

# An Introduction to General Relativistic Wave Differential Operator in the Quantum Field Theory

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## Abstract

This work introduces a unified operator framework for quantum field theory through the *generalized relativistic wave differential operator*  $D_{\alpha\beta}^{mn\lambda}$ . By appropriate parameter choices, this operator encompasses the Klein-Gordon, Dirac, and non-monogenic operators, revealing their fundamental connections. We propose a *quantum field tensor*  $\Psi_{\sigma\rho\tau}$  with binary activation indices for scalar, spinor, and gauge fields, generating all possible interactions while preserving Lorentz covariance. The framework incorporates gauge invariance through minimal coupling and yields both Proca and Maxwell equations as special cases. This approach provides a powerful operator-based unification of quantum field theory with applications to higher-spin theories.

**Keywords:** Generalized differential operator, Quantum field tensor, Relativistic wave operator, Klein-Gordon equation, Dirac equation, Proca equation, Non-monogenic operators, Tensor field representations.

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# 1 Introduction

The wave operator, or d'Alembertian, constitutes a fundamental object in mathematical physics, serving as the cornerstone of relativistic field theories. In special relativity, it generalizes the classical Laplacian to Minkowski spacetime, adopting the form  $\square = \partial^\mu \partial_\mu = g^{\mu\nu} \partial_\mu \partial_\nu$  in Cartesian coordinates, which explicitly becomes  $\square = -(1/c^2)\partial_t^2 + \nabla^2$  under the  $(-, +, +, +)$  metric convention. This Lorentz-invariant operator extends naturally to curvilinear coordinate systems through appropriate metric adaptations and acts component-wise on tensor fields in inertial frames. Its differential form representation,  $\square = d\delta + \delta d$ , provides an elegant geometric formulation. In general relativity, the operator generalizes further to the Laplace-Beltrami operator on curved spacetimes [1, 2].

Building upon this foundation, Bar, Ginoux, and Pfaffle provided a rigorous modern framework for understanding wave operators on Lorentzian manifolds. Their work establishes the wave operator as a second-order linear differential operator acting on sections of vector bundles over time-oriented spacetimes. Significantly, they characterized its local structure through inverse metric coefficients and smoothly varying matrix-valued coefficients, capturing essential analytic and geometric properties in a unified manner. This formulation has proven essential for studying hyperbolic partial differential equations on curved backgrounds, influencing subsequent developments in global analysis and mathematical relativity [4].

In this paper, we introduce a **generalized relativistic wave differential operator**,  $D_{\alpha\beta}^{mn\lambda}$ , designed to unify several fundamental operators in quantum field theory under a single analytic structure. By incorporating discrete indices and matrix-valued coefficients, this operator encapsulates a family of wave-like operators that include the Klein–Gordon, Dirac, and non-monogenic operators as special cases. Such a unification is not merely formal; it reveals underlying connections between seemingly distinct field theories and offers a versatile tool for constructing new models within the quantum field theoretic framework.

A key innovation of this work is the introduction of a **quantum field tensor**  $\Psi_{\sigma\rho\tau} = \varphi^\sigma \otimes \psi^\rho \otimes A_\mu^\tau$ , where the binary indices  $\sigma, \rho, \tau \in \{0, 1\}$  act as a "signature" that activates or deactivates the constituent fields—scalar, spinor, and gauge fields, respectively. This approach provides a compact and representation-theoretically natural description of field content and interactions, facilitating a unified treatment of field equations and coupling structures without recourse to ad hoc Lagrangian constructions.

Furthermore, we extend the operator formalism to include gauge interactions through a generalized minimal coupling prescription, leading to a gauge-covariant differential operator that naturally incorporates electromagnetic interactions for fields of various spins. This formulation allows for a direct derivation of field equations such as the Proca and Maxwell equations from a single operator-based

framework, highlighting the role of symmetry and consistency in the structure of quantum field theories.

