

A Primitive Notion Versus the non-Archimedean Continuum: Introducing Homogeneous Infinitesimals

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Introduction

The Archimedean axiom¹, often written as

$$na > b, \tag{1}$$

is a fundamental algebraic property where a multiple n of a is said to exceed b . However, this property is also said not to apply to infinitesimals (i.e. dx) and thus they are “non-Archimedean”. This has been said to change the inequality so that it is sometimes described using²

$$ndx \leq 1 \tag{2}$$

or stated as “the sum (of infinitesimals) is always infinitely small” [BBK+22]. If infinitesimals are still desired then they “enrich” real numbers and this is sometimes called a “non-Archimedean continuum”. In this paper I dispute Equation 2 via a new homogeneous concept of infinitesimals where their magnitudes and cardinality are only defined relative to each other. This will give us a direct counterargument of

$$n_a dx_a \leq n_b dx_b \tag{3}$$

where this equation defines *finite length A* relative to *B*, written as

$$A = n_a dx_a \leq n_b dx_b = B. \tag{4}$$

Here n is transfinite cardinality and dx is a new primitive notion of infinitesimal length. Using this argument and primitive notions of infinitesimal length,

area and volume I can write

$$n_a dx_1 = \text{homogeneous finite length}, \tag{5}$$

$$n_a(dx_1 dx_2) = \text{homogeneous finite area}, \tag{6}$$

and

$$n_a(dx_1 dx_2 dx_3) = \text{homogeneous finite volume}, \tag{7}$$

(and so forth) allowing new conceptual insights³.

I can now prove by contradiction, through an example of Evangelista Torricelli⁴, that Leibniz’s transcendental law of homogeneity [K+13] is fundamentally flawed due to not realizing this. Simply put, Leibniz’s technique gives an incorrect answer since he did not realize that area must be homogeneous (meaning that finite area must be composed of a transfinite number of elements of area) when using infinitesimals. His equation, written as ([Kle01] p. 147)

$$\Delta \text{area} = d(xy) = xdy + ydx + dxdy, \tag{8}$$

gives the incorrect third term $dxdy$ whereas this concept gives only the correct two,

$$\begin{aligned} \Delta \text{homogeneous area} &= d(n_a(dxdy)) \\ &= n_x dx(dy) + n_y dy(dx). \end{aligned} \tag{9}$$

This concept also explains the distinction of why both relative magnitudes such as dx and cardinality n is

³It may be helpful to think of infinitesimals as either having no shape or can conform to any shape.

⁴This example is well known enough to be the very first figure of [Ger16], a review within *AMS Notices* of a book that explores the history of infinitesimals

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¹see Equation 2.1 in [BBE+13]

²from Equation 4 in [BBE+13]

required to describe the relationship between finite measures. We shouldn't be counting the number of points, we should instead be counting the relative number of elements of length, area and volume. Whereas modern mathematics can not get past ⁵

$$(\exists dx > 0)(\forall n \in \mathbb{N})[dx \leq \frac{1}{n}], \quad (10)$$

this paper will demonstrate particular solutions that for the set (note that we use $dx_2 = dy$ etc.)

$$S = \{ \{dx_1, dx_2, dx_m\}_1, \{dx_1, dx_2, dx_m\}_2, \{dx_1, dx_2, dx_m\}_n \} \quad (11)$$

the

$$\sum_{dx_m \in S}^n dx_m \quad (12)$$

equals length for $m = 1$, area for $m = 2$, volume for $m = 3$ and so forth. The n (relative cardinality) is required since for example if all the elements are equal then the sum with the larger n is the greater length, area, volume etc.

