

Arguments in Favor of the Berger-Parker Index as an Effective Sample Size: The Only True Particle Counter

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Abstract

In many fields, including computational statistics, ecology, economics, and physics, normalized weights define a discrete probability mass function over a set of entities/samples. The effective sample size (ESS) quantifies the concentration of these weights, providing a measure of sample representativeness. In this work, we show that, among various ESS formulations, the Berger-Parker index uniquely preserves the relative proportions of the weights, acting as a true particle counter. Other commonly used ESS expressions tend to overestimate the effective sample size when only normalized weights are considered. Several examples and formal demonstration are provided.

Keywords: Effective sample size, Hill numbers, Berger-Parker Index, normalized weights, importance sampling, Sequential Monte Carlo

1. Introduction

In many application domains, such as computational statistics, ecology, economics, political science, and quantum physics, among others, one frequently encounters a set of normalized non-negative weights

$$\bar{w}_n, \quad n = 1, \dots, N, \quad \text{with} \quad \sum_{n=1}^N \bar{w}_n = 1,$$

which define a discrete probability mass function (pmf) over N entities, such as particles, samples, or other types of units (e.g., species in ecology or parties in political science) [13, 10, 16, 19, 21]. The degree of concentration or dispersion of this weight distribution provides essential information about the efficiency of the corresponding sample representation, and is a way to

measure the theoretical concept of effective sample size (ESS) [9, 5, 12, 21]. From a practical point of view, several formulas have been proposed to quantify the discrepancy between the probability mass function (pmf) defined by the normalized weights \bar{w}_i and the uniform pmf with equal weights, serving as potential ESS measures. When all weights are equal ($\bar{w}_i = 1/N$), the ESS formulas attain their maximum value of N ; conversely, when a single weight dominates all others, the ESS expressions reach their minimum value of 1 [21]. Hence, all the most relevant ESS measures satisfy $1 \leq \text{ESS} \leq N$.

Several formulations of ESS have been proposed in the literature [8, 19, 21]. The most commonly used expression is

$$\text{ESS} = \frac{1}{\sum_{i=1}^N \bar{w}_i^2}, \quad (1)$$

which has been independently proposed across different disciplines. In importance sampling (IS) and sequential Monte Carlo (SMC), it is widely employed for adaptive resampling schemes [6, 7, 15, 17]. In physics, it is referred to as the *inverse participation ratio* [26]. In ecology, it is known as the *Hill number of order 2* [13] or as the *inverse Simpson concentration* [25]. In economics, it appears as the *inverse Herfindahl-Hirschman index*. Finally, it is also referred to as *Kish's ESS* [14].

Another important expression, perhaps the second most well-known and used, is

$$\text{ESS} = \frac{1}{\max_n \bar{w}_n}, \quad (2)$$

which is known as the *Hill number of order ∞* [13], or as the *Berger-Parker index* in ecology [1, 19, 21]. Several other ESS formulations can be found in the literature, and some of them will also be presented in the following sections.

The most relevant ESS expressions fulfill five conditions that we will describe in Section 3. In this work, we demonstrate that, among the most relevant ESS formulations (i.e., those that are both proper and stable), the Berger-Parker index given in Eq. (2) stands out as the only genuine particle counter. Specifically, it is the sole expression that faithfully preserves the relative proportions among particle weights. In contrast, all other ESS formulas yield an increase of the effective sample size measure that cannot be justified based solely on the normalized weights \bar{w}_n , without invoking any additional information.