The paper is organized as follows: Section 2 reviews necessary preliminaries on Lorentzian geometry and operator theory in Minkowski space. Section 3 introduces the generalized wave operator and proves its reductions to standard operators in QFT. Section 4 extends the framework to include gauge interactions and discusses the resulting field equations. We conclude with a discussion of implications and future directions within quantum field theory.

## 2 Preliminaries

**Definition 2.1.** A *pseudo-Riemannian manifold*  $\mathcal{M}$  is a differentiable manifold, equipped with an everywhere non-degenerate metric  $g$ . We denote a manifold provided with a specific metric with the couple  $(\mathcal{M}, g)$  [3].

**Definition 2.2.** A *Lorentzian manifold* is a pseudo-Riemannian manifold  $\mathcal{M}$ , equipped with a metric  $g$  with signature  $(1, n - 1)$ , where  $n$  is the dimension of  $\mathcal{M}$  [3].

**Definition 2.3.** Let us consider the following metric tensor  $\eta$ :

$$g^{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (1)$$

A Lorentzian manifold  $\mathcal{M}$ , equipped with the metric  $g$ , is the *Minkowski space*. The metric  $g^{\alpha\beta}$  is called *Minkowski metric* [3].

## 3 Generalized relativistic wave operator

**Definition 3.1.** Let  $\alpha, \beta, \lambda, m, n \in \mathbb{N}$ . The *generalized relativistic wave differential operator* on Minkowski space is defined as:

$$D_{\alpha\beta}^{mn\lambda} = i^n (\gamma_{m+\alpha+\beta})^{n+\alpha+\lambda} \frac{\partial}{\partial x^{\alpha+m}} \left( \frac{\partial}{\partial x^\beta} \right)^\lambda (g^{\alpha\beta} - q^{\alpha\beta}) - i^n \gamma_{\alpha+m} q^{\alpha\beta} \frac{\partial}{\partial x^{m+\alpha+\beta}}, \quad (2)$$

where  $g^{\alpha\beta}$  denotes the Minkowski metric (1), and  $q^{\alpha\beta}$  is the auxiliary tensor defined by [5]:

$$q^{\alpha\beta} = \begin{cases} -1 & \alpha \neq \beta, \\ 0 & \alpha = \beta. \end{cases} \quad (3)$$

The symbols  $\gamma_{m+\alpha+\beta}$  and  $\gamma_{\alpha+m}$  represent matrix-valued coefficients.

**Definition 3.2.** From Definition 3.1 and equation (3), we define the relativistic energy relation:

$$D_{\alpha\beta}^{mn\lambda} + (m_0)^{\alpha+\lambda+n} = p_0^{\alpha+\lambda+n} (\gamma_{\alpha-\lambda})^{n+\alpha+\lambda}, \quad (4)$$

where  $p_0$  denotes the energy and  $m_0$  the rest mass.

According to the above definition, we demonstrate that for specific values of the parameters  $\alpha, \beta, \lambda, m, n \in \mathbb{N}$ , the generalized relativistic wave operator reduces to well-known operators: the Laplace operator, the Dirac operator, and the non-monogenic operator discussed in [5].

**Theorem 3.3.** For the parameter values  $\alpha = 1, \beta = 1, \lambda = 1, m = 0,$  and  $n = 0,$  the generalized relativistic wave operator (2) and (4) reduces to the Klein-Gordon operator:

$$-\frac{\partial^2}{\partial x^2} + m_0^2 = E^2, \quad (5)$$

where  $m_0$  denotes the rest mass and  $E$  represents the energy.

