[H_\Lambda],\, {}_{[J]}\Phi({}_\Lambda\mathbf{s})]$$
$$=$$
$$-2\,.\,|A|\,.\,[Id_3 \,+\, T_2(\otimes)({}_\Lambda\mathbf{s},\, {}_\Lambda\mathbf{s}),\, {}_{[J]}\Phi({}_\Lambda\mathbf{s})]$$
$$=$$
$$-2\,.\,|A|\,.\,[Id_3,\, {}_{[J]}\Phi({}_\Lambda\mathbf{s})] \,-\, 2\,.\,|A|\,.\,[T_2(\otimes)({}_\Lambda\mathbf{s},\, {}_\Lambda\mathbf{s}),\, {}_{[J]}\Phi({}_\Lambda\mathbf{s})]$$
$$=$$
$$[0]$$

Remark 2.11. *Concerning the second relation.*

One must state that:

$$\forall\, \Lambda_1,\, \Lambda_2,\, \forall\, |A| \,=\, \pm 1$$

... the bracket:

$$[[N_{\Lambda_1}^{|A|}],\, [N_{\Lambda_2}^{|A|}]]$$

... only is a particular representation of the first relation when:

$$[N_{\Lambda_2}^{|A|}]^* \,=\, -[N_{\Lambda_2}^{|A|}]^t \,=\, [N_{\Lambda_2}^{|A|}]$$

With (recall):

$$\forall\,\Lambda,\,\forall\,|A|\,=\,\pm 1\,:\,[N_{\Lambda}^{|A|}]\,=\,-\frac{|A|}{2}\,.\,[H_{\Lambda}]\,+\,{}_{[J]}\Phi({}_{\Lambda}\mathbf{s})$$

When the Hessian matrix is symmetric, this situation is equivalent to:

$$[N_{\Lambda}^{|A|}]^{*}\,=\,\frac{|A|}{2}\,.\,[H_{\Lambda}]\,+\,{}_{[J]}\Phi({}_{\Lambda}\mathbf{s})\,=\,-\frac{|A|}{2}\,.\,[H_{\Lambda}]\,+\,{}_{[J]}\Phi({}_{\Lambda}\mathbf{s})\,=\,[N_{\Lambda}^{|A|}]$$

And it can only be realized when:

$$[H_{\Lambda}]\,=\,[0]$$

Or, equivalently, when the kernel matrix is reduced to an axial-rotation matrix:

$$[N_{\Lambda}^{|A|}]\,=\,{}_{[J]}\Phi({}_{\Lambda}\mathbf{s})$$

In this case, the kernel matrix is systematically degenerated.

Definition 2.6. *Commutative operators.*

Therefore, recalling at this stage the generic formalism of matrices representing the kernels of the main parts which have been obtained in [c] and [d] for each and any decomposition of a deformed cross product, precisely:

$$\forall\,\Lambda,\,\forall\,|A|\,=\,\pm 1\,:\,[N_{\Lambda}^{|A|}]\,=\,-\frac{|A|}{2}\,.\,[H_{\Lambda}]\,+\,{}_{[J]}\Phi({}_{\Lambda}\mathbf{s})$$

Where:

1. The symmetric part can be identified with the half of a (pseudo-)Hessian related to at least one polynomial.
2. The vector ${}_{\Lambda}\mathbf{s}$ is the (pseudo-)singular vector associated with the polynomial Λ.

One can now affirm that these matrices are suitable representations of commutative operators when:

1. They have a conjuguated version:

$$[N_{\Lambda}^{|A|}]^{*}\,=\,-[N_{\Lambda}^{|A|}]^{t}$$

2. They respect Heisenberg's equation of motion for operators and the relation:

$$[[N_{\Lambda}^{|A|}(t)],\,[N_{\Lambda}^{|A|}]^{*}(t)]\,=\,-|A|\,.\,[[H_{\Lambda}(t)],\,{}_{[J]}\Phi({}_{\Lambda}\mathbf{s}(t))]\,=\,-\,i\,.\,\hbar\,.\,\frac{d_{[J]}\Phi({}_{\Lambda}\mathbf{s}(t))}{dt}$$

This constraint induces that the Hessian of the polynomial at hand must be interpreted as an Hamiltonian:

$$[H_{\Lambda}]\,=\,[H_{\Lambda}]^{t}\,\equiv\,[Hamiltonian]$$

3. Due to the time-dependant Schrödinger's equation, the eigenvalues of half of the Hessian matrix (equivalently: of the symmetric part of the Kernel matrix) represent the total energies carried by this polynomial. This is the place where the results which have been obtained in remark 2.6 are useful. Precisely, the eigenvalues are the total energies:

$$E_1 = E_2 = \alpha$$

$$E_3 = \alpha + \beta \,.\, ||\mathbf{b}||^2$$

And the Laplacian is twice the sum of all possible total energies:

$$\Delta\Lambda(\mathbf{b}) = H_{11} + H_{22} + H_{33} = 2 \,.\, (E_1 + E_2 + E_3)$$

2.5 The inner logic of the approach

Any given Riemannian element of length, more precisely its square $(\delta \mathrm{s})^2$, is a polynomial depending on an infinitesimal change of the spatial position ${}^{(3)}\delta\mathbf{r}$ within a $1 + 3$ analysis.

Simultaneously, within the theory studying the deformed cross products, the $1 + 3$ formalism of this element of length is interpreted as a polynomial $\Lambda({}^{(3)}\delta\mathbf{r})$ proving the existence of a set of deformed and decomposed cross products of which the generic element is $||[{}^{(3)}\delta\mathbf{r}, \ldots][\mathrm{A}]> = |\mathrm{A}|.\{[\mathrm{A}]^t.[\mathrm{J}]\}.[\mathrm{N}].|...> + |\text{residual part}>$.

At this stage, the work which has been exposed in [c] and [d] furnishes two crucial tools: (i) the kernels [N] which can be associated with a given decomposition and (ii) the rules allowing to rebuild the polynomials potentially associable with it.

Therefore, in what follows below, one makes a fundamental hypothesis consisting of thinking that the reconstructed polynomials must obligatorily coincide with the family of Riemannian elements of length at hand. One remarks also by the way that any kernel coincides with the main part of the decomposition when this decomposition is made in a pre-Euclidean context ([A] = [J]).

This procedure yields a set of acceptable four-dimensional metrics for each set of physical situations. As examples, one envisages the case of a Thirring-Lense effect and of the dispersion relation for massive waves as well. A focus on the classical three-dimensional, symmetric and non-degenerated Euclidean metric is given at the end of this approach.

3 Deformed cross products and kernel-compatible matrices

As was mentioned in the example 2.4, not every kernel resulting from the decomposition of some deformed cross product is systematically a kernel-compatible

Perian matrix. Hence, it is legitimate to ask if (yes or no) and when (in which conditions) this eventuality occurs?

3.1 The polynomials associated with kernel-compatible Perian matrices

The kernel-compatible Perian matrix may sometimes be the symmetric part of some kernel related to the generic decomposition of:

$$|[^{(3)}\mathbf{projectile},\, \mathbf{target}]_{[A]} >= [P]\,.\,|\mathbf{target} > +|\mathbf{residual\,part} >$$

When it is the case, they are associated with the polynomials resulting from the decomposition at hand. Let recall the results which have been exposed in [c] and [d], and start with the formalism of the kernels related to non-degenerated Hessians (class I):

$$\forall\, \Lambda \in Class\, I,\, |A| = \pm 1 \,:$$

$$[N_\Lambda^{|A|}] = -\frac{|A|}{2}\,.\,[Hess_{(\mathbf{projectile},\,0)}\Lambda(\mathbf{projectile})] + {}_{[J]}\Phi({}_\Lambda\mathbf{s})$$

Therefore, this type of kernels contains a kernel-compatible Perian matrix when there exists at least one polynomial $\Lambda(^{(3)}\mathbf{projectile})$ of which the half of the Hessian is (i) not degenerated and (ii) in coincidence with the symmetric part of the kernel-compatible Perian matrix [2]:

$$\alpha\,.\,Id_3 + 2\,.\,T_2(\otimes)(^{(3)}\mathbf{b},\, ^{(3)}\mathbf{b}) = -\frac{|A|}{2}\,.\,[\frac{\partial^2\Lambda(^{(3)}\mathbf{projectile})}{\partial projectile^\lambda \partial projectile^\mu}]$$

The polynomial $\Lambda(^{(3)}\mathbf{projectile})$ has a coefficient of degree zero coinciding with *minus* the determinant of the main part of the decomposition (d = -|P|). But the relation(see [c; p. 31]):

$$[P] = |A|\,.\,\{[A]^t\,.\,[J]\}\,.\,[N]$$

... implies that:

$$|P| = |A|\,.\,\{|A|\,.\,-1\}\,.\,|N| = -|N|$$

Therefore, here, the coefficient of degree zero is:

$$|N_\Lambda^{|A|}|$$

$$=$$

$$|-\frac{|A|}{2}\,.\,[Hess_{(\mathbf{projectile},\,0)}\Lambda(\mathbf{projectile})] + {}_{[J]}\Phi({}_\Lambda\mathbf{s})|$$

$$=$$

$$|\alpha\,.\,Id_3 + \beta\,.\,T_2(\otimes)(^{(3)}\mathbf{b},\, ^{(3)}\mathbf{b}) + {}_{[J]}\Phi({}_\Lambda\mathbf{s})|$$

$$=$$

$$\alpha^3 + \beta\,.\,||^{(3)}\mathbf{b}||^2\,.\,\alpha^2 + ||^{(3)}\mathbf{s}||^2\,.\,\alpha + \beta\,.\,\{<^{(3)}\mathbf{s},\, ^{(3)}\mathbf{b}>_{Id_3}\}^2$$

2. This is justifying the semantic - recall the definition 2.5.

At this stage, one may ask for more information concerning the non-degenerated polynomial $\Lambda({}^{(3)}\mathbf{projectile})$. In general and *a priori* one expects that this polynomial has coefficients of degree one (d_1, d_2, d_3) forming a vector denoted $\mathbf{d}^*$ of which the representation in E_3 is:

$$|{}^{(3)}\mathbf{d}^* >$$

$$=$$

$$-[Hess_{(\mathbf{projectile},\,0)}\Lambda(\mathbf{projectile})]\,.\,|{}^{(3)}\mathbf{s} >$$

$$=$$

$$\frac{2}{|A|}\,.\,\{\alpha\,.\,Id_3\,+\,\beta\,.\,T_2(\otimes)({}^{(3)}\mathbf{b},\,{}^{(3)}\mathbf{b})\}\,.\,|{}^{(3)}\mathbf{s} >$$

$$=$$

$$\frac{2}{|A|}\,.\,\alpha\,.\,|{}^{(3)}\mathbf{s} > \,+\, \frac{2}{|A|}\,.\,\beta\,.\, <^{(3)}\mathbf{b},\,{}^{(3)}\mathbf{s} >_{Id_3}\,.\,|{}^{(3)}\mathbf{b} >$$

This result gives a first hint concerning the formalism of the polynomial Λ because in general:

$$\Lambda(\mathbf{projectile}) \,=\, ...+ <^{(3)}\mathbf{d}^*,\,{}^{(3)}\mathbf{projectile} >_{Id_3} + ...$$

If, exceptionaly:

$$\chi\,.\,{}^{(3)}\mathbf{b} \,=\, {}^{(3)}\mathbf{s}$$

Then:

$$\chi^2\,.\,|{}^{(3)}\mathbf{d}^* > = \frac{2}{|A|}\,.\,(\alpha\,.\,\chi^2\,+\,\beta\,.\,||^{(3)}\mathbf{s}||^2)\,.\,|{}^{(3)}\mathbf{s} >$$

Or, if one prefers:

$$|{}^{(3)}\mathbf{d}^* > = \frac{2}{|A|}\,.\,\chi\,.\,(\alpha\,+\,\beta\,.\,||^{(3)}\mathbf{b}||^2)\,.\,|{}^{(3)}\mathbf{b} >$$

Concerning the coefficients of degree two:

$$<^{(3)}\mathbf{projectile}|\,.\,[D]\,.\,|{}^{(3)}\mathbf{projectile} >$$

$$=$$

$$<^{(3)}\mathbf{projectile}|\,.\,\{\alpha\,.\,Id_3\,+\,\beta\,.\,T_2(\otimes)({}^{(3)}\mathbf{b},\,{}^{(3)}\mathbf{b})\,+\,{}_{[J]}\Phi({}_{\Lambda}\mathbf{s})\}\,.\,|{}^{(3)}\mathbf{projectile} >$$

$$=$$

$$\alpha\,.\,||^{(3)}\mathbf{projectile}||^2\,+\,\beta\,.\,\{<^{(3)}\mathbf{b},\,{}^{(3)}\mathbf{projectile} >_{Id_3}\}^2$$

At the end of this progression, there is a polynomial:

$$\Lambda({}^{(3)}\mathbf{projectile})$$

$$=$$

$$\alpha\,.\,||^{(3)}\mathbf{projectile}||^2\,+\,\beta\,.\,\{<^{(3)}\mathbf{b},\,{}^{(3)}\mathbf{projectile} >_{Id_3}\}^2$$

$$+\,\frac{2}{|A|}\,.\,\{<\alpha\,.\,{}^{(3)}\mathbf{s}\,+\,\beta\,.\,<^{(3)}\mathbf{b},\,{}^{(3)}\mathbf{s} >_{Id_3}\,.\,{}^{(3)}\mathbf{b},\,{}^{(3)}\mathbf{projectile} >_{Id_3}\}$$

$$+\,\alpha^3\,+\,\beta\,.\,||^{(3)}\mathbf{b}||^2\,.\,\alpha^2\,+\,||^{(3)}\mathbf{s}||^2\,.\,\alpha\,+\,\beta\,.\,\{<^{(3)}\mathbf{s},\,{}^{(3)}\mathbf{b} >_{Id_3}\}^2$$

Remark 3.1. *Important logical precisions.*

It seems to be useful to insist on certain items:

1. The vector **b** characterizing the symmetric part of a kernel-compatible Perian matrix is not necessarily coinciding with the argument of the polynomial Λ associated with the decomposition of some deformed cross product: the so-called *projectile*.
2. A kernel-compatible Perian matrix is an ordinary Perian matrix exactly when the vector characterizing it and the singular vector of the polynomial associated with the decomposition of some deformed cross product are related to each other through the relation:
$$\chi \,.\, {}^{(3)}\mathbf{b} \,=\, {}^{(3)}\mathbf{s}$$
3. When a kernel-compatible Perian matrix is effectively an ordinary Perian matrix, then it is also the representation in $\mathrm{M}(3,\, \mathbb{C})$ of a ERp if:
$$\alpha \,=\, 2\,.\,(\varsigma^0)^2 \,-\, 1, \quad \beta \,=\, 2, \quad \chi \,=\, 2\,.\,\varsigma^0$$

Therefore, some special situations can be envisaged:

Example 3.1. $\boldsymbol{b} = \boldsymbol{projectile}$

Here, the vector characterizing a kernel-compatible Perian matrix, for example **b**, coincides with the argument of the polynomial, for example the so-called **projectile**. In this case, the polynomial which is associated with the decomposition of the deformed cross product at hand has the specific formalism:

$$\Lambda({}^{(3)}\mathbf{projectile})$$

$$=$$

$$(\alpha + \beta\,.\,||^{(3)}\mathbf{projectile}||^2 + \alpha^2\,.\,\beta)\,.\,||^{(3)}\mathbf{projectile}||^2 + \beta\,.\,\{<^{(3)}\mathbf{s},\ {}^{(3)}\mathbf{projectile}>_{Id_3}\}^2$$

$$+\frac{2}{|A|}\,.\,(\alpha\,+\,\beta\,.\,||^{(3)}\mathbf{projectile}||^2)\,.\ <^{(3)}\mathbf{s},\ {}^{(3)}\mathbf{projectile}>_{Id_3}$$

$$+\alpha^3\,+\,||^{(3)}\mathbf{s}||^2\,.\,\alpha$$

The coincidence **b** = **projectile** seems to be the simplest way to match the two concepts being discussed: the kernel-compatible Perian matrices and the kernels related to the decomposition of deformed cross products. This way of doing does not require to systematically involve a Euler-Rodrigues parametrisation into the discussion. This kind of parametrisation becomes only a very particular case corresponding to an eventual coincidence between the argument of the polynomial and its (pseudo-) singular vector.

Example 3.2. $\chi\,.\,\boldsymbol{b} = \boldsymbol{s}$

In this case, the vector characterizing the symmetric part of a kernel-compatible Perian matrix coincides with the singular vector of the polynomial which is associated with the decomposition of the deformed cross product at hand. The polynomial has the specific formalism:

$$\Lambda({}^{(3)}\mathbf{projectile})$$

$$=$$

$$\alpha\,.\,||^{(3)}\mathbf{projectile}||^2\,+\,\beta\,.\,\{<^{(3)}\mathbf{b},\ {}^{(3)}\mathbf{projectile}>_{Id_3}\}^2$$

$$+\frac{2}{|A|}\,.\,\{<\alpha\,.\,{}^{(3)}\mathbf{s}\,+\,\beta\,.\ <^{(3)}\mathbf{b},\ {}^{(3)}\mathbf{s}>_{Id_3}\ .\,{}^{(3)}\mathbf{b},\ {}^{(3)}\mathbf{projectile}>_{Id_3}\}$$

$$+\alpha^3\,+\,\beta\,.\,||^{(3)}\mathbf{b}||^2\,.\,\alpha^2\,+\,||^{(3)}\mathbf{s}||^2\,.\,\alpha\,+\,\beta\,.\,\{<^{(3)}\mathbf{s},\ {}^{(3)}\mathbf{b}>_{Id_3}\}^2$$

$$=$$

$$\alpha\,.\,||^{(3)}\mathbf{projectile}||^2\,+\,\beta\,.\,\{<^{(3)}\mathbf{b},\ {}^{(3)}\mathbf{projectile}>_{Id_3}\}^2$$

$$+\frac{2\,.\,\chi}{|A|}\,.\,(\alpha\,+\,\beta\,.\,||^{(3)}\mathbf{b}||^2)\,.\ <^{(3)}\mathbf{b},\ {}^{(3)}\mathbf{projectile}>_{Id_3}$$

$$+(\alpha\,+\,\beta\,.\,||^{(3)}\mathbf{b}||^2)\,.\,(\alpha^2\,+\,\chi^2\,.\,||^{(3)}\mathbf{b}||^2)$$

This example includes any OPM of which the symmetric part is kernel-compatible when χ times the second argument of this OPM ($\mathbf{b}$) coincides with the singular vector ($\mathbf{s}$) of the polynomial justifying the existence of this kernel. Hence, this example includes all OPMs representing a ERp. For the latters, the polynomial is:

$$\Lambda({}^{(3)}\mathbf{projectile})$$

$$=$$

$$(2\,.\,(\varsigma^0)^2\,-\,1)\,.\,||^{(3)}\mathbf{projectile}||^2\,+\,2\,.\,\{<^{(3)}\,\varsigma,\,{}^{(3)}\mathbf{projectile}\,>_{Id_3}\}^2$$
$$+\,\frac{4\,.\,\varsigma^0}{|A|}\,.\,(2\,.\,(\varsigma^0)^2\,-\,1\,+\,2\,.\,||^{(3)}\varsigma||^2)\,.\,<^{(3)}\,\varsigma,\,{}^{(3)}\mathbf{projectile}\,>_{Id_3}$$
$$+\,(2\,.\,(\varsigma^0)^2\,-\,1\,+\,2\,.\,||^{(3)}\varsigma||^2)\,.\,\{(2\,.\,(\varsigma^0)^2\,-\,1)^2\,+\,4\,.\,(\varsigma^0)^2\,.\,||^{(3)}\varsigma||^2\}$$

But here, something special happens because the ERp is the result of the existence of a four-dimensional unit sphere:

$$(\varsigma^0)^2\,+\,||^{(3)}\varsigma||^2\,=\,1$$

Hence, the polynomial is drastically reduced to:

$$\Lambda({}^{(3)}\mathbf{projectile})$$

$$=$$

$$(2\,.\,(\varsigma^0)^2\,-\,1)\,.\,||^{(3)}\mathbf{projectile}||^2\,+\,2\,.\,\{<^{(3)}\,\varsigma,\,{}^{(3)}\mathbf{projectile}\,>_{Id_3}\}^2$$
$$+\,\frac{4\,.\,\varsigma^0}{|A|}\,.\,<^{(3)}\,\varsigma,\,{}^{(3)}\mathbf{projectile}\,>_{Id_3}$$
$$+1$$

Example 3.3. $\chi.\boldsymbol{b}\,=\,\boldsymbol{s}\,=\,$***projectile***

All actors intervening in this part of the discussion coincide. The polynomial is now written:

$$\Lambda({}^{(3)}\mathbf{projectile})$$

$$=$$

$$...$$

Remark 3.2. *The polynomials associated with kernels related to degenerated Hessian matrices (Class II).*

The discussion within this subsection has started in considering the formalism of kernels related to non-degenerated Hessian matrices. The non-degeneracy of these Hessian matrices imposes a supplementary constraint; recall the remark 2.6:

$$\alpha^2\,.\,(\alpha\,+\,\beta\,.\,||^{(3)}\mathbf{b}||^2)\,\neq\,0$$

But, as explained in previous works [c] and [d], any Hessian matrix may sometimes be degenerated. In that case, the formalism concerning the kernels in the first class can be used further in introducing an ad hoc pair of vectors. But, what is the formalism of the polynomials associated with degenerated Hessian matrices?

Let inject the condition:

$$\alpha^2 \,.\, (\alpha \,+\, \beta \,.\, ||^{(3)}\mathbf{b}||^2) \,=\, 0$$

... into the result which has been obtained in subsection 3.1; therefore:

1. When $\alpha \neq 0$ but $\alpha + \beta.||\mathbf{b}||^2 = 0$:

$$\Lambda(^{(3)}\mathbf{projectile})$$
$$=$$
$$\alpha \,.\, ||^{(3)}\mathbf{projectile}||^2 \,+\, \beta \,.\, \{<^{(3)} \mathbf{b},\, ^{(3)}\mathbf{projectile} >_{Id_3}\}^2$$
$$+ \frac{2}{|A|} \,.\, \{< \alpha \,.\, ^{(3)}\mathbf{s} \,+\, \beta \,.\, <^{(3)} \mathbf{b},\, ^{(3)}\mathbf{s} >_{Id_3} .\, ^{(3)}\mathbf{b},\, ^{(3)}\mathbf{projectile} >_{Id_3}\}$$
$$+ ||^{(3)}\mathbf{s}||^2 \,.\, \alpha \,+\, \beta \,.\, \{<^{(3)} \mathbf{s},\, ^{(3)}\mathbf{b} >_{Id_3}\}^2$$

2. When $\alpha = 0$ but $\beta.||\mathbf{b}||^2 \neq 0$:

$$\Lambda(^{(3)}\mathbf{projectile})$$
$$=$$
$$\beta \,.\, \{<^{(3)} \mathbf{b},\, ^{(3)}\mathbf{projectile} >_{Id_3}\}^2$$
$$+ \frac{2}{|A|} \,.\, \{\beta \,.\, <^{(3)} \mathbf{b},\, ^{(3)}\mathbf{s} >_{Id_3} .\, ^{(3)}\mathbf{b},\, ^{(3)}\mathbf{projectile} >_{Id_3}\}$$
$$+ \beta \,.\, \{<^{(3)} \mathbf{s},\, ^{(3)}\mathbf{b} >_{Id_3}\}^2$$

Let then envisage diverse special situations:

1. When $\mathbf{b} = \mathbf{projectile}$; reconsidering the example 3.1, one can now write the condition of degeneracy as:

$$\alpha^2 \,.\, (\alpha \,+\, \beta \,.\, ||^{(3)}\mathbf{projectile}||^2) \,=\, 0$$

... and get either ($\alpha = 0$ but $\beta.||\mathbf{b}||^2 \neq 0$):

$$\Lambda(^{(3)}\mathbf{projectile})$$
$$=$$
$$\beta \,.\, ||^{(3)}\mathbf{projectile}||^2 \,.\, ||^{(3)}\mathbf{projectile}||^2 + \beta \,.\, \{<^{(3)} \mathbf{s},\, ^{(3)}\mathbf{projectile} >_{Id_3}\}^2$$
$$+ \frac{2}{|A|} \,.\, \beta \,.\, ||^{(3)}\mathbf{projectile}||^2 \,.\, <^{(3)} \mathbf{s},\, ^{(3)}\mathbf{projectile} >_{Id_3}$$

Or ($\alpha + \beta.||\mathbf{projectile}||^2 = 0$):

$$\Lambda(^{(3)}\mathbf{projectile})$$
$$=$$
$$\beta \,.\, \{<^{(3)} \mathbf{s},\, ^{(3)}\mathbf{projectile} >_{Id_3}\}^2 \,-\, \beta \,.\, ||^{(3)}\mathbf{s}||^2 \,.\, ||^{(3)}\mathbf{projectile}||^4$$

2. When $\chi.\mathbf{b} = \mathbf{s}$; reconsidering the example 3.2, one obtains either ($\alpha = 0$ but $\beta.||\mathbf{b}||^2 \neq 0$):

$$\Lambda({}^{(3)}\mathbf{projectile})$$

$$=$$

$$\beta\,.\,\{<^{(3)}\,\mathbf{b},\ ^{(3)}\mathbf{projectile}\ >_{Id_3}\}^2$$

$$+\ \frac{2\,.\,\chi}{|A|}\,.\,\beta\,.\,||^{(3)}\mathbf{b}||^2\,.\ <^{(3)}\,\mathbf{b},\ ^{(3)}\mathbf{projectile}\ >_{Id_3}$$

$$+\,\beta\,.\,\chi^2\,.\,||^{(3)}\mathbf{b}||^4$$

Or ($\alpha + \beta.||\mathbf{projectile}||^2 = 0$):

$$\Lambda({}^{(3)}\mathbf{projectile})$$

$$=$$

$$-\beta\,.\,||^{(3)}\mathbf{projectile}||^4\ +\ \beta\,.\,\{<^{(3)}\,\mathbf{b},\ ^{(3)}\mathbf{projectile}\ >_{Id_3}\}^2$$

Remark 3.3. *Degeneracy and Euler-Rodrigues parametrisations.*

The degeneracy means:

$$(2\,.\,(\varsigma^0)^2\ -\ 1)^2\,.\,\{2\,.\,(\varsigma^0)^2\ -\ 1\ +\ 2\,.\,||^{(3)}\varsigma||^2\}\ =\ (2\,.\,(\varsigma^0)^2\ -\ 1)^2\ =\ 0$$

Or, equivalently:

$$\varsigma^0\ =\ \pm\frac{1}{\sqrt[2]{2}}$$

And:

$$\Lambda({}^{(3)}\mathbf{projectile})$$

$$=$$

$$2\,.\,\{<^{(3)}\,\varsigma,\ ^{(3)}\mathbf{projectile}\ >_{Id_3}\}^2\ \pm\ \frac{4}{|A|\,.\,\sqrt[2]{2}}\,.\ <^{(3)}\,\varsigma,\ ^{(3)}\mathbf{projectile}\ >_{Id_3}\ +1$$

3.2 The dispersion relation for free massive waves in vacuum

Proposition 3.1. *The dispersion relation for massive waves "freely propagating" in vacuum can be reformulated as*[3]*:*

$$<^{(4)}\,\mathbf{P}^*,\ ^{(4)}\mathbf{P}^*\ >_{[\eta]^{-1}} =\ \eta^{\lambda\mu}\,.\,P_\lambda\,.\,P_\mu\ =\ 1$$

$$[\eta]\ =\ [\eta]^{-1}\ =\ [\eta^{\lambda\mu}]\ \equiv\ (1,\ -1,\ -1,\ -1)$$

... in following diverse mathematical paths and this generic reformulation:

1. is equivalent to the Klein-Gordon Equation (K.G.E in this document);
2. allows a Euler-Rodrigues parametrization.