2. Unnormalized weights as measure of relevance

In many scientific and computational frameworks, to each sample/particle, we have associated an unnormalized weight w_n that quantifies its relative relevance or importance, with $n = 1, \dots, N$. The specific definition of w_n depends on the scientific context (for instance, ecology, political science, physics, or statistics) as the weighting function reflects the underlying model or inferential objective [13, 10, 16, 19, 25]. To provide a unified formulation encompassing these diverse settings, let us assume that the unnormalized weights w_n are obtained as the application of a weighting function of the corresponding particles¹ $\mathbf{x}_n \in \mathbb{R}^d$, i.e.,

$$w_n = f(\mathbf{x}_n),$$

where the function $f(\cdot)$ represents the mechanism or rule by which the samples are evaluated. For example, in importance sampling, $f(\cdot)$ is defined as the ratio between the target and proposal distributions evaluated at \mathbf{x}_n [18, 19]. Note that, generally, this function can be very complex and even unbounded. However, we assume that the maximum attainable value M_{\max} of the function $f(\mathbf{x})$ exists and is finite, that is,

$$M_{\max} = \max f(\mathbf{x}) < \infty.$$

In general, this maximum value can be attained by several (possibly infinitely many) samples. For instance, suppose that $M_{\max} = f(\mathbf{x})$ for $\mathbf{x} \in \{\mathbf{x}', \mathbf{x}'', \mathbf{x}'''\}$. In this case, the three particles \mathbf{x}' , \mathbf{x}'' , and \mathbf{x}''' share the same level of relevance or importance, which is also the maximum possible. Clearly, a set composed of these three particles, $\{\mathbf{x}', \mathbf{x}'', \mathbf{x}'''\}$, should correspond to the highest attainable value of the effective sample size (ESS). Conversely, consider another particle \mathbf{z} such that $f(\mathbf{z}) < M_{\max}$. In this case, the set $\{\mathbf{x}', \mathbf{x}'', \mathbf{z}\}$ should be associated with a smaller ESS value compared to the previous set, $\{\mathbf{x}', \mathbf{x}'', \mathbf{x}'''\}$.

Therefore, given a set of N unnormalized weights $\{w_n\}_{n=1}^N$, in the absence of knowledge about the true maximum M_{\max} , one may use the empirical maximum $w_{\max} = \max w_n$ as a proxy for M_{\max} . Clearly, this approximation

¹Just for simplicity and without loss of generality, we are assuming that the samples \mathbf{x}_n are defined in \mathbb{R}^d .

becomes increasingly accurate as more different particles \mathbf{x}_n are evaluated and as $N \rightarrow \infty$. Finally, note that the normalized weights are obtained from the unnormalized ones according to the following expression:

$$\bar{w}_n = \frac{w_n}{\sum_{i=1}^N w_i} = \frac{w_n}{W}, \quad n = 1, \dots, N,$$

where $W = \sum_{i=1}^N w_i < \infty$ denotes the total sum of the weights. In the rest of this work, we assume the knowledge of only the normalized weights \bar{w}_n , $n = 1, \dots, N$ [8, 19, 21].

3. Properties of a generic ESS function

The ESS formulas proposed in the literature can be interpreted as measures of discrepancy between the probability mass defined by the normalized weights and a uniform probability mass. In order for an ESS expression to be considered suitable, five key conditions have been suggested in the literature [19, 21]. Given the vector of normalized weights $\bar{\mathbf{w}} = [\bar{w}_1, \dots, \bar{w}_N]$, any ESS formula can be written as:

$$\text{ESS}_N(\bar{\mathbf{w}}) = \text{ESS}_N(\bar{w}_1, \dots, \bar{w}_N) : \mathcal{S}_N \rightarrow [1, N], \quad (3)$$

where $\mathcal{S}_N \subset \mathbb{R}^N$ represents the *unit simplex* in \mathbb{R}^N . Namely, the variables $\bar{w}_1, \dots, \bar{w}_N$ are subjected to the following constraint $\sum_{n=1}^N \bar{w}_n = 1$. Denoting

$$\bar{\mathbf{w}}^* = \left[\frac{1}{N}, \dots, \frac{1}{N} \right], \quad (4)$$

and the vertices of the simplex \mathcal{S}_N as

$$\bar{\mathbf{w}}^{(j)} = [\bar{w}_1 = 0, \dots, \bar{w}_j = 1, \dots, \bar{w}_N = 0], \quad (5)$$

i.e., $\bar{w}_j = 1$ and $\bar{w}_n = 0$, for $n \neq j$ with $j \in \{1, \dots, N\}$, below we provide the five conditions that $\text{ESS}_N(\bar{\mathbf{w}})$ should satisfy:

- C1. **Symmetry:** ESS_N must be invariant under any permutation of the weights, i.e.,

$$\text{ESS}_N(\bar{w}_1, \bar{w}_2, \dots, \bar{w}_N) = \text{ESS}_N(\bar{w}_{j_1}, \bar{w}_{j_2}, \dots, \bar{w}_{j_N}), \quad (6)$$

for any possible set of indices $\{j_1, \dots, j_N\} = \{1, \dots, N\}$.

C2. **Max value.** The maximum value is N and it is reached (at least) at $\bar{\mathbf{w}}^*$ (see Eq. (4)), i.e.,

$$\text{ESS}_N(\bar{\mathbf{w}}^*) = N \geq \text{ESS}_N(\bar{\mathbf{w}}). \quad (7)$$

C3. **Min value.** The minimum value is 1 and it is reached (at least) at the vertices $\bar{\mathbf{w}}^{(j)}$ of the unit simplex in Eq. (5),

$$\text{ESS}_N(\bar{\mathbf{w}}^{(j)}) = 1 \leq \text{ESS}_N(\bar{\mathbf{w}}). \quad (8)$$

for all $j \in \{1, \dots, N\}$.

C4. **Unicity of the maximum and minima:** The maximum at $\bar{\mathbf{w}}^*$ is unique and the the minimum value 1 is reached *only* at the vertices $\bar{\mathbf{w}}^{(j)}$, for all $j \in \{1, \dots, N\}$.

C5. **Stability of the rate $\text{ESS}_N(\bar{\mathbf{w}})/N$:** Consider the vector of weights $\bar{\mathbf{w}} \in \mathbb{R}^N$ and the vector $\bar{\mathbf{v}} = [\bar{v}_1, \dots, \bar{v}_{MN}] \in \mathbb{R}^{MN}$, $M \geq 1$, obtained repeating and scaling by $\frac{1}{M}$ the entries of $\bar{\mathbf{w}}$, i.e.,

$$\bar{\mathbf{v}} = \frac{1}{M} \underbrace{[\bar{\mathbf{w}}, \bar{\mathbf{w}}, \dots, \bar{\mathbf{w}}]}_{M\text{-times}}. \quad (9)$$

The invariance condition is expressed as

$$\text{ESS}_N(\bar{\mathbf{w}}) = \frac{1}{M} \text{ESS}_{MN}(\bar{\mathbf{v}}), \quad (10)$$

for all $M \in \mathbb{N}^+$.

The first three conditions are strictly required to be considered a discrepancy measure with respect to the uniform pmf, and having $1 \leq \text{ESS}_N(\bar{\mathbf{w}}) \leq N$. The last condition C5 is related to the examples and the desired results given in Table 3. Given the five conditions above, a classification of the possible ESS functions is provided in Table 1, dividing the possible ESS formulas in different families depending on the conditions satisfied: all ESS formulas which satisfy at least the first four conditions are called *proper* functions. If all the conditions are fulfilled they are called *proper and stable*. From a practical point of view, we are interested proper and stable ESS expressions.

Table 1: Classification of ESS formulas.