*Proof.* Substituting the parameter values into equation (2), we obtain:

$$\begin{aligned} i^0(\gamma_{0+1+1})^{0+1+1} \frac{\partial}{\partial x^{1+0}} \left( \frac{\partial}{\partial x^1} \right)^1 (g^{11} - q^{11}) \\ - i^0 \gamma_{1+0} q^{11} \frac{\partial}{\partial x^{0+1+1}} + (m_0)^{1+1+0} = p_0^{1+1+0} (\gamma_{1-1})^{0+1+1}. \end{aligned}$$

Using the metric tensor (1) and the definition of  $q^{\alpha\beta}$  (3), this simplifies to:

$$(\gamma_2)^2 \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) + (m_0)^2 = p_0^2 \gamma_0^2.$$

Applying the property of Dirac gamma matrices  $\gamma_\nu^2 = -1, \gamma_0^2 = 1$  and identifying  $p_0$  with the energy  $E$ , we finally obtain:

$$-\frac{\partial^2}{\partial x^2} + m_0^2 = E^2,$$

which completes the proof of equation (5). □

Now, let us consider the Dirac's case in the following theorem

**Theorem 3.4.** In analogy to Theorem 3.3, the Dirac energy operator is obtained from the generalized relativistic wave operator (2) with the parameter values  $\alpha = 0,$   $\beta = 0,$   $m = k,$   $n = 1,$  and  $\lambda = 0.$  Specifically,

$$i\gamma^k \frac{\partial}{\partial x^k} + m_0 = E\gamma_0, \quad (6)$$

where  $\gamma^k, \gamma_0$  are the Dirac matrices for  $k = 1, 2, 3,$   $m_0$  is the rest mass, and  $E$  denotes the energy.

*Proof.* Substituting the parameter values into equations (2) and (4), we obtain:

$$i^1(\gamma_{k+0+0})^{1+0+0} \frac{\partial}{\partial x^{k+0}} \left( \frac{\partial}{\partial x^0} \right)^0 (g^{00} - q^{00}) - i^1 \gamma_{0+k} q^{00} \frac{\partial}{\partial x^{0+0+k}} + (m_0)^{0+0+1} = p_0^{0+0+1} (\gamma_{0-0})^{1+0+0}.$$

Using the metric tensor (1) and the definition of  $q^{\alpha\beta}$  (3), where  $g^{00} = -1$  and  $q^{00} = 0$ , this expression simplifies to:

$$-i\gamma_k \frac{\partial}{\partial x^k} + m_0 = p_0 \gamma_0.$$

Applying the properties of Dirac matrices and identifying  $p_0$  with the energy  $E$ , we obtain the final result:

$$-i\gamma^k \frac{\partial}{\partial x^k} + m_0 = E\gamma_0,$$

which completes the proof.  $\square$

Now we consider the case of the non-monogenic operator discussed in [5]. For the parameter values  $\alpha \neq \beta$ ,  $\lambda = 0$ ,  $\alpha = 0$ ,  $n = 1$ , and  $m, \beta \in \mathbb{N}$ , we establish the following result:

**Theorem 3.5.** *The non-monogenic energy operator is derived from the generalized relativistic wave operator (2) and (4) with parameter values  $\alpha \neq \beta$ ,  $\lambda = 0$ ,  $\alpha = 0$ ,  $n = 1$ , and  $m, \beta \in \mathbb{N}$ , yielding:*

$$ie_{m+\beta} \frac{\partial}{\partial x^m} + ie_m \frac{\partial}{\partial x^{m+\beta}} + m_0 = Ee_0, \quad (7)$$

where  $e_{m+\beta}$  and  $e_m$  are matrix-valued coefficients satisfying the relations (14) in [5],  $m_0$  denotes the rest mass, and  $E$  represents the energy.

*Proof.* For  $\alpha \neq \beta$ ,  $\lambda = 0$ ,  $\alpha = 0$ ,  $n = 1$ , and  $m, \beta \in \mathbb{N}$ , we substitute into equations (2) and (4) to obtain:

$$i^1(e_{m+0+\beta})^{1+0+0} \frac{\partial}{\partial x^{0+m}} \left( \frac{\partial}{\partial x^\beta} \right)^0 (g^{0\beta} - q^{0\beta}) - i^1 e_{0+m} q^{0\beta} \frac{\partial}{\partial x^{m+0+\beta}} + m_0^{0+0+1} = p_0^{0+0+1} (e_{0-0})^{1+0+0}.$$