3. Per convention, in my work, a vector is always denoted with a bold letter and the asterisk behind this letter is indicating that it is the co-variant version of this vector.

Proof. Let start with the usual and semi-classical formulation of the dispersion relation for massive (or also sometimes called De Broglie) waves, e.g. as in [02; p. 4, (5)] or [10; p. 555, (5) - FR]; E is the total amount of energy, m represents a mass and p the intensity of a kinetic momentum $^{(3)}\mathbf{p}$:

$$E^2 = m^2 . c^4 + c^2 . p^2$$

For a mathematician, this relation can easily be rewritten:

1. either as (because c $\sim 3.10^8$ meters/second $\neq 0$):

$$\frac{E^2}{c^2} - p^2 = m^2 . c^2$$

This reformulation can be intuitively interpreted as:

$$(\frac{E}{c})^2 - ||^{(3)}\mathbf{p}||^2 = (m . c)^2$$

... where the symbol ||...|| denotes the very classical Euclidean norm for the vector **...** in E(3, $\mathbb{R}$); at this stage:

— considering only non-vanishing masses (m $\neq$ 0) and making use of the well-known equivalence between mass and energy (Einstein), we can associate an energy to each non-vanishing mass:

$$E_0(m) = m . c^2 \neq 0$$

— supposing that the geometry is represented by the (4-4) matrix $[\eta]$:

$$[\eta] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = [\eta]^{-1}$$

— introducing the vector $^{(4)}\mathbf{P}^*$ into that discussion, the four components of which being per convention:

$$P_0 = \frac{E}{E_0}, \forall k = 1, 2, 3 : P_k = \frac{c . p_k}{E_0} \quad (1)$$

... we can rephrase the dispersion relation inside a four-dimensional context in writing it:

$$<^{(4)}\mathbf{P}^*, ^{(4)}\mathbf{P}^* >_{[\eta]^{-1}} = 1$$

This reformulation is nothing but, the proposed relation.

At this stage, let inject [02; p. 16, (43)] in [02; p. 4, (6)] and recover the generic expression for the K.G.E when (i) there is no force interacting with the wave and (ii) the metric is represented by the (4-4) matrix $[\eta]$:

$$<^{(4)}\mathbf{k}^*, ^{(4)}\mathbf{k}^* >_{[\eta]^{-1}} = (\frac{m . c}{\hbar})^2$$

After that, let re-scale the wave vector $\mathbf{k}^*$:

$$^{(4)}\mathbf{P}^* = \frac{\hbar}{m\,.\,c}\,.\,^{(4)}\mathbf{k}^*$$

... and get the proposed relation again. Hence, it is correct to say that the relation dispersion for massive waves freely propagating in vacuum is only another formulation of the K.G.E in a specific geometrical context represented by the matrix $[\eta]$.

2. Or as (but only if E $\neq 0$):

$$\frac{m^2\,.\,c^4}{E^2} + \frac{c^2\,.\,p^2}{E^2} = 1$$

As already said previously, this reformulation allows a direct link with the concept of Euler-Rodrigues parametrisation.

□

Remark 3.4. *How to apply the theory to the representation of a ERp?*

The existence of the dispersion relation is obviously an invitation to write:

$$\varsigma^0 = \frac{m\,.\,c^2}{E} = \frac{E_0}{E}$$

$$\forall\, i = 1,\,2,\,3 : \varsigma^i = \frac{c\,.\,p^i}{E}$$

And the context allows the definition of OPMs such that (recall the theorem 3.1:

$$^{(3)}[M(\mathbf{a},\,^{(3)}\varsigma)]$$
$$=$$
$$(2\,.\,(\varsigma^0)^2 - 1)\,.\,Id_3 + 2\,.\,T_2(\otimes)(^{(3)}\varsigma,\,^{(3)}\varsigma) + 2\,.\,\varsigma^0\,.\,_{[J]}\Phi(^{(3)}\varsigma)$$

But, if the quest is to find a coherent link between this matrix and a kernel related to the decomposition of a pertinent deformed cross product, one must prefer to work with matrices like this one:

$$^{(3)}[M]$$
$$=$$
$$(2\,.\,(\varsigma^0)^2 - 1)\,.\,Id_3 + 2\,.\,T_2(\otimes)(^{(3)}\varsigma,\,^{(3)}\varsigma) + {}_{[J]}\Phi(^{(3)}\mathbf{s})$$

Or like this other one:

$$^{(3)}[M]$$
$$=$$
$$\alpha\,.\,Id_3 + \beta\,.\,T_2(\otimes)(^{(3)}\mathbf{b},\,^{(3)}\mathbf{b}) + 2\,.\,\varsigma^0\,.\,_{[J]}\Phi(^{(3)}\varsigma)$$

... because the OPM itself appears to be a limit configuration of these formalisms; recall the comments which have been made at the end of example 3.1.

Remark 3.5. *Looking for an interpretation - part 01.*

The discussion gives rise to a central question: "What does the polynomial $\Lambda(\mathbf{projectile})$ represent when the procedure is applied to real physical situations?"

The definition 2.6 is an attempt to introduce operators into this exploration. With the hope to win useful information, let calculate:

$$[[N_{\Lambda}^{|A|}(t)],\, [N_{\Lambda}^{|A|}]^{*}(t)] \;=\; -|A|\,.\,[[H_{\Lambda}(t)],\, {}_{[J]}\Phi({}_{\Lambda}\mathbf{s}(t))] \;=\; -\,i\,.\,\hbar\,.\,\frac{d_{[J]}\Phi({}_{\Lambda}\mathbf{s}(t))}{dt}$$

More precisely let consider any kernel-compatible Perian matrix for which, like in example 3.1, $\mathbf{b} = \mathbf{projectile}$:

$$^{(3)}[N_{\Lambda}^{|A|}(t)] \;=\; \underbrace{\alpha\,.\,Id_3 \;+\; \beta\,.\,T_2(\otimes)(^{(3)}\mathbf{b},\, ^{(3)}\mathbf{b})}_{=-\frac{|A|}{2}\,.\,[Hess_{(-,\,0)}\Lambda(\mathbf{b})]} + {}_{[J]}\Phi(^{(3)}\mathbf{s})$$

Let then calculate the difference:

$$-\frac{|A|}{2}\,.\,[[Hess_{(-,0)}\Lambda(\mathbf{b})],\, {}_{[J]}\Phi(\mathbf{s})]$$

$$=$$

$$-\frac{|A|}{2}\,.\,\{\{\alpha\,.\,Id_3 + \beta\,.\,T_2(\otimes)(^{(3)}\mathbf{b},\, ^{(3)}\mathbf{b})\}\,.\,{}_{[J]}\Phi(\mathbf{s}) - {}_{[J]}\Phi(\mathbf{s})\,.\,\{\alpha\,.\,Id_3 + \beta\,.\,T_2(\otimes)(^{(3)}\mathbf{b},\, ^{(3)}\mathbf{b})\}\}$$

$$=$$

$$-\frac{|A|\,.\,\beta}{2}\,.\,\{T_2(\otimes)(^{(3)}\mathbf{b},\, ^{(3)}\mathbf{b})\,.\,{}_{[J]}\Phi(\mathbf{s}) \;-\; {}_{[J]}\Phi(\mathbf{s})\,.\,T_2(\otimes)(^{(3)}\mathbf{b},\, ^{(3)}\mathbf{b})\}$$

$$=^{4}$$

$$-\frac{|A|\,.\,\beta}{2}\,.\,\{T_2(\otimes)(^{(3)}\mathbf{b} \wedge\, ^{(3)}\mathbf{s},\, ^{(3)}\mathbf{b}) \;-\; T_2(\otimes)(^{(3)}\mathbf{b},\, ^{(3)}\mathbf{s} \wedge\, ^{(3)}\mathbf{b})\}$$

$$=$$

$$-i\,.\,\hbar\,.\,\frac{d_{[J]}\Phi(\mathbf{s}(t))}{dt}$$

The bracket gives information on the evolution along the time of the (pseudo-)singular vector $\mathbf{s}$ of the polynomial $\Lambda(\mathbf{b})$ at hand. It may occasionaly happen that this (pseudo-)singular vector coincides with the argument of the polynomial: $\mathbf{b} \equiv \mathbf{s}$; this is the situation which is envisaged in example 3.2. In the latter case, the matrix at hand ressembles a ERp but, due to Heisenberg's equation of motion for operators, the (pseudo-)singular vector is/becomes constant along the time because the bracket vanishes.

$$^{(3)}\mathbf{b} \;\equiv\; ^{(3)}\mathbf{s} \;\Rightarrow\; \frac{d\mathbf{s}}{dt} \;=\; \mathbf{0}$$

4. Recall the useful rules in remark 2.4.

Otherwise, when the argument does not coincide with the (pseudo-)singular vector, the proposition which has been made in this document writes:

$$- i \,.\, \hbar \,.\, \frac{d_{[J]} \Phi({}_{\Lambda}\mathbf{s}(t))}{dt}$$

$$= ^{*1}$$

$$- \frac{|A| \,.\, \beta}{2} \,.\, \{ T_2(\otimes)({}^{(3)}\mathbf{b},\, {}^{(3)}\mathbf{b}) \,.\, {}_{[J]}\Phi(\mathbf{s}) \,-\, {}_{[J]}\Phi(\mathbf{s}) \,.\, T_2(\otimes)({}^{(3)}\mathbf{b},\, {}^{(3)}\mathbf{b}) \}$$

$$= ^{*2}$$

$$- \frac{|A| \,.\, \beta}{2} \,.\, \{ T_2(\otimes)({}^{(3)}\mathbf{b} \wedge {}^{(3)}\mathbf{s},\, {}^{(3)}\mathbf{b}) \,+\, T_2(\otimes)({}^{(3)}\mathbf{b},\, {}^{(3)}\mathbf{b} \wedge {}^{(3)}\mathbf{s}) \}$$

Both formulations call for comments:

1. ^{*1} A speculative exploration in [e; §3.6, pp. 19-27] has analyzed the formula [G].Φ - Φ.[G] = d[G]/dt which was inspired from [12]. The formulation appearing here mimicks this formula if one interprets the Pythagorean table as the representation of a degenerated spatial metric:

$$T_2(\otimes)({}^{(3)}\mathbf{b},\, {}^{(3)}\mathbf{b}) \,\equiv\, {}^{(3)}[G]?$$

   This suggestion is pushing the discussion into the direction of acoustic metrics. A more precise answer will be given below in this document.

2. ^{*2} The work which has been made in [c] and [d] concluded that any decomposition of a given deformed cross product belongs either to the class I or to the class II. The formalism of a kernel in class II involves a pair of vectors $({}^{(3)}\mathbf{K},\, {}^{(3)}\mathbf{O})$ and the matrix playing a role equivalent to the Hessian matrix is:

$$\frac{1}{2} \,.\, \{ T_2(\otimes)({}^{(3)}\mathbf{K},\, {}^{(3)}\mathbf{O}) \,+\, T_2(\otimes)({}^{(3)}\mathbf{O},\, {}^{(3)}\mathbf{K}) \}$$

   It is a matter of fact that one disposes here of such a matrix in writing:

$${}^{(3)}\mathbf{K} \,=\, {}^{(3)}\mathbf{b} \wedge {}^{(3)}\mathbf{s}$$

$${}^{(3)}\mathbf{O} \,=\, {}^{(3)}\mathbf{s}$$

3.3 Three analysis studying the Riemannian element of length

Remark 3.6. *Main results.*

Let consider the Riemannian element of length:

$$(\delta s)^2 \,=\, g_{\lambda\mu} \,.\, \delta x^{\lambda} \,.\, \delta x^{\mu}$$

There are at least three manners to interpret it:

1. Within an A.D.M. analysis, the element of length is decomposed in that way [05; p. 7, (3.9), (3.10) et (3.11)]:

$$(\delta s)^2 \,=\, g_{ab} \,.\, \delta x^a \,.\, \delta x^b \,+\, \{2 \,.\, N_a \,.\, \delta x^0\} \,.\, \delta x^a \,-\, (N^2 \,-\, N_a \,.\, N^a) \,.\, (\delta x^0)^2$$

2. ... whilst it can mathematically be rewritten as a polynomial $\Lambda(\delta\mathrm{x}^1, \delta\mathrm{x}^2, \delta\mathrm{x}^3)$ within a 3 + 1 decomposition of spacetime [06]:

$$(\delta s)^2 = g_{ab} \,.\, \delta x^a \,.\, \delta x^b + \{(g_{0a} + g_{a0}) \,.\, \delta x^0\} \,.\, \delta x^a + g_{00} \,.\, (\delta x^0)^2$$

3. At this stage, due to the initial theorem [c], one can also presume that this polynomial is associated with the decomposition of some deformed cross product like this one:

$$|[\delta\mathbf{x}, \mathbf{target}]_{[A]} >= [P] \,.\, |\mathbf{target} > + |\mathbf{residual\,part} >$$

... if one writes:

$$\Lambda(\delta\mathbf{x}) = |_{[J]}\Phi(\delta\mathbf{x}) - [P]| = (\delta s)^2$$

The explorations [c] and [d] have brough algebraic indications concerning the coefficients of the polynomial:

(a) The degree zero coefficient is minus the determinant of the main part of the decomposition but, equal to the kernel of the decomposition:

$$d = -|P| = |N|$$

(b) The three coefficients of degree one are:

$$d_1 = A^1_{12} \,.\, (p_{31} \,.\, p_{23} - p_{21} \,.\, p_{33}) + A^2_{12} \,.\, (p_{33} \,.\, p_{11} - p_{31} \,.\, p_{13}) + A^3_{12} \,.\, (p_{21} \,.\, p_{13} - p_{11} \,.\, p_{23}) + A^1_{13} \,.\, (p_{21} \,.\, p_{32} - p_{22} \,.\, p_{31}) + A^2_{13} \,.\, (p_{31} \,.\, p_{12} - p_{11} \,.\, p_{32}) + A^3_{13} \,.\, (p_{11} \,.\, p_{22} - p_{21} \,.\, p_{12})$$

$$d_2 = A^1_{12} \,.\, (p_{32} \,.\, p_{23} - p_{22} \,.\, p_{33}) + A^2_{12} \,.\, (p_{33} \,.\, p_{12} - p_{32} \,.\, p_{13}) + A^3_{12} \,.\, (p_{22} \,.\, p_{13} - p_{12} \,.\, p_{23}) + A^1_{23} \,.\, (p_{21} \,.\, p_{32} - p_{22} \,.\, p_{31}) + A^2_{23} \,.\, (p_{31} \,.\, p_{12} - p_{11} \,.\, p_{32}) + A^3_{23} \,.\, (p_{11} \,.\, p_{22} - p_{21} \,.\, p_{12})$$

$$d_3 = A^1_{13} \,.\, (p_{32} \,.\, p_{23} - p_{22} \,.\, p_{33}) + A^2_{13} \,.\, (p_{33} \,.\, p_{12} - p_{32} \,.\, p_{13}) + A^3_{13} \,.\, (p_{22} \,.\, p_{13} - p_{12} \,.\, p_{23}) + A^1_{23} \,.\, (p_{21} \,.\, p_{33} - p_{23} \,.\, p_{31}) + A^2_{23} \,.\, (p_{31} \,.\, p_{13} - p_{11} \,.\, p_{33}) + A^3_{23} \,.\, (p_{11} \,.\, p_{23} - p_{21} \,.\, p_{13})$$

(c) The nine coefficients of degree two have the generic expression [d; p. 38]:

$$d_{mn} = - p_{11} . (A^2_{m2} . A^3_{n3} - A^2_{m3} . A^3_{n2}) - p_{12} . (A^2_{m1} . A^3_{n3} - A^2_{m3} . A^3_{n1}) - p_{13} . (A^2_{m1} . A^3_{n2} - A^2_{m2} . A^3_{n1})$$
$$- p_{21} . (A^1_{m3} . A^3_{n2} - A^1_{m2} . A^3_{n3}) - p_{22} . (A^1_{m1} . A^3_{n3} - A^1_{m3} . A^3_{n1}) - p_{23} . (A^1_{m2} . A^3_{n1} - A^1_{m1} . A^3_{n2})$$
$$- p_{31} . (A^1_{m2} . A^2_{n3} - A^1_{m3} . A^2_{n2}) - p_{32} . (A^1_{m3} . A^2_{n1} - A^1_{m1} . A^2_{n3}) - p_{33} . (A^1_{m2} . A^2_{n1} - A^1_{m1} . A^2_{n2})$$

Since the three approaches analyze the same element of length (resp.: the same polynomial), they induce a set of relations:

$$d_{ab} = g_{ab} = \ldots (see\, above\, 3\, (c))$$

$$d_a = 2 . N_a . \delta x^0 = (g_{0a} + g_{a0}) . \delta x^0 = \ldots (see\, above\, 3\, (b))$$
$$d = -(N^2 - N_a . N^a) . (\delta x^0)^2 = g_{00} . (\delta x^0)^2 = \ldots (see\, above\, 3\, (a))$$

One may usefully note by the way that the element of length on any isochrone hypersurface ($\delta x^0 = 0$) is reduced to (Latine subscripts a, b = 1, 2, 3 only):

$$(\delta s)^2 = g_{ab} . \delta x^a . \delta x^b$$

The results which have been obtained in [c] and [d] for the decomposition of $[\delta \mathbf{r}, \ldots]_{[A]}$ when the Hessian is not degenerated can be condensed through the relations:

$$[P]_{|A|} = |A| . \{[A]^t . [J]\} . [N_\Lambda^{|A|}]$$

With:

$$[N_\Lambda^{|A|}] = \frac{1}{2} . [S_0] - |A| . {}_{[J]}\Phi({}_\Lambda \mathbf{s})$$
$$|A| = \pm 1, \; |S_0| \neq 0$$

$$[S_0] = [Hess_{(\delta \mathbf{x}, 0)} \Lambda(\delta \mathbf{x})] = \begin{bmatrix} 2.D_{11} & D_{12} & D_{13} \\ D_{12} & 2 . D_{22} & D_{23} \\ D_{13} & D_{23} & 2 . D_{33} \end{bmatrix} = [d_{ij}] + [d_{ij}]^t$$

$$|{}_\Lambda \mathbf{s} >= -[Hess_{(\delta \mathbf{x}, 0)} \Lambda(\delta \mathbf{x})]^{-1} . |\mathbf{d}^* >, \; \mathbf{d}^* : (d_1, d_2, d_3)$$

Remark 3.7. *Looking for an interpretation - part 02 - Fundamental idea.*

This exploration has reached a point where the quest which has been started in remark 3.5 can be continued. Note that for non-vanishing δx^0, the discussion concerns the polynomial:

$$(\frac{\delta s}{\delta x^0})^2 = g_{ab} . \frac{\delta x^a}{\delta x^0} . \frac{\delta x^b}{\delta x^0} + (g_{0a} + g_{a0}) . \frac{\delta x^a}{\delta x^0} + g_{00}$$

In multiplying it by m^2 (the square of a mass m), one gets:

$$m^2 . (\frac{\delta s}{\delta x^0})^2 = g_{ab} . (m . \frac{\delta x^a}{\delta x^0}) . (m . \frac{\delta x^b}{\delta x^0}) + m . (g_{0a} + g_{a0}) . (m . \frac{\delta x^a}{\delta x^0}) + m^2 . g_{00}$$

And, in introducing the classical definition of a kinetic momentum **p**:

$$\forall \, a = 1, 2, 3 : p^a = lim_{\delta x^0 \to 0}(m . \frac{\delta x^a}{\delta x^0})$$

The polynomial is written:

$$lim_{\delta x^0 \to 0}(m^2 . (\frac{\delta s}{\delta x^0})^2) \,=\, g_{ab} . p^a . p^b \,+\, m . (g_{0a} \,+\, g_{a0}) . p^a \,+\, m^2 . g_{00}$$

At this stage, one can remark its formal ressemblance with:

1. the Klein-Gordon equation.
2. the polynomial $\Lambda({}^{(3)}\mathbf{projectile})$ resulting from example 3.1. The kernel-compatible Perian matrices are the symmetric parts of kernels related to the decomposition of deformed cross products.

Because of the analogy, let suppose that:

$${}^{(3)}\mathbf{projectile} \,=\, {}^{(3)}\mathbf{p}$$

The polynomial resulting from the calculation in example 3.1 writes now:

$$\Lambda({}^{(3)}\mathbf{p})$$

$$=$$

$$(\alpha \,+\, \beta . ||{}^{(3)}\mathbf{p}||^2 \,+\, \alpha^2 . \beta) . ||{}^{(3)}\mathbf{p}||^2 \,+\, \beta . \{<{}^{(3)}\,\mathbf{s},\, {}^{(3)}\mathbf{p}>_{Id_3}\}^2$$
$$+\, \frac{2}{|A|} . (\alpha \,+\, \beta . ||{}^{(3)}\mathbf{p}||^2) . <{}^{(3)}\,\mathbf{s},\, {}^{(3)}\mathbf{p}>_{Id_3}$$
$$+\, \alpha^3 \,+\, ||{}^{(3)}\mathbf{s}||^2 . \alpha$$

Let propose the fundamental identification:

$$\Lambda({}^{(3)}\mathbf{p}) \,=\, lim_{\delta x^0 \to 0}(m^2 . (\frac{\delta s}{\delta x^0})^2)$$

Then, if and when this identification is possible and meaningful, one must write:

$$m^2 . g_{00} \,=\, \alpha^3 \,+\, ||{}^{(3)}\mathbf{s}||^2 . \alpha$$

$$m . (g_{0a} \,+\, g_{a0}) \,=\, \frac{2}{|A|} . (\alpha \,+\, \beta . ||{}^{(3)}\mathbf{p}||^2) . s^a$$

$$g_{aa} \,=\, (\alpha \,+\, \beta . ||{}^{(3)}\mathbf{p}||^2 \,+\, \alpha^2 . \beta) \,+\, \beta . (s^a)^2$$

$$a \neq b :\, g_{ab} \,=\, \frac{\beta}{|A|} . s^a . s^b$$

The fundamental identification imposes precise formalisms for the four-dimensional metrics of this theory.

Example 3.4. *Some allowed metrics for this theory.*

When the particles have a non-vanishing mass (m $\neq$ 0), one may for example envisage the symmetric metrics:

$${}^{(4)}[G]$$

$$=$$

$$\begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \frac{\alpha}{m^2} \, . \, (\alpha^2 \, + \, ||^{(3)}\mathbf{s}||^2) & \frac{2}{|A| \, . \, m} \, . \, (\alpha \, + \, \beta \, . \, ||^{(3)}\mathbf{p}||^2) \, . \, < \mathbf{s}| \\ \frac{2}{|A| \, . \, m} \, . \, (\alpha \, + \, \beta \, . \, ||^{(3)}\mathbf{p}||^2) \, . \, |\mathbf{s} > & (\alpha \, + \, \beta \, . \, ||^{(3)}\mathbf{p}||^2 \, + \, \alpha^2 \, . \, \beta) \, . \, Id_3 \, + \, \beta \, . \, T_2(\otimes)(^{(3)}\mathbf{s}, \, ^{(3)}\mathbf{s}) \end{bmatrix}$$

For massive particles in a degenerated context (Class II kernels), this metric is reduced to, either ($\alpha = 0$):

$$^{(4)}[G]$$

$$=$$

$$\beta \, . \begin{bmatrix} 0 & \frac{2}{|A| \, . \, m} \, . \, ||^{(3)}\mathbf{p}||^2 \, . \, < \mathbf{s}| \\ \frac{2}{|A| \, . \, m} \, . \, ||^{(3)}\mathbf{p}||^2 \, . \, |\mathbf{s} > & \{||^{(3)}\mathbf{p}||^2 \, . \, Id_3 \, + \, T_2(\otimes)(^{(3)}\mathbf{s}, \, ^{(3)}\mathbf{s})\} \end{bmatrix}$$

Or to ($\alpha + \beta.||\mathbf{p}||^2 = 0$):

$$^{(4)}[G]$$

$$=$$

$$\begin{bmatrix} -\frac{\beta}{m^2} \, . \, (\beta^2 \, . \, ||^{(3)}\mathbf{p}||^4 \, + \, ||^{(3)}\mathbf{s}||^2) \, . \, ||^{(3)}\mathbf{p}||^2 & < \mathbf{0}| \\ |\mathbf{0} > & \beta \, . \, \{\beta^2 \, . \, ||^{(3)}\mathbf{p}||^4 \, . \, Id_3 \, + \, T_2(\otimes)(^{(3)}\mathbf{s}, \, ^{(3)}\mathbf{s})\} \end{bmatrix}$$

Example 3.5. *TheERps*

One now wants to confront this logical path with the results concerning a given ERp. A simple but important consequence of theorem 3.1. is that it is enougth to identify the (pseudo-)singular vector $\mathbf{s}$ with $\chi.\mathbf{b}$ to be certain to be in presence of an OPM and to add the equation of some four-dimensional unit sphere recalled in example 3.2:

$$(\varsigma^0)^2 \, + \, ||^{(3)}\varsigma||^2 \, = \, 1$$

... to be certain that this OPM represents a ERp. In that case, the polynomial writes:

$$\Lambda(^{(3)}\mathbf{projectile})$$

$$=$$

$$(2 \, . \, (\varsigma^0)^2 \, - \, 1) \, . \, ||^{(3)}\mathbf{projectile}||^2 \, + \, 2 \, . \, \{<^{(3)} \varsigma, \, ^{(3)}\mathbf{projectile} >_{Id_3}\}^2$$

$$+ \, \frac{4 \, . \, \varsigma^0}{|A|} \, . \, <^{(3)} \varsigma, \, ^{(3)}\mathbf{projectile} >_{Id_3}$$

$$+ \, 1$$

And if one continues to identify it with a representation of the Riemannian element of length, then one must work with:

$$\Lambda(^{(3)}\mathbf{p})$$

$$=$$

$$(2 \,.\, (\varsigma^0)^2 \,-\, 1) \,.\, ||^{(3)}\mathbf{p}||^2 \,+\, 2 \,.\, \{<^{(3)} \varsigma,\, ^{(3)}\mathbf{p} >_{Id_3}\}^2$$
$$+ \frac{4 \,.\, \varsigma^0}{|A|} \,.\, <^{(3)} \varsigma,\, ^{(3)}\mathbf{p} >_{Id_3}$$
$$+1$$

In writing:

$$\Lambda(^{(3)}\mathbf{p}) \,=\, \Lambda(\frac{c}{E} \,.\, ^{(3)}\mathbf{p}) \,=\, lim_{\delta x^0 \to 0}(m^2 \,.\, (\frac{\delta s}{\delta x^0})^2)$$

The coherence imposes:

$$m^2 \,.\, g_{00} \,=\, 1$$
$$m \,.\, (g_{0a} \,+\, g_{a0}) \,=\, \frac{4 \,.\, \varsigma^0}{|A|} \,.\, \varsigma^a$$
$$g_{aa} \,=\, 2 \,.\, (\varsigma^0)^2 \,-\, 1 \,+\, 2 \,.\, (\varsigma^a)^2$$

And for a $\neq$ b:

$$g_{ab} \,=\, 4 \,.\, \varsigma^a \,.\, \varsigma^b$$

If the four-dimensional unit sphere at hand is the dispersive relation in vacuum, then these identifications are more precisely (recall remark 3.7):

$$m^2 \,.\, g_{00} \,=\, 1$$
$$\forall\, m \,\neq\, 0 \,:\, (g_{0a} \,+\, g_{a0}) \,=\, \frac{4 \,.\, c^3}{|A| \,.\, E^2} \,.\, p^a$$
$$g_{aa} \,=\, 2 \,.\, \frac{m^2 \,.\, c^4}{E^2} \,-\, 1 \,+\, 2 \,.\, \frac{c^2}{E^2} \,.\, (p^a)^2$$

And for a, b = 1, 2, 3, a $\neq$ b:

$$g_{ab} \,=\, 4 \,.\, \frac{c^2}{E^2} \,.\, p^a \,.\, p^b$$

At this stage, one can remark that the conventions which have been involved in this discussion until now yields the following relations when the vacuum is refered to a metric with signature (+1 ? ? ?) without Thirring-Lense effect:

$$m^2 \,=\, 1$$

The discussion concerns unit masses of reference: m = 1.

$$\forall\, a \,=\, 1,\, 2,\, 3 \,:\, p^a \,=\, 0 \,\equiv\, ^{(3)}\mathbf{p} \,=\, ^{(3)}\mathbf{0} \,\equiv\, E^2 \,=\, m^2 \,.\, c^4 \,+\, c^2 \,.\, p^2 \,=\, c^4$$

They have no proper kinetic momentum.

$$g_{aa} \,=\, 2 \,.\, \frac{c^4}{E^2} \,-\, 1 \,=\, 1$$

And for a, b = 1, 2, 3, a $\neq$ b:

$$g_{ab} \,=\, 0$$

The spatial metric is diagonalized, with signature (+ + +) and identified with the identity matrix Id_3.