Class of ESS	C1	C2	C3	C4	C5
Degenerate	✓	✓	✓	x	x
Proper	✓	✓	✓	✓	x
Degenerate and Stable	✓	✓	✓	x	✓
Proper and Stable	✓	✓	✓	✓	✓

4. Huggins-Roy ESS family

The Huggins-Roy ESS family introduced in [11] is defined as

$$\text{ESS-H}_N^{(\beta)}(\bar{\mathbf{w}}) = \left(\frac{1}{\sum_{n=1}^N \bar{w}_n^\beta} \right)^{\frac{1}{\beta-1}}, \quad (11)$$

$$= \left(\sum_{n=1}^N \bar{w}_n^\beta \right)^{\frac{1}{1-\beta}}, \quad \beta \geq 0. \quad (12)$$

contains all proper and stable ESS functions, except for $\beta \rightarrow 0$ (that is degenerate and stable). The Huggins-Roy formula is also known in ecology as the *Hill number of order β* [13, 19]. Table 3 provides some examples showing as two well-known Huggins-Roy formulas, for $\beta = 2$ and $\beta \rightarrow \infty$, fulfill the condition C5. Moreover, Table 2 shows below that the Huggins-Roy family contains all the most important ESS functions introduced in literature. Other proper and stable ESS functions can be found beyond the Huggins-Roy ESS family (see [19, 21]).

Table 2: Relevant special cases contained in the Huggins-Roy family.

$\beta = 1/2$	$\beta \rightarrow 1$	$\beta = 2$	$\beta \rightarrow \infty$
$\left(\sum_{n=1}^N \sqrt{\bar{w}_n} \right)^2$	$\exp \left(- \sum_{n=1}^N \bar{w}_n \log \bar{w}_n \right)$	$\frac{1}{\sum_{n=1}^N \bar{w}_n^2}$	$\frac{1}{\max \bar{w}_n}$
[21]	[4, 24]	[15]	[1, 21]

Table 3: Examples (with $N = 5$) related to the stability condition C5. In this cases, the well-known formulas $\frac{1}{\sum_{n=1}^N \bar{w}_n^2}$ ($\beta = 2$) and $\frac{1}{\max \bar{w}_n}$ ($\beta \rightarrow \infty$), provide the desired results. Generally, all the formulas in Huggins-Roy family (i.e., for any $\beta > 0$) yield the expected results, since they are all proper and stable.

$\bar{\mathbf{w}}$	$[1, 0, 0, 0, 0]$	$[\frac{1}{2}, \frac{1}{2}, 0, 0, 0]$	$[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0]$	$[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0]$	$[\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}]$
$\frac{1}{\sum_{n=1}^5 \bar{w}_n^2}$	1	2	3	4	5
$\frac{1}{\max \bar{w}_n}$	1	2	3	4	5

5. A suitable particle counter

In the following, we examine three distinct scenarios that serve as illustrative examples. Then, we will provide a general demonstration of a desired property satisfied by the Berger-Parker index.

Scenario 1. Let us consider the following setting with the unnormalized and normalized vectors of weights:

$$\mathbf{w} = [1, 1, 1, 1, 0], \quad \text{which yields} \quad \bar{\mathbf{w}} = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0 \right]. \quad (13)$$

This situation is graphically represented in Figure 1(a). We expect from an ESS measure to give 4 and it is the case for all the proper and stables formulas. Indeed, for instance, we have

$$\text{ESS-H}_N^{(2)}(\bar{\mathbf{w}}) = \frac{1}{\sum_{i=1}^N \bar{w}_i^2} = 4, \quad (14)$$

$$\text{ESS-H}_N^{(\infty)}(\bar{\mathbf{w}}) = \frac{1}{\max \bar{w}_n} = 4, \quad (15)$$

then both formulas perform equally well. This is exactly what we expect from all the formulas satisfying the stability condition C5.