Using the metric tensor (1) and the definition of  $q^{\alpha\beta}$  (3), where  $g^{0\beta} = 0$  for  $\beta \neq 0$  and  $q^{0\beta} = -1$  for  $\alpha \neq \beta$ , this expression simplifies to:

$$i(e_{m+\beta})^1 \frac{\partial}{\partial x^m} (0 - (-1)) - ie_m (-1) \frac{\partial}{\partial x^{m+\beta}} + m_0 = p_0 e_0.$$

Simplifying further:

$$ie_{m+\beta} \frac{\partial}{\partial x^m} + ie_m \frac{\partial}{\partial x^{m+\beta}} + m_0 = p_0 e_0.$$

Identifying  $p_0$  with the energy  $E$ , we obtain the final result:

$$ie_{m+\beta} \frac{\partial}{\partial x^m} + ie_m \frac{\partial}{\partial x^{m+\beta}} + m_0 = Ee_0,$$

which completes the proof of Theorem 3.5.  $\square$

One may inquire about the action of the operator (2). The answer to this question is provided by the following definition:

**Definition 3.6.** Building upon Definition 2.2, we define a *quantum field tensor* over a Lorentzian manifold as follows:

$$\Psi_{\sigma\rho\tau} = \varphi^\sigma \otimes \psi^\rho \otimes A_\mu^\tau, \quad (8)$$

where  $\sigma, \rho, \tau \in \mathbb{N}$ .

A particularly illustrative case is when these indices are binary (0 or 1), acting as a "signature" that specifies which fundamental fields are present in the tensor product. In this framework, the scalar field  $\varphi$  is activated when  $\sigma = 1$ , the spinor field  $\psi$  is activated when  $\rho = 1$ , and the electromagnetic potential  $A_\mu$  is activated when  $\tau = 1$ . A value of 0 for any index signifies that the corresponding field is not included in the product (effectively represented by the scalar 1). This provides a compact notation for representing fundamental fields and their interactions. The following table enumerates the non-trivial cases for binary indices  $(\sigma, \rho, \tau)$ .

Table 1: Field tensor signatures and physical interpretations

Tensor	Fields	Interpretation
$\Psi_{001}$	$A_\mu$	Fundamental electromagnetic field
$\Psi_{010}$	$\psi$	Fundamental spinor field
$\Psi_{100}$	$\varphi$	Fundamental scalar field
$\Psi_{101}$	$\varphi A_\mu$	Scalar-EM field coupling
$\Psi_{110}$	$\varphi\psi$	Scalar-spinor coupling
$\Psi_{011}$	$\psi A_\mu$	Spinor-EM coupling (QED vertex)
$\Psi_{111}$	$\varphi\psi A_\mu$	Triple field interaction

From the above definitions we can rewrite the relativistic wave equation for some values of  $\sigma, \rho, \tau \in \mathbb{N}$ .

**Proposition 3.7.** Taking into account the relation 4 and the Theorems 3.3 and 3.4 for  $\sigma = 1, \rho = 0, \tau = 0$ , the Klein - Gordon, and Dirac equation can be write as

$$D_{11}^{001} \Psi_{100} = 0,$$

and for  $\sigma = 0, \rho = 1$ , and  $\tau = 0$  the Dirac equation is given by:

$$D_{00}^{k10} \Psi_{010} = 0.$$

## 4 Application to quantum field theory

For the study in this section, it is necessary to consider the following definition.

**Definition 4.1.** Let  $\alpha, \beta, \lambda, \mu \in \{0, 1, 2, 3\}$  be Lorentz indices. The *general quantum field differential operator* is defined as:

$$D_\mu^{\alpha\beta\lambda} = i^{\alpha+\lambda}(\gamma_{\beta+\mu})^{\alpha+\lambda}\partial^{\mu+\beta}(\partial_\mu)^\lambda + p^{\alpha\lambda}(m_0)^{\lambda+\alpha}, \quad (9)$$

where  $p^{\alpha\lambda} = g^{\alpha\beta} - q^{\alpha\lambda}$  denotes a metric perturbation tensor with components:

$$p^{\alpha\lambda} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \quad (10)$$

$m_0$  represents the rest mass  $\gamma_\mu$  are the Dirac gamma matrices, and  $\partial_\mu$  denotes the partial derivative operator.