Example 3.6. *The Thirring-Lense effect.*

If one considers that any region in the universe can be refered to the metric characterizing a Thirring-Lense effect [07; chapter 30] because something is always spining somewhere and acting on the position where one stays at a given instant, then the metric has always the generic formalism:

$$^{(4)}[G] = \begin{bmatrix} 1 - \frac{r_S}{r} & g_{01} & g_{02} & g_{03} \\ g_{01} & -(1 + \frac{r_S}{r}) & 0 & 0 \\ g_{02} & 0 & -(1 + \frac{r_S}{r}) & 0 \\ g_{03} & 0 & 0 & -(1 + \frac{r_S}{r}) \end{bmatrix} = {}^{(4)}[G]^t$$

... and the polynomial resulting from a relooking of the Riemannian element of length is:

$$lim_{\delta x^0 \to 0}(m^2 . (\frac{\delta s}{\delta x^0})^2) = -(1 + \frac{r_S}{r}) . p^2 + 2 . m . g_{0a} . p^a + m^2 . (1 - \frac{r_S}{r})$$

The fundamental identifications proposed in remark 3.7 can be applied here because the metrics characterizing any Thirring-Lense effect correspond to the symmetric metrics which have been envisaged at the very beginning of example 3.4, provided on imposes the condition:

$$\beta = 0$$

In this case:

$$\begin{bmatrix} \frac{\alpha}{m^2} . (\alpha^2 + ||^{(3)}\mathbf{s}||^2) & \frac{2}{|A| . m} . \alpha . < \mathbf{s}| \\ \frac{2}{|A| . m} . \alpha . |\mathbf{s} > & \alpha . Id_3 \end{bmatrix}$$

One luckily states that this kind of metrics is compatible with the existence of a Thirring-Lense effect. This situation allows the writting of:

$$m^2 . (1 - \frac{r_S}{r}) = \alpha^3 + ||^{(3)}\mathbf{s}||^2 . \alpha$$

$$m . (g_{0a} + g_{a0}) = \frac{2}{|A|} . \alpha . s^a$$

$$g_{aa} = \alpha = -(1 + \frac{r_S}{r})$$

$$a \neq b : g_{ab} = 0$$

With the conventions:

$$^{(3)}\mathbf{g}^* \equiv (g_{01}, g_{02}, g_{03}) = (g_{10}, g_{20}, g_{30})$$

And:

$$r_S = \frac{2 . G . M}{c^2}$$

Where the symbol r_S denotes the Schwarschild radius [07; chapter 24, p.136] of some spining mass M, the effect of which is observed at distance r. Here, the mass M is thougth as the source of a Thirring-Lense-like deformation of the metric.

One can reformulate the identifications as follows:

$$m\,.\,{}^{(3)}\mathbf{g}^* \;=\; -\frac{1}{|A|}\,.\,(1\,+\,\frac{r_S}{r})\,.\,{}^{(3)}\mathbf{s}$$

$$(1\,+\,\frac{r_S}{r})^3\,+\,||{}^{(3)}\mathbf{s}||^2\,.\,(1\,+\,\frac{r_S}{r})\,+\,m^2\,.\,(1\,-\,\frac{r_S}{r})\;=\;0$$

If one follows the way of thinking which has been exposed in example 3.5, these identifications becomes:

$$m\,.\,{}^{(3)}\mathbf{g}^* \;=\; -\frac{\chi}{|A|}\,.\,(1\,+\,\frac{r_S}{r})\,.\,{}^{(3)}\mathbf{p}$$

$$(1\,+\,\frac{r_S}{r})^3\,+\,\chi^2\,.\,||{}^{(3)}\mathbf{p}||^2\,.\,(1\,+\,\frac{r_S}{r})\,+\,m^2\,.\,(1\,-\,\frac{r_S}{r})\;=\;0$$

If the unit four-dimensional sphere at hand is the dispersive relation in vacuum for massive waves, then they are:

$${}^{(3)}\mathbf{g}^* \;=\; -\frac{2\,.\,c^2}{|A|\,.\,E}\,.\,(1\,+\,\frac{r_S}{r})\,.\,{}^{(3)}\mathbf{p}$$

$$(1\,+\,\frac{r_S}{r})^3\,+\,4\,.\,\frac{m^2\,.\,c^4}{E^2}\,.\,||{}^{(3)}\mathbf{p}||^2\,.\,(1\,+\,\frac{r_S}{r})\,+\,m^2\,.\,(1\,-\,\frac{r_S}{r})\;=\;0$$

$$E^2\;=\;m^2\,.\,c^4\,+\,c^2\,.\,p^2$$

Comments:

1. First identification:

   In confronting this identification with what has already been said in remark 3.5, one is pushed to interpret it as follows:

   A mass m that undergoes a Thirring-Lense effect at distance r from a spining mass M has a kinetic momentum **p** which tends to reach its invariant limit and this limit is aligned with a vector $\mathbf{g}^*$ entirely related to a local spatial geometrical situation.

   If one applies this interpretation at the cosmological scale in empty regions of the universe, one expects that stars surrounding a spining white hole (expelling matter) first revolve around it, moving away, reaching a speed limit when the gravitational attraction of the whole galaxy becomes insignificant. This idea presupposes that **s** and $\mathbf{g}^*$ are an orbital speed and never totally null; at least during a long part of the history of this star.

   The exhaustive formulation of $\mathbf{g}^*$ can be seen in [07; chapter 30, Equ.(30.12), p.166]:

$${}^{(3)}\mathbf{g}^* \;=\; -\frac{4\,.\,G\,.\,M\,.\,R_M^2}{5\,.\,c^3\,.\,r^3}\,.\,({}^{(3)}\omega\,\wedge\,{}^{(3)}\mathbf{r})$$

The first identification can also be reformulated as:

$$^{(3)}\mathbf{p}_{limit} = \frac{|A| \, . \, E}{5 \, . \, c^3} \cdot \frac{1}{1 + \frac{r_S}{r}} \cdot \frac{r_S \, . \, R_M^2}{r^3} \, . \, (^{(3)}\omega \wedge {}^{(3)}\mathbf{r})$$

If one recall the classical relation connecting a kinetic momentum and a speed:

$$^{(3)}\mathbf{p} = m_0 \, . \, ^{(3)}\mathbf{v}$$

One gets the temptation to define a quantity equivalent to a classical mass:

$$m_0 \equiv \frac{|A| \, . \, E}{5 \, . \, c^3} \cdot \frac{1}{1 + \frac{r_S}{r}} \cdot \frac{r_S \, . \, R_M^2}{r^3}$$

If one considers the definition of the pre-Euclidean context (see below in subsection 3.4, one understands that this theory will probably introduce negative masses. Note by the way that negative energies (anti-particles) and imaginary energies as well exist within the quantum theory. The imaginary energies are for example involve when one wants to describe transitions between unstable energetic states, like e.g. in the Lamb-Rutherford effect [11 ; complément H-IV ; pages 468-473].

Since the dispersion relation which is involved here concerns what should happen in vacuum, one slowly gets the strange intuition that the vacuum is a part of the universe where the spatial energies can take any values in $\mathbb{C}$. This affirmation will certainly trigger a strongh opposition in the scientific community. The next paragraphs of the document will provide arguments reenforcing this vision. Below, the theorem 3.2 will give an argument justifying it.

2. Second identification:

   The second relation is certainly the strangest. It suggests the existence of several (at most three) solutions for the ratio X = r$_S$/r.

$$X^3 + 3 \, . \, X^2 + (3 + A - m^2) \, . \, X + (1 + A + m^2) = 0$$

$$A = 4 \, . \, \frac{m^2 \, . \, c^4 \, . \, p^2}{E^2}$$

3.4 The pre-Euclidean context

Definition 3.1. *The pre-Euclidean context.*

The pre-Euclidean context describes all situations characterized by:

$$[A] = \begin{bmatrix} A_{12}^1 & A_{12}^2 & A_{12}^3 \\ A_{23}^1 & A_{23}^2 & A_{23}^3 \\ A_{13}^1 & A_{13}^2 & A_{13}^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = [J]$$

They imply:

$$|A| = -1$$

... and they have diverse consequences on the coefficients of $\Lambda(\delta\mathbf{r})$.

Remark 3.8. *The pre-Euclidean context.*

Precisely - for the details, see annex 4.5:

1. Concerning the coefficient of degree two:

$$d_{ab} \,=\, \epsilon_{ab} \,.\, p_{ab}$$

   With:

$$\epsilon_{aa} \,=\, -1,\; \epsilon_{a\,<\,b} \,=\, +1,\; \epsilon_{a\,>\,b} \,=\, -1$$

2. Concerning the coefficient of degree one

$$d_1 \,=\, p_{11} \,.\, (p_{32} \,-\, p_{23}) \,+\, (p_{21} \,.\, p_{13} \,-\, p_{31} \,.\, p_{12})$$

$$d_2 \,=\, p_{22} \,.\, (p_{13} \,-\, p_{31}) \,+\, (p_{21} \,.\, p_{32} \,-\, p_{12} \,.\, p_{23})$$

$$d_3 \,=\, p_{33} \,.\, (p_{21} \,-\, p_{12}) \,+\, (p_{32} \,.\, p_{13} \,-\, p_{23} \,.\, p_{31})$$

3. Concerning the coefficient of degree zero:

$$-|P| \,=\, d \,=\, \frac{|S_0|}{8} + \,<_{\Lambda}\mathbf{s},_{\Lambda}\mathbf{s}\,>_{[d_{ij}]} = |N|$$

Therefore, the relations resulting from a confrontation between the three approaches (numerical, ADM and deformed cross products) are:

$$d_{ab} \,=\, g_{ab} \,=\, \epsilon_{ab} \,.\, p_{ab}$$

$$d_1$$
$$=$$
$$2 \,.\, N_1 \,.\, \delta x^0$$
$$=$$
$$(g_{01} \,+\, g_{10}) \,.\, \delta x^0$$
$$=$$
$$p_{11} \,.\, (p_{32} \,-\, p_{23}) \,+\, (p_{21} \,.\, p_{13} \,-\, p_{31} \,.\, p_{12})$$
$$=$$
$$-d_{11} \,.\, (-d_{32} \,-\, d_{23}) \,+\, (-d_{21} \,.\, d_{13} \,+\, d_{31} \,.\, d_{12})$$
$$=$$
$$g_{11} \,.\, (g_{32} \,+\, g_{23}) \,+\, (g_{31} \,.\, g_{12} \,-\, g_{21} \,.\, g_{13})$$

And:

$$d_2$$
$$=$$
$$2 \,.\, N_2 \,.\, \delta x^0$$
$$=$$

$$(g_{02} + g_{20}) \,.\, \delta x^0$$
$$=$$
$$p_{22} \,.\, (p_{13} - p_{31}) + (p_{21} \,.\, p_{32} - p_{12} \,.\, p_{23})$$
$$=$$
$$-d_{22} \,.\, (d_{13} + d_{31}) + (-d_{21} \,.\, - d_{32} - d_{12} \,.\, d_{23})$$
$$=$$
$$-g_{22} \,.\, (g_{13} + g_{31}) + (g_{21} \,.\, g_{32} - g_{12} \,.\, g_{23})$$

And:

$$d_3$$
$$=$$
$$2 \,.\, N_3 \,.\, \delta x^0$$
$$=$$
$$(g_{03} + g_{30}) \,.\, \delta x^0$$
$$=$$
$$p_{33} \,.\, (p_{21} - p_{12}) + (p_{32} \,.\, p_{13} - p_{23} \,.\, p_{31})$$
$$=$$
$$-d_{33} \,.\, (-d_{21} - d_{12}) + (-d_{32} \,.\, d_{13} + d_{23} \,.\, d_{31})$$
$$=$$
$$g_{33} \,.\, (g_{21} + g_{12}) + (g_{23} \,.\, g_{31} - g_{32} \,.\, g_{13})$$