1 Background

Infinitesimals, no longer in vogue in mathematics due to the success of real analysis (and in spite of non-standard analysis[Rob74]), have always had some paradoxes associated with them. Some might say they have been “purged”[HT15] and “banished”[Kei12] in part due to these unanswered questions. One of the original paradoxes was the homogeneous/heterogeneous viewpoint of the 17th century. This argument centered on whether geometrical sums were made of elements of the same dimension (homogeneous) or one less (heterogeneous). The simplest example was whether lines were made of infinitesimal one-dimensional segments of lines (homogeneous ([J+15] p.4)) or made of non-dimensional points (heterogeneous)⁶. In the same vein, it was also debated whether area would be composed of infinitesimally

⁵see Equation 2.4 in [BBE+13]

⁶Note that this in contrast to the philosophical view that a line exists and a point “lies on” that line.

thin slices of area versus one dimensional stacked lines and whether volume was made of infinitesimally thin sheets of volume versus stacked two-dimensional planes. Evangelista Torricelli (1608-1647), a brilliant scientist and inventor in his own right and well known to Galileo, is known in these debates for his talent at taking a difficult concept and explaining it in many different ways. This has been said to have enabled the transfer of fundamental concepts more so than the voluminous writings of Cavalieri. Torricelli's analysis of the heterogeneous/homogeneous debate [DG87] landed him firmly on the infinitesimal segment side as recent authors have pointed out([Ale14] and [J+15] p. 125).

Francois De Gandt, recognized as a leading expert on Torricelli's work, has called one particular example of Torricelli's a “condensed” “fundamental example” for the Italian's view on the heterogeneous/homogeneous paradox([DG87] p.164). Let us take his example and start off presenting the historical viewpoint so that we have a firm footing for the homogeneous/heterogeneous argument from which we can develop a proof by contradiction.

Historical Viewpoint of Torricelli's Parallelogram Paradox

We could initially recreate the historical explanation for Torricelli's parallelogram involving lines of non-zero infinitesimal width but let's first consider the simplest observation, how to determine the number of points in a line.

Instead of a line having points on it, assume that a line is made of points and that the number of points in a line determine the length. Two lines that are of the same length have the same number of points. A shorter line has less points and a longer line has more points.

Now take a parallelogram with the four corner points labeled A,B,C, and D as shown in Figure 1. Draw a line BD down the diagonal of it. Let us make a point E on the diagonal line BD. Now draw perpendicular lines from E to a point F on AD, and a second line to a point G on CD. Move these two lines point

by point simultaneously so that E moves toward B until they meet, keeping the lines EF and EG always parallel to AB and BC respectively. When we move the lines EF and EG, we are moving their ends simultaneously from point to point on AD, CD and BD.

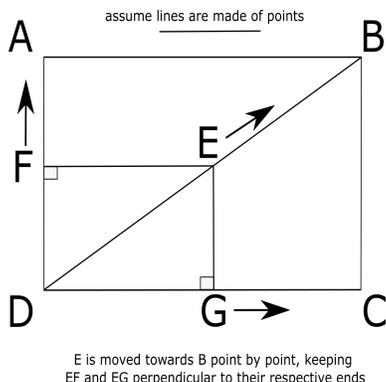


Figure 1: Torricelli: moving perpendicular lines point by point

If line AD is shorter than the line CD, the number of points that line AD contains is less than the number of points that line CD contains. However, this creates a paradox. Since we moved the lines point by point and with both points F and G ending up together at point B then this shows that lines AD and CD must also have the same number of points as shown in the equations in Figure 2.

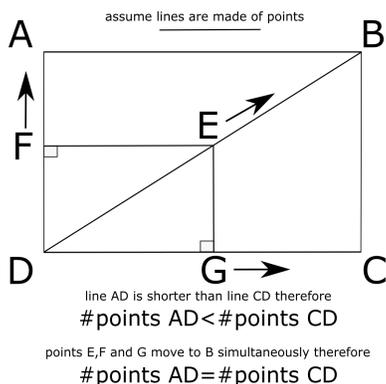


Figure 2: Torricelli: number of points equal or not?

The simplest interpretation of Torricelli’s meaning is that the lines AD and CD must be made up of infinitesimal segments (and not dimensionless points) and that these segments must consist of the same number in each line even if they are not of the same magnitude.

The current most advanced explanation ([J+15] p. 125) for this paradox is the “understanding that the amount of elements of an infinite set, that is its cardinality, is different from its measure.” While I very much agree with these properties, let us introduce a new tool to gain more insight.