**Definition 4.2.** Let  $\alpha, \beta, \lambda, \mu$  be Lorentz indices, and let  $e$  denote the electric charge coupling constant. The *gauge-covariant differential operator*  $D_\mu^{\alpha+\beta}$  is defined through the minimal coupling substitution [6, 7]

$$\partial^{\mu+\beta} \longrightarrow D^{\mu+\beta} = \partial^{\mu+\beta} + ieA^{\mu+\beta}, \quad (11)$$

and at the vector level:

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + ieA_\mu. \quad (12)$$

The general operator acting on a tensor quantum field (8) is given by:

$$D_\mu^{\alpha\beta\lambda}\Psi_{\sigma\rho\tau} = i^{\alpha+\lambda}(\gamma_{\beta+\mu})^\alpha D^{\mu+\beta}\Psi_{\sigma\rho\tau} + p^{\alpha\lambda}(m_0)^{\lambda+\alpha}\Psi_{\sigma\rho\tau}, \quad (13)$$

where  $A_\mu$  is the electromagnetic four-potential,  $\gamma_{\rho+\mu}$  denotes a Dirac gamma matrix with shifted index, and  $p^{\alpha\lambda}$  is a metric perturbation tensor.

**Remark 4.3.** This definition generalizes the minimal coupling prescription  $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$  of quantum electrodynamics to the case of higher-index differential operators  $\partial^{\mu+\beta} \rightarrow D^{\mu+\beta}$ , enabling a unified description of electromagnetic interactions for fields of various spins.

**Definition 4.4.** Let  $A_\mu$  be the electromagnetic four-potential and  $D_\mu = \partial_\mu + ieA_\mu$  the gauge-covariant derivative. The generalized differential operator (13) acting on  $A_\mu$  is defined as:

$$D_\mu^{\alpha\beta\lambda}A_\mu = i^{\alpha+\lambda}(\gamma_{\beta+\mu})^\alpha D^{\mu+\beta}(\gamma_{\beta+\mu})^\lambda (D_\mu)^\lambda A_\mu + p^{\alpha\lambda}(m_0)^{\lambda+\alpha}A_\mu \quad (14)$$

where  $\gamma_{\rho+\mu}$  are Dirac matrices, and  $p^{\alpha\lambda}$  is the metric perturbation tensor (10).

**Theorem 4.5.** *Let  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  be the electromagnetic field tensor and  $j^\nu$  the conserved current. The generalized operator  $D_\mu^{101} A_\mu$  can be expressed in terms of standard electromagnetic quantities as:*

$$D_\mu^{101} A_\mu = D_\mu D^\mu A_\mu + \mu_0 j^\nu - \partial_\nu F^{\mu\nu}, \quad (15)$$

where  $\mu_0$  is the vacuum permeability. This expression establishes a direct connection between the abstract operator formalism and concrete electromagnetic field equations.

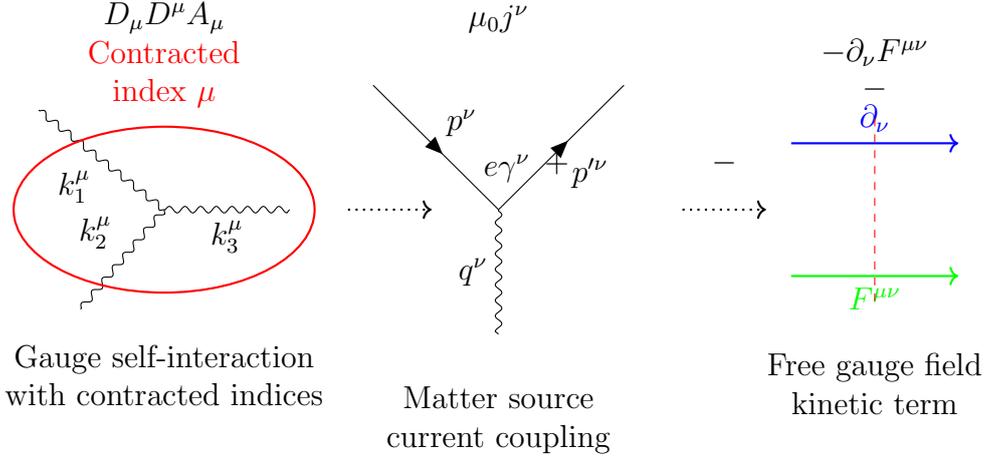
*Proof.* According to the classification in Table 1 and the general definition in Eq. (14), the operator  $D_\mu^{\alpha\beta\lambda}$  reduces to known physical equations for specific index values.