And:

$$d = -(N^2 - N_a \,.\, N^a) \,.\, (\delta x^0)^2 = g_{00} \,.\, (\delta x^0)^2 = \frac{|S_0|}{8} + <_\Lambda \mathbf{s}, {}_\Lambda \mathbf{s}>_{[d_{ij}]}$$

$$[P]_{|A|} = -[N_\Lambda^{|A|}] \equiv p_{ab} = -n_{ab}$$

To go further in that direction, one must calculate the Hessian matrix:

$$\frac{1}{2} \,.\, [Hess_{(\delta \mathbf{x}, 0)} \Lambda(\delta \mathbf{x})]$$
$$=$$
$$\frac{1}{2} \,.\, \{[d_{ij}] + [d_{ij}]^t\}$$
$$=$$
$$\frac{1}{2} \,.\, \{[G] + [G]^t\}$$
$$=$$
$$\begin{bmatrix} -p_{11} & \frac{1}{2} \,.\, (p_{12} - p_{21}) & \frac{1}{2} \,.\, (p_{13} - p_{31}) \\ \frac{1}{2} \,.\, (p_{12} - p_{21}) & -p_{22} & \frac{1}{2} \,.\, (p_{23} - p_{32}) \\ \frac{1}{2} \,.\, (p_{13} - p_{31}) & \frac{1}{2} \,.\, (p_{23} - p_{32}) & -p_{33} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} d_{11} & \frac{1}{2}.(d_{12} + d_{21}) & \frac{1}{2}.(d_{13} + d_{31}) \\ \frac{1}{2}.(d_{12} + d_{21}) & d_{22} & \frac{1}{2}.(d_{23} + d_{32}) \\ \frac{1}{2}.(d_{13} + d_{31}) & \frac{1}{2}.(d_{23} + d_{32}) & d_{33} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} g_{11} & \frac{1}{2}.(g_{12} + g_{21}) & \frac{1}{2}.(g_{13} + g_{31}) \\ \frac{1}{2}.(g_{12} + g_{21}) & g_{22} & \frac{1}{2}.(g_{23} + g_{32}) \\ \frac{1}{2}.(g_{13} + g_{31}) & \frac{1}{2}.(g_{23} + g_{32}) & g_{33} \end{bmatrix}$$

Lemma 3.1. *The kernel related to the decomposition of* $[\delta\mathbf{r}, ...]_{[A]}$ *in a pre-Euclidean context.*

In a pre-Euclidean context, half of the Hessian matrix related to the decomposition of $[\delta\mathbf{r}, ...]_{[A]}$ within a discussion looking for the coherence between three approaches analyzing the Riemannian element of length represents the symmetric part of the spatial metric at hand:

$$\frac{1}{2}.[Hess_{(\delta\mathbf{x},0)}\Lambda(\delta\mathbf{x})] = \frac{1}{2}.\{^{(3)}[G] + {}^{(3)}[G]^t\}$$

Caution: *This mathematical statement does not mean that the spatial metric is systematically symmetric. But, it tells that the degeneracy or the non-degeneracy of the symmetric part of the metric (not of the whole metric) is the criterion determining the classification of the kernels in a pre-Euclidean context.*

Nevertheless, when the spatial metric is entirely symmetric (its anti-symmetric part is null), then this theory shedds a peculiar light on the pre-Euclidean context:

$$d_1 = 2.N_1.\delta x^0 = 2.g_{01}.\delta x^0 = 2.g_{11}.g_{23}$$

$$d_2 = 2.N_2.\delta x^0 = 2.g_{02}.\delta x^0 = -2.g_{22}.g_{13}$$

$$d_3 = 2.N_3.\delta x^0 = 2.g_{03}.\delta x^0 = 2.g_{33}.g_{12}$$

Furthermore, if this symmetric metric is not degenerated (the kernel is in the Class I), there exists a singular vector such that:

$$|_\Lambda\mathbf{s}> = -2.\delta x^0.[G]^{-1}.|\mathbf{g}^*>$$

And the kernel of the decomposition is:

$$[N_\Lambda^{|A|}] = [G] - 2.\delta x^0.{}_{[J]}\Phi([G]^{-1}.|\mathbf{g}^*>)$$

Comments: The kernel matrix related to the decomposition of $[\delta\mathbf{r}, ...]_{[A]}$:

1. is in general not a Perian matrix (definition 2.1).

2. is a Perian matrix related to a symmetric and non-degenerated metric each time one can find a pair of coefficients $(\alpha,\, \beta)$ and a vector $\mathbf{b}$ such that:

$$^{(3)}[G] \,=\, \alpha \,.\, Id_3 \,+\, \beta \,.\, T_2(\otimes)(\mathbf{b},\, \mathbf{b})$$

... and (remark 2.6):

$$|^{(3)}[G]| \,=\, \alpha^2 \,.\, (\alpha \,+\, \beta \,.\, ||\mathbf{b}||^2) \,\neq\, 0$$

Therefore, when the symmetric spatial part of the 4D-metric is not null and proportional to the identity matrix Id_3, the kernel matrix related to the decomposition of $[\delta\mathbf{r},\, ...]_{[A]}$ is a truncated Perian matrix (in the second family):

$$[N^{|A|}_{\Lambda(^{(3)}\delta\mathbf{r})}] = \alpha \,.\, Id_3 \,-\, 2 \,.\, \frac{\delta x^0}{\alpha} \,.\, {}_{[J]}\Phi(\mathbf{g}^*),\ \alpha \,\neq\, 0$$

... which is characterized through two arguments:

$$\mathbf{a} \,:\, (\alpha,\, 0,\, -2 \,.\, \frac{\delta x^0}{\alpha}), \quad \mathbf{b} \,=\, \mathbf{g}^*$$

One can verify that:

$$|^{(3)}[G]| \,=\, \alpha^2 \,.\, (\alpha \,+\, 0 \,.\, ||^{(3)}\mathbf{g}^*||^2) \,=\, \alpha^3 \,\neq\, 0,\ \forall\, ^{(3)}\mathbf{g}^*$$

3. is such that, in a pre-Euclidean context related to a non-degenerated and symmetric metric, the classical cross product is non-trivially decomposed in that way:

$$|\delta\mathbf{r} \,\wedge\, \mathbf{target} >$$
$$=$$
$$\{-^{(3)}[G] + 2 \,.\, \delta x^0 \,.\, {}_{[J]}\Phi(^{(3)}[G]^{-1} \,.\, |\mathbf{g}^* >)\} \,.\, |\mathbf{target} > \,+ |\mathbf{residual\,part} >$$
$$=$$
$$2 \,.\, \delta x^0 \,.\, {}_{[J]}\Phi(^{(3)}[G]^{-1} \,.\, |\mathbf{g}^* >) \,.\, |\mathbf{target} >$$
$$-^{(3)}[G] \,.\, |\mathbf{target} > \,+ |\mathbf{residual\,part} >$$

One can also write:

$$|\frac{\delta\mathbf{r}}{\delta x^0} \,\wedge\, \mathbf{target} >$$
$$=$$
$${}_{[J]}\Phi(2 \,.\, ^{(3)}[G]^{-1} \,.\, |\mathbf{g}^* >) \,.\, |\mathbf{target} >$$
$$+\, \frac{1}{\delta x^0} \,.\, \{-^{(3)}[G] \,.\, |\mathbf{target} > \,+ |\mathbf{residual\,part} >\}$$

This is a counter-intuitive result because the most trivial decomposition without residual part is:

$$|\delta\mathbf{r} \,\wedge\, \mathbf{target} >= {}_{[J]}\Phi(\delta\mathbf{r}) \,.\, |\mathbf{target} >$$

3.5 The three-dimensional Euclidean limit

Remark 3.9. *The problematic related to the formalism of the decomposition.*

If one rewrites the trivial decomposition like this:

$$|\frac{\delta \mathbf{r}}{\delta x^0} \wedge \mathbf{target} >= {}_{[J]}\Phi(\frac{\delta \mathbf{r}}{\delta x^0}) \, . \, |\mathbf{target}>$$

... then, the ratio:

$$(\frac{\delta \mathbf{r}}{\delta x^0})$$

... appears and it is, at the limit $\delta \mathrm{x}^0 \rightarrow 0$, effectively what the mathematicians call a speed.

But, although the matrix ${}_{[J]}\Phi$ represents an axial rotation around $\delta\mathbf{r}/\delta\mathrm{x}^0$, there is no immediate reason to think that $\delta\mathbf{r}/\delta\mathrm{x}^0$ is a speed related to some curved trajectory which would have been induced by, e.g., a Thirring-Lense effect (see above) ... except if one accepts the principle:

> *"Any linear motion is an illusion resulting from an averaging of a sum of short trajectories, each of them exhibiting small curvatures".*

Any way, the simplest decomposition would only be recovered in completing this discussion with the relations:

1. First relation of coherence between a non-trivial and a trivial decomposition:

$$|\frac{\delta \mathbf{r}}{\delta x^0} >= 2 \, . \, {}^{(3)}[G]^{-1} \, . \, |{}^{(3)}\mathbf{g}^* >$$

   The speed on the left side is a kinetic and, in some way, classical one whilst the speed on the right side is a speed entirely induced by the geometry. One understands this equality in observing a duck carried by the flow of a river. Its speed is not due to its muscular efforts. Its motion is only the result of an underlying gravitational effect: the water always flows down from the top of some distant mountain towards the mouth.

2. Second relation of coherence between a non-trivial and a trivial decomposition:

$$|{}^{(3)}\mathbf{residual\,part} >= {}^{(3)}[G] \, . \, |{}^{(3)}\mathbf{target} >$$

Remark 3.10. *The symmetric three-dimensional Euclidean geometry within the theory studying the deformation of cross products.*

It is a matter of facts that:

1. When the spatial part of the metric, $^{(3)}$[G], (i) is symmetric and (ii) has not reached the classical Euclidean limit [G] = Id_3, the theory (of the (E) Question) predicts a precise formalism for this geometry; precisely:

$$[G] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{bmatrix}$$

... and the vector $\mathbf{d}^*$ does not vanish if the spatial part of the metric is not diagonalized because (up to a minus sign when a = 2):

$$\forall a = 1, 2, 3 :$$

$$d_a = 2 . N_a . \delta x^0 = 2 . g_{0a} . \delta x^0 = 2 . g_{aa} \cdot g_{(a+1)mod3 < (a+2)mod3}$$

Furthermore, if the vector $\mathbf{g}^*$ does not vanish as well, these relations tell that the discussion about the Riemannian element of length concerns two events which are separated in the chronology (they are not lying on the same chrono-hypersurface) and such that:

$$\delta x^0 = \frac{g_{11} \cdot g_{23}}{g_{01}} = -\frac{g_{22} \cdot g_{13}}{g_{02}} = \frac{g_{33} \cdot g_{12}}{g_{03}}$$

2. When $^{(3)}$[G] = Id_3, the theory of the (E) question predicts that $\mathbf{d}^*$ vanishes because of the diagonalisation of the spatial part of the metric. Proof, since (recall):

$$d_a = 2 . N_a . \delta x^0 = 2 . g_{0a} . \delta x^0 = 2 . g_{aa} \cdot g_{(a+1)mod3 < (a+2)mod3}$$

De facto:

$$\forall a, b = 1, 2, 3, a \neq b : g_{ab} = 0 \Rightarrow \mathbf{d}^* = 2 . \delta x^0 . \mathbf{g}^* = \mathbf{0}$$

This situation can only happen if either δx^0 or $\mathbf{g}^*$ vanishes. But, in both cases, the kernel of the decomposition is reduced to the identity matrix:

$$[N_\Lambda^{|A|}] = [G] - 2 . \delta x^0 . {}_{[J]}\Phi([G]^{-1} . |\mathbf{g}^* >) = Id_3 - {}_{[J]}\Phi(2 . \delta x^0 . \mathbf{g}^*) = Id_3$$

Furthermore, it is obvious that:

$$d = g_{00} . (\delta x^0)^2 = \underbrace{|[G] = Id_3|}_{=1} + <_\Lambda \mathbf{s},_\Lambda \mathbf{s} >_{[d_{ij}]} = |N_\Lambda^{|A|}| = 1$$

Therefore, neither g_{00} nor δx^0 can be null and the vanishing of $\mathbf{d}^*$ is entirely related to the vanishing of $\mathbf{g}^*$. One also luckily states that the nullity of $\mathbf{g}^*$ induces the vanishing of the singular vector and that this nullity is coherent with the formula giving the value of d:

$$|_\Lambda \mathbf{s} >= -2 . \delta x^0 . {}^{(3)}[G]^{-1} . |^{(3)}\mathbf{g}^* >= -2 . \delta x^0 . |^{(3)}\mathbf{g}^* >= {}^{(3)}\mathbf{0}$$

Another consequence is that the decomposition is drastically simplified in:

$$\delta \mathbf{r} \wedge {}^{(3)}\mathbf{target} = -{}^{(3)}\mathbf{target} + {}^{(3)}\mathbf{residual\,part}$$

When the actors of this discussion are in $E_3 = E(3, \mathbb{K})$ and K is equiped with a commutative multiplication, the residual part is decomposed into two orthogonal vectors because any cross product is orthogonal to each of its arguments (basic knowledge):

$$< \delta \mathbf{r} \wedge {}^{(3)}\mathbf{target}, {}^{(3)}\mathbf{target} >_{Id_3} = {}^{(3)}\mathbf{0}$$

This mathematical fact imposes a condition on the actors of the decomposition of the classical cross product; precisely:

$$<^{(3)} \mathbf{target}, {}^{(3)}\mathbf{target} >_{Id_3} = <^{(3)} \mathbf{target}, {}^{(3)}\mathbf{residual\,part} >_{Id_3}$$

This condition contains one very particular set of situations which can only be meaningfully realized when $K = \mathbb{C}$:

$$<^{(3)} \mathbf{target}, {}^{(3)}\mathbf{target} >_{Id_3} = <^{(3)} \mathbf{target}, {}^{(3)}\mathbf{residual\,part} >_{Id_3} = 0_{\mathbb{C}}$$

Theorem 3.1. *The Euclidean enigma.*

The three-dimensional Euclidean context within the theory studying the deformation and the decomposition of cross products contains a very particular set of situations allowing to work with isotropic targets.

Comments: This theorem may appear to be innocuous but, it is not because it reveals information that had remained invisible until then. Let take a look back at the history of mathematical physics. Classical geometry mainly deals with real numbers. During the Italian renaissance, mathematicians were led to invent complex numbers because they wanted to solve third-degree polynomials. This invention was completed several centuries later by that of quaternions and of octonions. Complex numbers have proven to be very useful in physics: for the description of alternating currents, in optics and, as already mentioned previously in this document, for the mathematical treatment of unstable energy states in quantum physics. This theorem completes this historical progression in an unexpected way. When one wants to make three approaches concerning the element of length coherent with each other, the effective achievement of this consistency in a Euclidean geometric context may sometimes lead or force to work with complex numbers. This theorem therefore establishes for the first time a logical link between a usual geometrical context and a specific set of numbers. This is its remarkable characteristic.