Scenario 2. Now let us consider the scenario where

$$\mathbf{w} = \left[1, 1, 1, \frac{1}{5}, 0 \right], \quad \text{so that} \quad \bar{\mathbf{w}} = \left[\frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{1}{16}, 0 \right]. \quad (16)$$

This scenario is graphically represented in Figure 1(b). From the weight vector \mathbf{w} above (without any additional information), we can state with certainty that the ESS must be smaller than 4 but greater than 3 (just looking the unnormalized weights). Moreover, one particle should be regarded as five times *less significant* (or *smaller in weight*) than the other three particles with non-zero weights. Namely, in the absence of further information, we can assume each of the three particles with non-zero weights represents 100% of an *entire* particle, whereas the particle with unnormalized weight $w = 1/5$ corresponds to 20% of a full particle. Thus, we would expect an ESS of 3.2. Computing the ESS formulas for $\beta = \infty$ and $\beta = 2$ (as an example of alternative), we obtain

$$\text{ESS-H}_N^{(2)}(\bar{\mathbf{w}}) = \frac{1}{\sum_{i=1}^N \bar{w}_i^2} = 3.37, \quad (17)$$

$$\text{ESS-H}_N^{(\infty)}(\bar{\mathbf{w}}) = \frac{1}{\max \bar{w}_n} = 3.2, \quad (18)$$

i.e., only $\text{ESS-H}_N^{(\infty)}(\bar{\mathbf{w}})$ provides the desired result.

Furthermore, since it is possible to prove that [19]

$$\text{ESS-H}_N^{(\beta)}(\bar{\mathbf{w}}) > \text{ESS-H}_N^{(\infty)}(\bar{\mathbf{w}}), \quad 0 < \beta < \infty, \quad (19)$$

hence the Berger-Parker index $\text{ESS-H}_N^{(\infty)}(\bar{\mathbf{w}})$ is the unique ESS formula within Huggins-Roys family that is actually a particle counter. It is important to remark that, in this second example, there is no logical justification, without any extra information, for considering an ESS value greater than 3.2.

Conceptually, the true ESS value (independent of any specific theoretical definition) may be smaller if the true maximum weight M_{\max} is underestimated, that is, when $w_{\max} < M_{\max}$. Even in this situation, the Berger-Parker index $\text{ESS-H}_N^{(\infty)}(\bar{\mathbf{w}})$ remains the most appropriate choice, as it provides a lower bound for the entire Huggins-Roy family, as demonstrated by inequality (19). It is also worth noting that, by definition of the true maximum M_{\max} , this value cannot be overestimated by its empirical counterpart $w_{\max} = \max_n w_n \leq M_{\max}$.

Scenario 3. Now let us consider the scenario where

$$\mathbf{w} = \left[1, 1, 1, \frac{1}{2}, \frac{1}{2} \right], \quad \text{so that} \quad \bar{\mathbf{w}} = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8} \right]. \quad (20)$$

This context is graphically represented in Figure 1(c). For the same arguments, without any additional information, we would expect $ESS = 4$, since two particles seems to be as “half particles” of an entire one. Computing the two ESS measures for $\beta = \infty$ and $\beta = 2$ (as an example of alternative)

$$ESS-H_N^{(2)}(\bar{\mathbf{w}}) = \frac{1}{\sum_{i=1}^N \bar{w}_i^2} = 4.57, \quad (21)$$

$$ESS-H_N^{(\infty)}(\bar{\mathbf{w}}) = \frac{1}{\max \bar{w}_n} = 4, \quad (22)$$

and again the Berger-Parker index provides the desired result. The increase in ESS produced by other Huggins-Roy formulations (such as an excess of 0.57 for $\beta = 2$) lacks justification in the absence of additional information. Other examples are given in Table 4.