2 Axiom of N-M Choice

First let us define a common starting paradigm. Imagine that you could have a single one-dimensional finite line segment and it is itself composed solely of “adjacent” “infinitesimal” line segments SEG and that the sum of their magnitudes $|SEG|$ is defined as the “length” of that finite line segment. This is very similar to the original geometric concept in Calculus of taking a singular line segment and dividing it into smaller and smaller segments an “infinite” number of times until these segments of length become “infinitesimals” (but of course their sum is still the original finite length). We define that the magnitude of the infinitesimals of this line is in agreement with Eudoxus’ theory of proportions in that their *magnitudes are only measurable relative to another infinitesimal* (see equations 14 and 15). Thus their sums, finite line lengths, are also in agreement with Bernhard Riemann’s definition [Rie73] that the length of every line is “measurable by every other line” (although his definition does not include any mention of the infinitesimals of which we are considering), and we can write

$$\sum |SEG| \equiv \text{line length.} \quad (13)$$

In the interest of utilizing a graphical teaching method let us use Figure 3 to represent this line segment.

Suppose that the magnitude $|SEG^n|$ of a segment n could either be of equal relative magnitude (as in Equation 14) to another segment $n - 1$ or could have



Figure 3: Graphical segments of length representing magnitudes of infinitesimals of length: Intrinsically flat

a different value (as in Equation 15) *even within the line itself*. Let us call this concept, defined via

$$|SEG^n| - |SEG^{n-1}| = 0, \quad (14)$$

intrinsically flat.

Let us define a “point” as simply an infinitesimal SEG that is of null magnitude in the direction along the line so that we can also understand that Euclid’s definition of a straight line (Euclid’s Elements, Book I, Definition 4) is one that “lies evenly with the points upon itself” and in this case both terms are equal or even (with a point between the two segments and at their respective ends). We could then propose that infinitesimals can have non-equal relative magnitude such and use

$$|SEG^n| - |SEG^{n-1}| \neq 0 \quad (15)$$

to describe this so that the points would no longer be equally spaced as represented by Figure 4. Let us call this *intrinsically curved*.



Figure 4: Graphical segments of unequal length representing unequal infinitesimal magnitudes: Intrinsically curved.

Now also suppose that we can examine the number of segments in one line versus another and we call this number the *relative cardinality* (n) so that we can write (in the simplifying case that the line is intrinsically flat) the equation

$$\sum |SEG| = n * |SEG| \equiv \text{line length}. \quad (16)$$

Let us assume that n has similar properties to Cantor’s transfinite numbers[Jou55], in that n is an “infinite” cardinal number, but that cardinality can be

greater than, equal to or less than the value of another n . Let us now rename $|SEG|$ to dx ,

$$dx \equiv |SEG| \quad (17)$$

so that we have

$$\sum_1^n |SEG| = n * dx \equiv \text{line length}. \quad (18)$$

This is where we arrive at the discrepancy with the concept of infinitesimals being non-Archimedean. I am not aware of any research which has considered the ramifications of letting n be transfinite. By this I mean the following equations can be proven to be logically true for mapping real numbers onto sums of homogeneous infinitesimals. This means we can write

$$n_a dx_a \leq n_b dx_b \quad (19)$$

which we will assume for now can be used to represent the real number equations

$$5 = 5 \quad (20)$$

or

$$4.5 < 9. \quad (21)$$

It is important to understand that if we instead wrote equation 19 as

$$A = n_a dx_a, \quad (22)$$

$$B = n_b dx_b, \quad (23)$$

$$A \leq B, \quad (24)$$

then this would mask the relativity of their constituent infinitesimals which I will call the *Axiom of N-M Choice* (ANMC). In words, what this axiom states is that there is an inherent choice made when two sums are compared and that is the relative choice of the Number n of elements and the Magnitude M of those elements. Note that I state *elements* since there is not only the primitive notion (see Section 4) of “infinitesimal length” but also “infinitesimal area”, “infinitesimal volume” and so forth.