In particular, for the electromagnetic field case  $\Psi_{001} = A_\mu$  with indices  $\alpha = 1$ ,  $\lambda = 1$ ,  $\beta = 0$ , we obtain:

$$\begin{aligned} D_\mu^{101} \Psi_{001} = D_\mu^{101} A_\mu &= -\gamma_\mu D^\mu \gamma_\mu D_\mu A_\mu + p^{11} (m_0)^2 A_\mu, \\ &= D^\mu D_\mu A_\mu + (m_0)^2 A_\mu, \end{aligned}$$

which corresponds to the massive Proca equation for the electromagnetic potential [9]. This equation describes a massive vector field where the mass term  $(m_0)^2$  explicitly breaks the gauge symmetry, consistent with known results in quantum field theory. Comparing with the inhomogeneous Proca equation  $\partial_\nu F^{\mu\nu} + (m_0 c)^2 A^\mu = \mu_0 j^\mu$ , we arrive at the stated relationship after appropriate term rearrangement and identification of the current coupling, obtaining (15).  $\square$

Let us consider the particular case when  $D_\mu^{101} A_\mu = 0$ , which corresponds to the field equation for the electromagnetic potential. This equation encapsulates three fundamental physical processes: gauge self-interactions [7, 11], matter couplings, and free propagation, as represented by the Feynman diagram below.



**Feynman Diagram for the Complete Gauge Field Equation:**

$$D_\mu D^\mu A_\mu + \mu_0 j^\nu - \partial_\nu F^{\mu\nu} = 0$$

Representing gauge self-interactions, matter couplings, and free propagation

### 4.1 Diagram Interpretation

The Feynman diagram illustrates the complete structure of the gauge field equation:

$$D_\mu D^\mu A_\mu + \mu_0 j^\nu - \partial_\nu F^{\mu\nu} = 0$$

The term  $D_\mu D^\mu A_\mu$  in electromagnetism represents the quantum propagation of the photon field  $A_\mu$  through the covariant derivative  $D_\mu = \partial_\mu + ieA_\mu$  [7]. However, in quantum electrodynamics (QED), which is an abelian  $U(1)$  gauge theory, the triple photon vertex vanishes identically due to the commuting nature of the gauge group. The momentum conservation  $k_1^\mu + k_2^\mu + k_3^\mu = 0$  would describe photon self-interactions, but in QED photons do not interact directly with each other. The contracted  $\mu$  indices indicate a Lorentz-invariant combination, but the absence of non-abelian structure means no genuine triple photon vertex exists. Instead, this term contributes to the photon propagator and quantum corrections through loop diagrams where photons interact indirectly via virtual electron-positron pairs, maintaining the fundamental principle that photons in QED only interact with charged matter, not with themselves.