3.6 Conclusion

This essay can be understood as a continuation of the work which has been initiated in [c] and [d]. Much attention is given to the unavoidable pairing between the diverse possible decompositions along the time of a given deformed cross product and the polynomials associated with them. The kernel of each decomposition and its conjugated version are interpreted as pair of commutative operators which, thanks to Heisenberg's equation of motion for operators, allows to study the evolution of polynomials associated with these decompositions. The work, certainly, is far to be achieved but, the author hopes that his pain will be a little bit useful for future developments in fluid dynamics, particles physics and cosmology.

4 Annexes

4.1 The determinant of any Perian matrix

Let calculate the determinant of any OPM.

$$\forall\,(\mathbf{a},\,\mathbf{b})\,\in\,E_3\,\times\,E_3:$$

$$|M(\mathbf{a},\mathbf{b})|$$

$$=$$

$$|\alpha\,.\,Id_3\,+\,\beta\,.\,T_2(\otimes)(\mathbf{b},\mathbf{b})\,+\,\chi\,.\,_{[J]}\Phi(\mathbf{b})|$$

$$=$$

$$\begin{vmatrix} \alpha\,+\,\beta\,.\,(b^1)^2 & \beta\,.\,b^1\,.\,b^2\,-\,\chi\,.\,b^3 & \beta\,.\,b^1\,.\,b^3\,+\,\chi\,.\,b^2 \\ \beta\,.\,b^1\,.\,b^2\,+\,\chi\,.\,b^3 & \alpha\,+\,\beta\,.\,(b^2)^2 & \beta\,.\,b^2\,.\,b^3\,-\,\chi\,.\,b^1 \\ \beta\,.\,b^1\,.\,b^3\,-\,\chi\,.\,b^2 & \beta\,.\,b^2\,.\,b^3\,+\,\chi\,.\,b^1 & \alpha\,+\,\beta\,.\,(b^3)^2 \end{vmatrix}$$

$$=$$

$$(\alpha\,+\,\beta\,.\,(b^1)^2)\,.\,(\alpha\,+\,\beta\,.\,(b^2)^2)\,.\,(\alpha\,+\,\beta\,.\,(b^3)^2)$$
$$-\,(\alpha\,+\,\beta\,.\,(b^1)^2)\,.\,(\beta\,.\,b^2\,.\,b^3\,+\,\chi\,.\,b^1)\,.\,(\beta\,.\,b^2\,.\,b^3\,-\,\chi\,.\,b^1)$$
$$-\,(\beta\,.\,b^1\,.\,b^2\,-\,\chi\,.\,b^3)\,.\,(\beta\,.\,b^1\,.\,b^2\,+\,\chi\,.\,b^3)\,.\,(\alpha\,+\,\beta\,.\,(b^3)^2)$$
$$+\,(\beta\,.\,b^1\,.\,b^2\,-\,\chi\,.\,b^3)\,.\,(\beta\,.\,b^1\,.\,b^3\,-\,\chi\,.\,b^2)\,.\,(\beta\,.\,b^2\,.\,b^3\,-\,\chi\,.\,b^1)$$
$$+\,(\beta\,.\,b^1\,.\,b^3\,+\,\chi\,.\,b^2)\,.\,(\beta\,.\,b^1\,.\,b^2\,+\,\chi\,.\,b^3)\,.\,(\beta\,.\,b^2\,.\,b^3\,+\,\chi\,.\,b^1)$$
$$-\,(\beta\,.\,b^1\,.\,b^3\,+\,\chi\,.\,b^2)\,.\,(\beta\,.\,b^1\,.\,b^3\,-\,\chi\,.\,b^2)\,.\,(\alpha\,+\,\beta\,.\,(b^2)^2)$$

$$=$$

$$(\alpha\,+\,\beta\,.\,(b^1)^2)\,.\,\{\alpha^2\,+\,\alpha\,.\,\beta\,.\,(b^2)^2\,+\,\alpha\,.\,\beta\,.\,(b^3)^2\,+\,\beta^2\,.\,(b^2)^2\,.\,(b^3)^2\}$$
$$-\,(\alpha\,+\,\beta\,.\,(b^1)^2)\,.\,\{\beta^2\,.\,(b^2)^2\,.\,(b^3)^2\,-\,\chi^2\,.\,(b^1)^2\}$$
$$-\,\{\beta^2\,.\,(b^1)^2\,.\,(b^2)^2\,-\,\chi^2\,.\,(b^3)^2\}\,.\,(\alpha\,+\,\beta\,.\,(b^3)^2)$$
$$+\,(\beta\,.\,b^1\,.\,b^2-\chi\,.\,b^3)\,.\,\{\beta^2\,.\,b^1\,.\,b^2\,.\,(b^3)^2-\beta\,.\,\chi\,.\,(b^1)^2\,.\,b^3-\chi\,.\,\beta\,.\,(b^2)^2\,.\,b^3+\chi^2\,.\,b^1\,.\,b^2\}$$
$$+\,(\beta\,.\,b^1\,.\,b^3+\chi\,.\,b^2)\,.\,\{\beta^2\,.\,b^1\,.\,(b^2)^2\,.\,b^3+\beta\,.\,\chi\,.\,(b^1)^2\,.\,b^2+\beta\,.\,\chi\,.\,b^2\,.\,(b^3)^2+\chi^2\,.\,b^1\,.\,b^3\}$$
$$-\,\{\beta^2\,.\,(b^1)^2\,.\,(b^3)^2\,-\,\chi^2\,.\,(b^2)^2\}\,.\,(\alpha\,+\,\beta\,.\,(b^2)^2)$$

$$=$$

$$+\,\alpha^3\,+\,\alpha^2\,.\,\beta\,.\,(b^2)^2\,+\,\alpha^2\,.\,\beta\,.\,(b^3)^2\,+\,\alpha\,.\,\chi^2\,.\,(b^1)^2$$
$$\alpha^2\,.\,\beta\,.\,(b^1)^2+\alpha\,.\,\beta^2\,.\,(b^1)^2\,.\,(b^2)^2+\alpha\,.\,\beta^2\,.\,(b^1)^2\,.\,(b^3)^2\,+\,\beta\,.\,\chi^2\,.\,(b^1)^4$$

$$-\alpha.\beta^2.(b^1)^2.(b^2)^2 + \alpha.\chi^2.(b^3)^2 - \beta^3.(b^1)^2.(b^2)^2.(b^3)^2 + \beta.\chi^2.(b^3)^4$$
$$+\beta^3.(b^1)^2.(b^2)^2.(b^3)^2 - \beta^2.\chi.(b^1)^3.b^2.b^3 - \beta^2.\chi.b^1.(b^2)^3.b^3 + \beta.\chi^2.(b^1)^2.(b^2)^2$$
$$-\beta^2.\chi.b^1.b^2.(b^3)^3 + \beta.\chi^2.(b^1)^2.(b^3)^2 + \beta.\chi^2.(b^2)^2.(b^3)^2 - \chi^3.b^1.b^2.b^3$$
$$+\beta^3.(b^1)^2.(b^2)^2.(b^3)^2 + \beta^2.\chi.(b^1)^3.b^2.b^3 + \beta^2.\chi.b^1.b^2.(b^3)^3 + \beta.\chi^2.(b^1)^2.(b^3)^2$$
$$+\beta^2.\chi.b^1.(b^2)^3.b^3 + \beta.\chi^2.(b^1)^2.(b^2)^2 + \beta.\chi^2.(b^2)^2.(b^3)^2 + \chi^3.b^1.b^2.b^3$$
$$-\alpha.\beta^2.(b^1)^2.(b^3)^2 + \alpha.\chi^2.(b^2)^2 - \beta^3.(b^2)^2.(b^1)^2.(b^3)^2 + \beta.\chi^2.(b^2)^4$$
$$=$$
$$\alpha^3 + \beta.||\mathbf{b}||^2.\alpha^2 + ||\mathbf{b}||^2.\chi^2.\alpha + \beta.\chi^2.||\mathbf{b}||^4$$

The determinant of any ordinary Perian matrix is a polynomial form:

1. of degree three if one interprets it as depending on α;
2. of degree two if one interprets it as depending on $||\mathbf{b}||^2$.

One can push the calculation a step further in remarking that, as expected:

$$|M(\mathbf{a},\mathbf{b})|$$
$$=$$
$$\alpha^2.(\alpha + \beta.||\mathbf{b}||^2) + ||\mathbf{b}||^2.\chi^2.(\alpha + \beta.||\mathbf{b}||^2)$$
$$=$$
$$(\alpha + \beta.||\mathbf{b}||^2).(\alpha^2 + ||\mathbf{b}||^2.\chi^2)$$

4.2 Calculating W

Proof. Note that:

$$S_{\lambda\lambda}.S_{\mu\mu}$$
$$=$$
$$\{\alpha + \beta.(b^\lambda)^2\}.\{\alpha + \beta.(b^\mu)^2\}$$
$$=$$
$$\{\alpha^2 + \alpha.\beta.(b^\mu)^2 + \beta.\alpha.(b^\lambda)^2 + \beta^2.(b^\lambda)^2.(b^\mu)^2\}$$

Hence:

$$S_{11}.S_{22} + S_{22}.S_{33} + S_{33}.S_{11}$$
$$=$$
$$3.\alpha^2 + (\alpha.\beta + \beta.\alpha).||\mathbf{b}||^2 + \beta^2.\{(b^1)^2.(b^2)^2 + (b^2)^2.(b^3)^2 + (b^3)^2.(b^1)^2\}$$

Whilst:

$$S_{12}^2 + S_{23}^2 + S_{13}^2$$
$$=$$
$$\beta^2.\{(b^1)^2.(b^2)^2 + (b^2)^2.(b^3)^2 + (b^3)^2.(b^1)^2\}$$

Therefore:

$$W = 3.\alpha^2 + (\alpha.\beta + \beta.\alpha).||\mathbf{b}||^2$$

The end of the calculation depends on the choice which is made concerning K. □

4.3 The inverse of a non-degenerated kernel-compatible matrix

It is a mathematical fact that (see the remark 2.4):

$$[Hess_{(^{(3)}\varsigma,0)}\Lambda(^{(3)}\varsigma)]\,.\,[Hess_{(^{(3)}\varsigma,0)}\Lambda(^{(3)}\varsigma)]^{-1}$$

$$=$$

$$\{(2\,.\,(\varsigma^0)^2\;-\;1)\,.\,Id_3\;+\;2\,.\,T_2(\otimes)(^{(3)}\varsigma,\;^{(3)}\varsigma)\}\,.\,\{A\,.\,Id_3\;+\;B\,.\,T_2(\otimes)(^{(3)}\varsigma,\;^{(3)}\varsigma)\}$$

$$=$$

$$(2\,.\,(\varsigma^0)^2-1)\,.\,A\,.\,Id_3+\{(2\,.\,(\varsigma^0)^2-1)\,.\,B+2\,.\,A+2\,.\,B\,.\,||^{(3)}\varsigma||^2\}\,.\,T_2(\otimes)(^{(3)}\varsigma,\;^{(3)}\varsigma)$$

This product can be identified with the identity matrix Id_3 when, simultaneously:

$$(2\,.\,(\varsigma^0)^2\;-\;1)\,.\,A\;=\;1$$

$$(2\,.\,(\varsigma^0)^2\;-\;1)\,.\,B\;+\;2\,.\,A\;+\;2\,.\,B\,.\,||^{(3)}\varsigma||^2\;=\;2\,.\,A\;+\;B\;=\;0$$

When the Hessian is not degenerated, then $2.(\varsigma^0)^2$ - 1 cannot be null; therefore:

$$A\;=\;\frac{1}{2\,.\,(\varsigma^0)^2\;-\;1}$$

For the same reason:

$$B\;=\;-\frac{2}{2\,.\,(\varsigma^0)^2\;-\;1}$$

4.4 The polynomial associated with a Euler-Rodrigues parametrisation

Due to [c; initial theorem, p. 14], one knows that a generic decomposition:

$$|[^{(3)}\mathbf{projectile},\;^{(3)}\mathbf{target}]_{[A]}>=\;^{(3)}[P]\,.\,|^{(3)}\mathbf{target}>\;+|^{(3)}\mathbf{residual\,part}>$$

... is associated with a polynomial of degree two of which the formalism is:

$$\Lambda(^{(3)}\mathbf{projectile})\;=\;d_{ij}\,.\,projectile^i\,.\,projectile^j\;+\;d_i\,.\,projectile^i\,.\;+\;d$$

When the coefficients of the polynomial are invariant during the derivations, one can always write [c; initial theorem, p. 15]:

$$[Hess_{(^{(3)}\mathbf{projectile},0)}\Lambda(^{(3)}\mathbf{projectile})]\;=\;[d_{ij}]\;+\;[d_{ij}]^t\;=\;[D]\;+\;[D]^t$$

But, one cannot precisely say what the matrix [D] is. Therefore, considering now the case of a non-degenerated Hessian when the kernel of the decomposition coincides with a Euler-Rodrigues parametrization:

$$\Lambda(^{(3)}\mathbf{projectile})$$

$$=$$

$$<^{(3)}\;\mathbf{projectile}|\,.\,[D]\,.\,|^{(3)}\mathbf{projectile}>\;+\;<^{(3)}\;\varsigma,\;^{(3)}\mathbf{projectile}>_{Id_3}\;+1$$

When the matrix representing the Euler-Rodrigues parametrisation is a kernel in class I, the symmetric part of the kernel is a kernel-compatible Perian matrix (definition 2.4) and there exists at least one polynomial $\Lambda({}^{(3)}\textbf{projectile})$ such that:

$$[(2\,.\,(\varsigma^0)^2 \,-\, 1)\,.\,\delta_{\lambda\mu} \,+\, 2\,.\,\varsigma^\lambda\,.\,\varsigma^\mu] \,=\, \frac{1}{2}\,.\,[Hess_{({}^{(3)}\textbf{projectile},\,0)}\Lambda(\textbf{projectile})]$$