What is commonly considered to be a line⁷ is defined as the sum of infinitesimal elements of length, area is defined as the sum of infinitesimal elements of area, volume is defined as the sum of infinitesimal elements of volume, etc. The sums (length, area, volume) are homogeneous in that their composition is strictly made of elements of the same “dimension”. Area cannot be composed of elements of length as will be logically proven from the primitive notions and postulates. This is what defines a sum as being “homogeneous”. Thus we have

$$n_a dx_a \leq n_b dx_b \quad (25)$$

for sums of elements of length (there are n_a elements of length on the left and n_b elements of length on the right),

$$n_a(dx_{1a}dx_{2a}) \leq n_b(dx_{1b}dx_{2b}) \quad (26)$$

for sums of elements of area and

$$n_a(dx_{1a}dx_{2a}dx_{3a}) \leq n_b(dx_{1b}dx_{2b}dx_{3b}). \quad (27)$$

for sums of volumes (there are n_a elements of volume on the left and n_b elements of volume on the right). If the elements are all of the same magnitude, then the sum with the most elements has the longer length, larger area, larger volume etc. I will limit ourselves to three terms for now (due to the nature of the problem at hand), but by no means is this limited to only three. Note that in each case, n_a represents the total number of elements and not the constituent elemental directional magnitudes dx . Thus we can write for set

$$S = \{ \{dx_1, dx_2, dx_m\}_1, \{dx_1, dx_2, dx_m\}_2, \{dx_1, dx_2, dx_m\}_n \} \quad (28)$$

the

$$\sum_{dx_m \in S}^n dx_m \quad (29)$$

equals length for $m = 1$, area for $m = 2$, volume for $m = 3$ and so forth. The n is required since for

⁷I state “commonly” because there are “lineal” lines, “areal” lines, “voluminal” lines, etc. I propose to redefine n-forms and Calculus using these but I do not expand upon it further since this is an introductory paper.

example if all the elements are equal then the sum with the larger n is the greater length, area, volume etc.

For a square we could write

$$n_a(dx_{1a}dx_{2a}) = (n_{1a}dx_{1a})(n_{2a}dx_{2a}) = (n_{1a2a}dx_a)^2 \quad (30)$$

since

$$n_a = (n_{1a})(n_{2a}) = (n_{1a2a})^2. \quad (31)$$

The two terms $(n_{1a2a}dx_a)$ would represent the length of the sides of the square since

$$(n_{1a}dx_{1a}) = (n_{2a}dx_{2a}). \quad (32)$$

I have named this overall concept and the resulting philosophy the *Calculus, Philosophy and Notation of Axiomatic Homogeneous Infinitesimals* (CPNAHI). Although it is a mouthful, CPNAHI seems to adequately cover the breadth of the concept. Also note that I do not use standard mathematical notation such as R^n for “real space” since there appears to be a conceptual distinction between CPNAHI and real analysis that I have yet to explore.

2.1 Definition of Lines

Allow me to briefly define concepts called lineal lines, areal lines, voluminal lines etc. A lineal line is defined as a path of adjacent infinitesimal elements of length and a lineal line point is just a null infinitesimal (0 dimensions). An areal line is defined as a path of adjacent infinitesimal elements of area. An areal line point is defined as an element of area that is null perpendicular to the path (1 dimension). A voluminal line is defined as a path of adjacent infinitesimal elements of volume. A voluminal line point is an element that is null perpendicular to the path (2 dimensions). This understanding allows us a new method to examine Evangelista Torricelli’s parallelogram paradox([Ale14] and [J⁺15]).