The complete expression  $D_\mu D^\mu A_\mu + \mu_0 j^\nu - \partial_\nu F^{\mu\nu} = 0$  encapsulates the full structure of electromagnetic theory: the first term describes the quantum dynamics of the photon field, the second term  $\mu_0 j^\nu$  represents the coupling to matter currents through the minimal coupling prescription, and the third term  $-\partial_\nu F^{\mu\nu}$  gives the classical Maxwell equations in the absence of quantum corrections. This formulation highlights how quantum field theory generalizes classical electromagnetism

while preserving gauge invariance and Lorentz covariance. The term  $\mu_0 j^\nu$  describes the fundamental interaction between matter currents and gauge fields through minimal coupling, where the conserved Noether current  $j^\nu = \bar{\psi}\gamma^\nu\psi$  couples directly to the gauge field  $A_\nu$ . In the Feynman diagram representation, this corresponds to a fermion-gauge vertex characterized by momentum transfer  $q^\nu = p^\nu - p'^\nu$ , ensuring energy-momentum conservation at the interaction point. The coupling strength is governed by vertex factors that depend on the specific gauge theory: for quantum electrodynamics (QED), the vertex factor is  $ie\gamma^\nu$ , representing the electromagnetic coupling between electrons and photons [6, 10], while for non-abelian theories like quantum chromodynamics (QCD), the vertex factor becomes  $igT^a\gamma^\nu$ , incorporating the gauge group generators  $T^a$  that account for the additional color degrees of freedom [6, 12]. This term embodies the core principle of gauge theories where matter fields interact with force carriers through locally conserved currents derived from symmetry principles.

The term  $-\partial_\nu F^{\mu\nu}$  encodes the free propagation dynamics of gauge fields in the absence of external sources, representing the left-hand side of the inhomogeneous Maxwell equations that govern classical electromagnetic phenomena. Diagrammatically, this corresponds to the gauge field propagator where the differential operator  $\partial_\nu$  acts upon the field strength tensor  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ , which for the abelian case of electromagnetism captures the fundamental antisymmetric structure of the electromagnetic field. This term describes how gauge fields evolve and propagate through spacetime independently of matter interactions, embodying the wave-like nature of electromagnetic radiation and the self-dynamics of the gauge field itself. In the context of the complete equation, this free field term provides the classical foundation upon which quantum corrections and matter interactions are built, ensuring that in the source-free limit the theory reduces to the well-established Maxwellian electrodynamics.

## 5 Discussion and further works

The generalized relativistic wave differential operator introduced in Definition 3.1,  $D_{\alpha\beta}^{mn\lambda}$ , provides a powerful unifying framework that encompasses several fundamental operators in mathematical physics. As demonstrated in Theorems 3.3, 3.4 and 3.5 this operator reduces to the Klein-Gordon operator, Dirac operator, and non-monogenic operator for specific parameter values. This unification is not merely formal but reveals deep connections between seemingly distinct physical theories. The parameter space  $(\alpha, \beta, \lambda, m, n) \in \mathbb{N}$  creates a rich landscape where different physical regimes emerge naturally, suggesting that these operators are not independent entities but special cases of a more fundamental mathematical structure.

The reduction to the Klein-Gordon operator (Theorem 3.3) for  $\alpha = 1$ ,  $\beta = 1$ ,  $\lambda = 1$ ,  $m = 0$ ,  $n = 0$  demonstrates how massive scalar fields emerge from the general framework. Similarly, the recovery of the Dirac operator (Theorem 3.4) for  $\alpha = 0$ ,  $\beta = 0$ ,  $m = k$ ,  $n = 1$ ,  $\lambda = 0$  shows the incorporation of spinorial degrees of freedom. Most notably, Theorem 3.5 reveals how non-monogenic operators—previously considered mathematical curiosities—find a natural place within this extended framework, suggesting potential applications in describing exotic field configurations or higher-spin theories. The quantum field tensor  $\Psi_{\sigma\rho\tau} = \varphi^\sigma \otimes \psi^\rho \otimes A_\mu^\tau$  introduced in Definition 3.6 represents a fundamental advancement in field quantization. The binary index structure  $(\sigma, \rho, \tau) \in \{0, 1\}$  acts as a genetic code for quantum fields, where different combinations generate either fundamental fields or interaction vertices. This approach transcends the traditional Lagrangian formulation by providing a first-principles derivation of interaction terms rather than postulating them ad hoc.