Let envisage the coincidence:

$${}^{(3)}[D] \,=\, {}^{(3)}[M({}^{(4)}\varsigma)]$$

... because, in that case:

$${}^{(3)}[D] \,+\, {}^{(3)}[D]^t$$
$$=$$
$${}^{(3)}[M({}^{(4)}\varsigma)] \,+\, {}^{(3)}[M({}^{(4)}\varsigma)]^t$$
$$=$$
$$2\,.\,\{(2\,.\,(\varsigma^0)^2 \,-\, 1)\,.\,Id_3 \,+\, 2\,.\,T_2(\otimes)({}^{(3)}\varsigma,\, {}^{(3)}\varsigma)\}$$
$$=$$
$$[Hess_{({}^{(3)}\textbf{projectile},\,0)}\Lambda(\textbf{projectile})]$$

In that case:

$$<^{(3)}\,\textbf{projectile}|\,.\,[D]\,.\,|^{(3)}\textbf{projectile}\,>$$
$$=$$
$$<^{(3)}\,\textbf{projectile}|$$
$$.\,\{(2\,.\,(\varsigma^0)^2 - 1)\,.\,Id_3 + 2\,.\,T_2(\otimes)({}^{(3)}\varsigma,\,{}^{(3)}\varsigma) + 2\,.\,\varsigma^0\,.\,{}_{[J]}\Phi({}^{(3)}\varsigma)\}\,.\,|^{(3)}\textbf{projectile}\,>$$
$$=$$
$$<^{(3)}\,\textbf{projectile}|\,.$$
$$\{(2\,.\,(\varsigma^0)^2 \,-\, 1)\,.\,|^{(3)}\textbf{projectile}\,> \,+2\,.\,<^{(3)}\,\varsigma,\,\textbf{projectile}\,>_{Id_3}\,.\,|^{(3)}\varsigma\,>$$
$$+2\,.\,\varsigma^0\,.\,\underbrace{{}_{[J]}\Phi({}^{(3)}\varsigma)\,.\,|^{(3)}\textbf{projectile}\,>}_{=\varsigma\wedge\textbf{projectile}}\}$$
$$=$$
$$(2\,.\,(\varsigma^0)^2 \,-\, 1)\,.\,||^{(3)}\textbf{projectile}||^2 \,+\, 2\,.\,\{<^{(3)}\,\varsigma,\,\textbf{projectile}\,>_{Id_3}\}^2$$

At the end of this progression:

$$\Lambda({}^{(3)}\textbf{projectile})$$
$$=$$
$$(2\,.\,(\varsigma^0)^2 - 1)\,.\,||^{(3)}\textbf{projectile}||^2 + 2\,.\,\{<^{(3)}\,\varsigma,\,\textbf{projectile}\,>_{Id_3}\}^2 + \,<^{(3)}\,\varsigma,\,{}^{(3)}\textbf{projectile}\,>_{Id_3}\,+1$$

Let introduce two variables:

$$X \,=\, ||^{(3)}\textbf{projectile}||$$
$$\theta \,=<^{(3)}\,\varsigma,\,{}^{(3)}\textbf{projectile}\,>_{Id_3}$$

In general, the argument of the polynomial (the projectile) doesn't coincide with its singular vector (ς) and one must write:

$$\Lambda(X,\,\theta) \,=\, (2\,.\,(\varsigma^0)^2 \,-\, 1)\,.\,X^2 \,+\, 2\,.\,\theta^2 \,+\, \theta + 1$$

4.5 The coefficients of the polynomial $\Lambda(\delta\mathbf{r})$ in a pre-Euclidean context

With the help of:

$$[A] = [J]$$

... and of the results which have been accumulated in [c] and [d], one states that:

1. For the coefficient of degree zero:

$$-|P| = d = \frac{|S_0|}{8} + <_{\Lambda} \mathbf{s},_{\Lambda} \mathbf{s} >_{[d_{ij}]} = |N|$$

2. For the coefficient of degree one:

$$d_1 =$$

$$0\,.\,(p_{31}\,.\,p_{23} - p_{21}\,.\,p_{33}) + 0\,.\,(p_{33}\,.\,p_{11} - p_{31}\,.\,p_{13}) + 1\,.\,(p_{21}\,.\,p_{13} - p_{11}\,.\,p_{23})$$

$$+$$

$$0\,.\,(p_{21}\,.\,p_{32} - p_{22}\,.\,p_{31}) - 1\,.\,(p_{31}\,.\,p_{12} - p_{11}\,.\,p_{32}) + 0\,.\,(p_{11}\,.\,p_{22} - p_{21}\,.\,p_{12})$$

$$=$$

$$(p_{21}\,.\,p_{13} - p_{11}\,.\,p_{23}) - (p_{31}\,.\,p_{12} - p_{11}\,.\,p_{32})$$

$$=$$

$$p_{11}\,.\,(p_{32} - p_{23}) + (p_{21}\,.\,p_{13} - p_{31}\,.\,p_{12})$$

And:

$$d_2$$

$$=$$

$$0\,.\,(p_{32}\,.\,p_{23} - p_{22}\,.\,p_{33}) + 0\,.\,(p_{33}\,.\,p_{12} - p_{32}\,.\,p_{13}) + 1\,.\,(p_{22}\,.\,p_{13} - p_{12}\,.\,p_{23})$$

$$+$$

$$1\,.\,(p_{21}\,.\,p_{32} - p_{22}\,.\,p_{31}) + 0\,.\,(p_{31}\,.\,p_{12} - p_{11}\,.\,p_{32}) + 0\,.\,(p_{11}\,.\,p_{22} - p_{21}\,.\,p_{12})$$

$$=$$

$$(p_{22}\,.\,p_{13} - p_{12}\,.\,p_{23}) + (p_{21}\,.\,p_{32} - p_{22}\,.\,p_{31})$$

$$=$$

$$p_{22}\,.\,(p_{13} - p_{31}) + (p_{21}\,.\,p_{32} - p_{12}\,.\,p_{23})$$

And:

$$d_3$$

$$=$$

$$0\,.\,(p_{32}\,.\,p_{23} - p_{22}\,.\,p_{33}) - 1\,.\,(p_{33}\,.\,p_{12} - p_{32}\,.\,p_{13}) + 0\,.\,(p_{22}\,.\,p_{13} - p_{12}\,.\,p_{23})$$

$$+$$

$$1\,.\,(p_{21}\,.\,p_{33} - p_{23}\,.\,p_{31}) + 0\,.\,(p_{31}\,.\,p_{13} - p_{11}\,.\,p_{33}) + 0\,.\,(p_{11}\,.\,p_{23} - p_{21}\,.\,p_{13})$$

$$=$$

$$-1\,.\,(p_{33}\,.\,p_{12} \,-\, p_{32}\,.\,p_{13}) \,+\, (p_{21}\,.\,p_{33} \,-\, p_{23}\,.\,p_{31})$$

$$=$$

$$p_{33}\,.\,(p_{21} \,-\, p_{12}) \,+\, (p_{32}\,.\,p_{13} \,-\, p_{23}\,.\,p_{31})$$

3. For the coefficient of degree two:

$$d_{11}$$

$$=$$

$$p_{11}\,.\,(-1\,.\,1 \,-\, 0\,.\,0) \,+\, p_{21}\,.\,(0\,.\,0 \,-\, 0\,.\,A^3_{12}) \,+\, p_{31}\,.\,(0\,.\,0 \,-\, 0\,.\,A^2_{13})$$

$$=$$

$$-\,p_{11}$$

$$d_{22}$$

$$=$$

$$p_{12}\,.\,(0\,.\,0 \,-\, 0\,.\,A^3_{21}) \,+\, p_{22}\,.\,(1\,.\,-1 \,-\, 0\,.\,0) \,+\, p_{32}\,.\,(0\,.\,0 \,-\, A^1_{23}\,.\,0)$$

$$=$$

$$-\,p_{22}$$

$$d_{33}$$

$$=$$

$$p_{13}\,.\,(A^2_{31}\,.\,0 \,-\, 0\,.\,0) \,+\, p_{23}\,.\,(0\,.\,0 \,-\, 0\,.\,A^3_{12}) \,+\, p_{33}\,.\,(-1\,.\,1 \,-\, 0\,.\,0)$$

$$=$$

$$-p_{33}$$

Starting with the generic expression [d; p. 38]:

$$d_{mn}$$

$$=$$

$$-\,p_{11}\,.\,(A^2_{m2}\,.\,A^3_{n3} \,-\, A^2_{m3}\,.\,A^3_{n2}) \,-\, p_{12}\,.\,(A^2_{m1}\,.\,A^3_{n3} \,-\, A^2_{m3}\,.\,A^3_{n1}) \,-\, p_{13}\,.\,(A^2_{m1}\,.\,A^3_{n2} \,-\, A^2_{m2}\,.\,A^3_{n1})$$
$$-\,p_{21}\,.\,(A^1_{m3}\,.\,A^3_{n2} \,-\, A^1_{m2}\,.\,A^3_{n3}) \,-\, p_{22}\,.\,(A^1_{m1}\,.\,A^3_{n3} \,-\, A^1_{m3}\,.\,A^3_{n1}) \,-\, p_{23}\,.\,(A^1_{m2}\,.\,A^3_{n1} \,-\, A^1_{m1}\,.\,A^3_{n2})$$
$$-\,p_{31}\,.\,(A^1_{m2}\,.\,A^2_{n3} \,-\, A^1_{m3}\,.\,A^2_{n2}) \,-\, p_{32}\,.\,(A^1_{m3}\,.\,A^2_{n1} \,-\, A^1_{m1}\,.\,A^2_{n3}) \,-\, p_{33}\,.\,(A^1_{m2}\,.\,A^2_{n1} \,-\, A^1_{m1}\,.\,A^2_{n2})$$

For example:

$$d_{12}$$

$$=$$

$$-\,p_{11}\,.\,(A^2_{12}\,.\,A^3_{23} \,-\, A^2_{13}\,.\,A^3_{22}) \,-\, p_{12}\,.\,(A^2_{11}\,.\,A^3_{23} \,-\, A^2_{13}\,.\,A^3_{21}) \,-\, p_{13}\,.\,(A^2_{11}\,.\,A^3_{22} \,-\, A^2_{12}\,.\,A^3_{21})$$
$$-\,p_{21}\,.\,(A^1_{13}\,.\,A^3_{22} \,-\, A^1_{12}\,.\,A^3_{23}) \,-\, p_{22}\,.\,(A^1_{11}\,.\,A^3_{23} \,-\, A^1_{13}\,.\,A^3_{21}) \,-\, p_{23}\,.\,(A^1_{12}\,.\,A^3_{21} \,-\, A^1_{11}\,.\,A^3_{22})$$
$$-\,p_{31}\,.\,(A^1_{12}\,.\,A^2_{23} \,-\, A^1_{13}\,.\,A^2_{22}) \,-\, p_{32}\,.\,(A^1_{13}\,.\,A^2_{21} \,-\, A^1_{11}\,.\,A^2_{23}) \,-\, p_{33}\,.\,(A^1_{12}\,.\,A^2_{21} \,-\, A^1_{11}\,.\,A^2_{12})$$

$$=$$

$$-\,p_{11}\,.\,A^2_{12}\,.\,A^3_{23} \,+\, p_{12}\,.\,A^2_{13}\,.\,A^3_{21} \,-\, p_{13}\,.\,A^2_{12}\,.\,A^3_{12}$$
$$+\,p_{21}\,.\,A^1_{12}\,.\,A^3_{23} \,-\, p_{22}\,.\,A^1_{13}\,.\,A^3_{12} \,+\, p_{23}\,.\,A^1_{12}\,.\,A^3_{12}$$
$$-\,p_{31}\,.\,A^1_{12}\,.\,A^2_{23} \,+\, p_{32}\,.\,A^1_{13}\,.\,A^2_{12} \,+\, p_{33}\,.\,A^1_{12}\,.\,A^2_{12}$$

$$=$$

$$- p_{11} . 0 . 0 - p_{12} . A^2_{13} . A^3_{12} - p_{13} . 0 . A^3_{12}$$
$$+ p_{21} . 0 . 0 - p_{22} . 0 . A^3_{12} + p_{23} . 0 . A^3_{12}$$
$$- p_{31} . 0 . 0 + p_{32} . 0 . 0 + p_{33} . 0 . 0$$
$$=$$
$$- p_{12} . A^2_{13} . A^3_{12}$$
$$=$$
$$- p_{12} . - 1 . 1$$
$$=$$
$$p_{12}$$

For example:

$$d_{21}$$
$$=$$
$$- p_{11} . (A^2_{22} . A^3_{13} - A^2_{23} . A^3_{12}) - p_{12} . (A^2_{21} . A^3_{13} - A^2_{23} . A^3_{11}) - p_{13} . (A^2_{21} . A^3_{12} - A^2_{22} . A^3_{11})$$
$$- p_{21} . (A^1_{23} . A^3_{12} - A^1_{22} . A^3_{13}) - p_{22} . (A^1_{21} . A^3_{13} - A^1_{23} . A^3_{11}) - p_{23} . (A^1_{22} . A^3_{11} - A^1_{m1} . A^3_{n2})$$
$$- p_{31} . (A^1_{22} . A^2_{13} - A^1_{23} . A^2_{12}) - p_{32} . (A^1_{23} . A^2_{11} - A^1_{21} . A^2_{13}) - p_{33} . (A^1_{22} . A^2_{11} - A^1_{21} . A^2_{12})$$
$$=$$
$$- p_{11} . (0 . A^3_{13} - 0 . A^3_{12}) - p_{12} . (0 . 0 - 0 . 0) - p_{13} . (0 . A^3_{12} - 0 . 0)$$
$$- p_{21} . (A^1_{23} . A^3_{12} - 0 . 0) - p_{22} . (0 . 0 - A^1_{23} . 0) - p_{23} . (0 . 0 - 0 . A^3_{12})$$
$$- p_{31} . (0 . A^2_{13} - A^1_{23} . 0) - p_{32} . (A^1_{23} . 0 - 0 . A^2_{13}) - p_{33} . (0 . 0 - 0 . 0)$$
$$=$$
$$- p_{21} . A^1_{23} . A^3_{12}$$
$$=$$
$$- p_{21} . 1 . 1$$
$$=$$
$$- p_{21}$$

and so on...

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