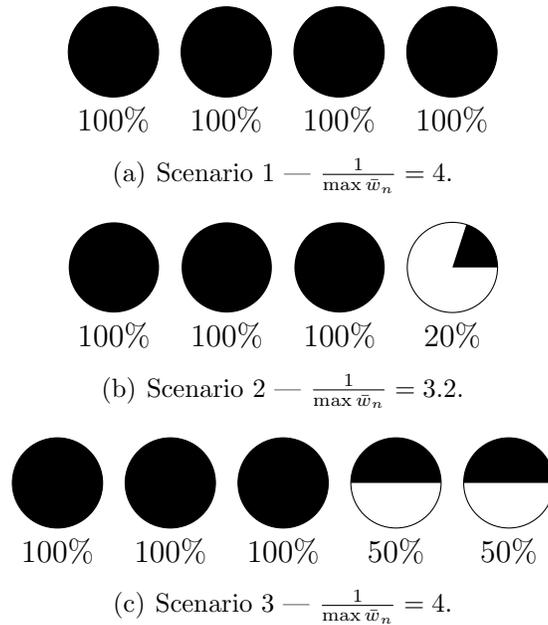


Figure 1: Illustration of normalized weights in the three different scenarios. The full-particles are the samples with assigned the maximum weight.

6. A general demonstration

It could be argued that the examples presented above were deliberately selected to produce outcomes favorable to the Berger-Parker index. However,

Table 4: Examples (with $N = 5$). We can observe that the Berger-Parker index $\frac{1}{\max \bar{w}_n}$ provides the expected results according to the particle percents (with respect to the maximum value of the weights). Whereas the other formula $\frac{1}{\sum_{n=1}^5 \bar{w}_n^2}$ returns always an excess of ESS measure.

$\bar{\mathbf{w}}$	$[\frac{5}{11}, \frac{5}{11}, \frac{1}{11}, 0, 0]$	$[\frac{1}{2}, \frac{2}{5}, \frac{1}{20}, \frac{1}{20}, 0]$ $= [\frac{10}{20}, \frac{8}{20}, \frac{1}{20}, \frac{1}{20}, 0]$	$[\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}]$ $= [\frac{2}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}]$
Particle percent (w.r.t. max)	100%, 100%, 20%, 0%, 0%	100%, 80%, 10%, 10%, 0%	100%, 50%, 50%, 50%, 50%
$\frac{1}{\sum_{n=1}^5 \bar{w}_n^2}$	2.3725	2.4096	4.5
$\frac{1}{\max \bar{w}_n}$	2.2	2	3

this is not the case. Below, we provide a formal demonstration showing that the results above naturally arise from an intrinsic property of the Berger-Parker index.

Proof. Let define $w_{\max} = \max_n w_n$, where w_n are the unnormalized weights. Then, we can write

$$\text{ESS-H}_N^{(\infty)}(\bar{\mathbf{w}}) = \frac{1}{\max \bar{w}_n} = \frac{\sum_{i=1}^N w_i}{\max w_n} \quad (23)$$

$$= \sum_{i=1}^N \frac{w_i}{w_{\max}}, \quad (24)$$

$$= \sum_{i=1}^N \tilde{w}_i, \quad (25)$$

where we have defined the *relative weights*, $\tilde{w}_i = \frac{w_i}{w_{\max}} \in [0, 1]$. Thus, the Berger-Parker index is the sum of the relative weights of each particle. Each relative weight represents proportion with respect to the maximum (unnormalized) weight w_{\max} .

Another interesting observation is that $\text{ESS-H}_N^{(\infty)}(\bar{\mathbf{w}}) = \frac{W}{w_{\max}}$ where $W = \sum_{i=1}^N w_i$. The total sum of the weights W , or its scaled version $\frac{1}{N}W$, is used for the marginal likelihood approximation and in distributed/parallel schemes [2, 3, 20, 23, 22]. However, note that the computation of $\text{ESS-H}_N^{(\infty)}(\bar{\mathbf{w}}) = \frac{1}{\max w_n}$ does not require the knowledge of the unnormalized weights.

7. Conclusions

In this work, we have shown that the Berger-Parker index is the only effective sample size measure that accurately preserves the relative contributions of individual weights, acting as a true particle counter. In contrast, other ESS measures tend to systematically overestimate sample representativeness, which can lead to biased or misleading interpretations in practical applications. Through several illustrative examples and formal demonstrations, we have highlighted the robustness, interpretability, and practical relevance of the Berger-Parker index across diverse domains, including computational statistics, ecology, economics, and physics.

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