2.2 Analysis of Lines in Torricelli's Parallelogram Using the ANMC

2.2.1 Intrinsically Flat Lineal Lines Curved With Respect To Another Line

Using the ANMC we can assign equations as a description of Torricelli's parallelogram. Since Torricelli's example states that lines BD, CD and AD are each traversed one segment at a time, then the number of segments in each can be thought of as equal on a one to one basis and we can write

$$n_{BD} = n_{CD} = n_{AD}. \quad (33)$$

Since the segments in any line are all equal, then all adjacent segments in each line are equal and we can write

$$dx_{BD}^n - dx_{BD}^{n-1} = 0 \quad (34)$$

$$dx_{CD}^n - dx_{CD}^{n-1} = 0 \quad (35)$$

$$dx_{AD}^n - dx_{AD}^{n-1} = 0. \quad (36)$$

Since the length of line BD is greater than lines CD which is greater than line AD we can observe that the magnitudes of the infinitesimal segments of each line in relation to each other can be written as

$$dx_{BD} > dx_{CD} > dx_{AD}. \quad (37)$$

Thus the segments of line BD have a magnitude that is *relatively* larger than CD and similarly for AD.

Figure 5 is a visual aid for understanding the previous equations. If by the property of congruence we can lay the lines BD, CD and AD next to each other and they are of unequal length, let us then imagine that we can use the vertical dividing lines to help denote the infinitesimal segments within each line. Torricelli's example is represented so that we can understand that the magnitudes of the segments within BD, CD and AD are all the same (intrinsically flat) within their respective lines. However, the magnitudes of the segments within BD must not be the same as CD, nor AD. Again, this is what Torricelli meant when he said that points are indistinguishable whereas segments can differ by their magnitude. They are intrinsically curved relative to each

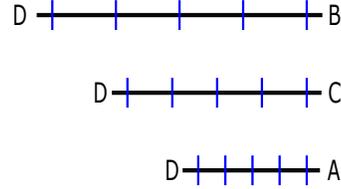
other. We can then also understand that the cardinality within AD must be the same as BD as well as for CD which was chosen when we opted to move the perpendicular lines "point by point". Thus for example line *BD* would be written as

$$S_{BD} = \{\{dx_1\}_1, \{dx_1\}_2, \{dx_1\}_n\} \quad (38)$$

and comparing the sum of *BD* versus *AD* would give

$$\sum_{dx_1 \in S_{BD}}^{n_{BD}} dx_1 > \sum_{dx_1 \in S_{AD}}^{n_{AD}} dx_1. \quad (39)$$

Note that the dx_1 in each of the sums are not the same magnitude as shown in Equation 37.



- Same # of segments within lines BD, CD and AD
- segment magnitude equivalent within each line
- segment magnitude differs between each line
- each line is intrinsically flat

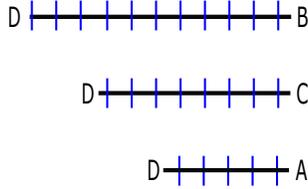
Figure 5: Intrinsically Flat Lineal Lines With Equal Cardinality And Relative Curvature

2.2.2 Intrinsically Flat Lineal Lines With Relative Flatness

Imagine now that we have again taken line BD, CD and AD and laid them next to each. We again traverse the segments on the line one for one but this time we do so in equal magnitude increments as shown in Figure 6. Line BD is still longer but it is longer since the relative cardinality (number of segments) is the greatest as in Figure 6.

We can then write

$$n_{BD} > n_{CD} > n_{AD} \quad (40)$$



- Differing # of segments within lines BD, CD and AD
- segment magnitude equivalent within each line
- segment magnitude the same between each line
- each line is intrinsically flat

Figure 6: Intrinsically Flat Lines With Differing Cardinality And Relative Flatness

$$dx_{BD}^n - dx_{BD}^{n-1} = 0 \quad (41)$$

$$dx_{CD}^n - dx_{CD}^{n-1} = 0 \quad (42)$$

$$dx_{AD}^n - dx_{AD}^{n-1} = 0 \quad (43)$$

$$dx_{BD} = dx_{CD} = dx_{AD} \quad (44)$$

2.2.3 Relative Curvature and Flatness via the ANMC

From these two simple examples we can observe the axiom in that we have the choice to make the inequalities

$$n_{AD} * dx_{AD} < n_{CD} * dx_{CD} < n_{BD} * dx_{BD} \quad (45)$$

true by either make n all equal and vary dx or vice versa. This is the same meaning using summation notation (and just using the subscript for their respective lines),

$$\sum_{dx_{AD} \in S_{AD}}^{n_{AD}} dx_{AD} < \sum_{dx_{CD} \in S_{CD}}^{n_{CD}} dx_{CD} < \sum_{dx_{BD} \in S_{BD}}^{n_{BD}} dx_{BD}. \quad (46)$$

2.3 The Defining of Scale Factors: Sum of Cardinality vs Sums of HIs

Notation provides economy of thought but that notation can be of a bad value if it mischaracterizes the

underlying geometry it represents. Let's see what that means by examining sums of infinitesimals and defining scale factors.