The significance of this tensor product structure becomes particularly evident when considering the representation theory of the Lorentz group. Each component field transforms under specific representations:  $\varphi$  under  $(0, 0)$ ,  $\psi$  under  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ , and  $A_\mu$  under  $(\frac{1}{2}, \frac{1}{2})$ . The tensor product then automatically generates the correct transformation properties for composite systems, ensuring Lorentz invariance without additional constraints. This mathematical naturalness suggests that the tensor product approach may be fundamental to understanding why certain interactions are allowed while others are forbidden in quantum field theory. Section 4 extends this framework to incorporate gauge theories through the general quantum field differential operator  $D_\mu^{\alpha\beta\lambda}$  (Definition 4.1) and its gauge-covariant generalization (Definition 4.2). The minimal coupling substitution  $\partial^{\mu+\beta} \rightarrow D^{\mu+\beta} = \partial^{\mu+\beta} + ieA^{\mu+\beta}$  generalizes the standard prescription to higher-index differential operators, enabling a unified treatment of electromagnetic interactions for fields of various spins.

Theorem 4.5 represents a cornerstone result, establishing the connection between the abstract operator formalism and concrete electromagnetic field equations. The expression  $D_\mu^{101}A_\mu = D_\mu D^\mu A_\mu + \mu_0 j^\nu - \partial_\nu F^{\mu\nu}$  shows how the generalized operator naturally incorporates both the massive Proca equation and Maxwell's equations within a single framework. This result has profound implications for understanding mass generation mechanisms and gauge symmetry breaking, as the mass term  $(m_0)^2$  explicitly breaks the gauge symmetry in a controlled manner.

The Feynman diagram presented in Section 4 provides a visual representation of the complete gauge field equation, illustrating how the three fundamental processes gauge self-interactions, matter couplings, and free propagation emerge from the operator formalism. While the triple photon vertex in QED vanishes due to the abelian nature of U(1) gauge theory, its inclusion in the general framework sug-

gests possible extensions to non-abelian theories where such vertices are physical. The term  $\mu_0 j^\nu$  embodies the fundamental interaction between matter currents and gauge fields, with the vertex factor depending on the specific gauge theory:  $ie\gamma^\nu$  for QED and  $igT^a\gamma^\nu$  for non-abelian theories. This universality demonstrates how the framework naturally accommodates different gauge groups through appropriate choices of coupling constants and generators. Respect to future works, the intentional generality of our definitions with indices ranging over  $\mathbb{N}$  rather than being restricted to binary values opens numerous research avenues. Higher index values could correspond to field derivatives, enabling the description of higher-derivative theories that may be relevant for quantum gravity or effective field theories. The framework also naturally accommodates higher-spin fields [8] through appropriate extensions of the tensor product structure.

Moreover, the metric perturbation tensor  $p^{\alpha\lambda}$  introduced in Definition 4.1 suggests connections to curved spacetime and gravitational interactions. Future work could explore how this formalism generalizes to Riemannian manifolds, potentially providing a unified framework for coupling quantum fields to gravity.

The non-monogenic operators discussed in Theorem 3.5, while mathematically well-defined, warrant further investigation for their physical significance. They may describe exotic field configurations or provide insights into theories beyond the Standard Model, particularly in contexts where conventional monogenic functions are insufficient.

## 6 Conclusions

The unified framework developed in Sections 3 and 4 represents a significant step toward a more fundamental understanding of quantum field theory. By demonstrating how diverse physical theories emerge from a single mathematical structure, we provide evidence for the existence of deeper organizing principles governing relativistic quantum fields. The tensor product approach not only offers mathematical elegance but also suggests concrete pathways for extending quantum field theory beyond its current limitations, particularly in the direction of higher-spin theories, non-abelian gauge interactions, and potentially quantum gravity.

The compatibility of this framework with established results while simultaneously suggesting new generalizations demonstrates its robustness and potential for future developments in theoretical physics. As we continue to explore the rich structure revealed by these generalized operators, we may uncover deeper connections between seemingly disparate areas of physics, ultimately leading to a more unified understanding of the fundamental laws of nature.

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