2.3.1 Sums of Infinitesimals

If line BD is twice as long as AD then from Figure 5 we could view line BD as the summing of two lines AD where the magnitudes of each element in one AD was summed with the magnitude of a corresponding element in the other AD ($dx_{AD} + dx_{AD} = dx_{BD}$ element by element). From Figure 6 we could view line BD as the summing of two lines AD where the cardinality of one line AD was summed with the cardinality of the other line AD creating a longer line, BD, with larger cardinality ($n_{AD} + n_{AD} = n_{BD}$).

Simply put, this just means we have two choices here for summing of line lengths. Sum either their elements so that the cardinality stays the same but each element is bigger or keep the elements the same magnitude and double their number.

2.3.2 Euclidean Scale Factor

Now let's consider the common geometric conception that if $A = 4$ is written as the representation of the length of a line, then it is said that a scale factor $k = 3$ would give

$$kA = 12. \quad (47)$$

One could say that if we have another line $B = 12$ then k could be defined as

$$k = \frac{B}{A} \quad (48)$$

as can be seen in Figure 7. However, this hides some-

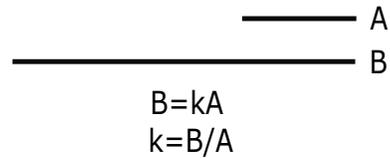


Figure 7: Common understanding of scale factor for simple one-dimensional line

thing geometrical that can be fleshed out with the ANMC.

2.3.3 Euclidean Scale Factor Defined As A Quotient of Relative Cardinalities

Let us take the equation 25 and relabel it so that I am comparing the length of a line to a “reference” line,

$$n_a dx_a \leq n_{ref} dx_{ref}. \quad (49)$$

If

$$n_a dx_a = 3 = A \quad (50)$$

and

$$n_{ref} dx_{ref} = 12 = B \quad (51)$$

and

$$dx_a = dx_{ref}. \quad (52)$$

then we can define a *relative cardinality* scale factor k as

$$k_{RC} = \frac{n_{ref}}{n_a} = 3. \quad (53)$$

Thus, writing

$$(k_{RC} n_a) dx_a = 12 = B \quad (54)$$

means that line A has $\frac{1}{3}$ the cardinal number of elements of magnitude as the referenced line B as shown in Figure 8. Line B is scaled 3 times the length of line A.



Figure 8: Scaling of relative cardinality

2.3.4 Scale Factor Defined As A Quotient of Relative Magnitudes

The opposing case allowed by the ANMC here is scaling of magnitude dx . Again, if I have the equation 25 and relabel it so that I am comparing the length of a line to a “reference” line,

$$n_a dx_a \leq n_{ref} dx_{ref}. \quad (55)$$

If

$$n_a dx_a = 3 = A \quad (56)$$

and

$$n_{ref} dx_{ref} = 12 = B \quad (57)$$

but instead this time

$$n_a = n_{ref}. \quad (58)$$

then I can define a *relative magnitude* scale factor j_{RM} as

$$j_{RM} = \frac{dx_{ref}}{dx_a} = 3. \quad (59)$$

Thus, writing

$$n_a (j_{RM} dx_a) = 12 = B \quad (60)$$

means that line A has infinitesimal elements that $\frac{1}{3}$ the magnitude as the referenced line B as shown in Figure 9. Again, line B is scaled 3 times the length of line A.



Figure 9: Scaling of relative cardinality

2.3.5 ANMC and Area

Before we analyze Torricelli’s parallelogram further, let us understand the ramifications of scale factors with area. As with Figures 5 and 6 hopefully it is obvious we can sum together numbers of elements of length to create a longer line or sum together the magnitudes of elements of length to create a longer line.

In the same vein there are two methods for summing together two areas, either by their cardinality of elements or by the magnitude of the elements (or some combination of both) as shown in Figure 10 (note that the figures are not exact cardinality wise as this is an inherent flaw in using graphical proofs for homogeneous infinitesimals). For the top sum of area this is a sum of cardinalities and we could write

$$n_a(dx_1 dx_2) + n_b(dx_1 dx_2) = (n_a + n_b)(dx_1 dx_2). \quad (61)$$

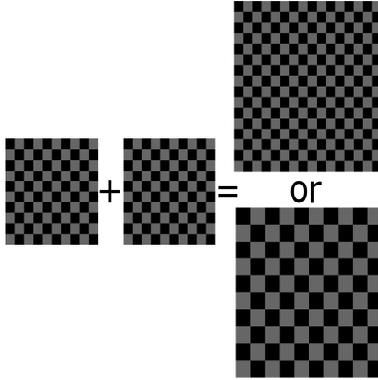


Figure 10: Summation of areas either via cardinality or magnitude

The bottom sum of area is a sum of magnitudes and could be written as

$$n(dx_{a1}dx_{a2}) + n(dx_{b1}dx_{b2}) = n(dx_{a1}dx_{a2} + dx_{b1}dx_{b2}). \quad (62)$$

Note that summation notation

$$\sum_{dx_1, dx_2 \in S}^n dx_1, dx_2 \quad (63)$$

doesn't really give us a good way to annotate whether we are summing the magnitudes of elements of area or their cardinality so we will stick with the previous form in order to flesh out a difference with Leibniz's notation further in the paper.

As with just elements of length, we could view this as scaling of either the cardinality of elements of area or of their magnitudes such that we could write

$$kn(dx_1dx_2) = (n_1 + n_2 + n_k)(dx_1dx_2) \quad (64)$$

using k to scale the cardinal number of the elements or j in

$$n(j(dx_{a1}dx_{a2})) = n((dx_{a1}dx_{a2})_1 + (dx_{b1}dx_{b2})_2 + (dx_{b1}dx_{b2})_j) \quad (65)$$

to scale the magnitudes of each of the elements of area.

2.4 Analysis of the Area of Torricelli's Parallelogram Using ANMC

Now that we have an understanding of area, let's take Torricelli's parallelogram, remove some of the notation and split up the opposing triangles as in Figure 11. We have left point E so that we still have some visual reference to the diagonal line in the parallelogram. Without proof, assume that $Area1 = Area2$ and that $Area3 = Area4$.

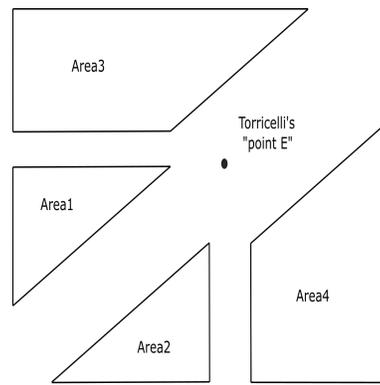


Figure 11: Torricelli's parallelogram divided up with areas labeled

We might mentally understand that as E "moves" toward the top right, that the area in $Area3$ and $Area4$ decreases while the area in $Area1$ and $Area2$ increases. We might also understand that the previous equalities still hold in that $Area1 = Area2$ and that $Area3 = Area4$. Let's add a concept from ANMC to our graphic to help us understand why this is.

In Figure 12 We can obtain the concept that infinitesimal slices of area are being moved from $Area3$ to $Area1$ and from $Area4$ to $Area2$. Since the magnitude relationships between the top and bottom areas are constant, then these slices of area in the top and bottom must be equal also.

Let's now rotate the top areas as a triangle so that the slices of area for both are vertical in our graphic as in Figure 13. If we view these slice of area instead as columns of elements of area, then we can draw Figure 14.

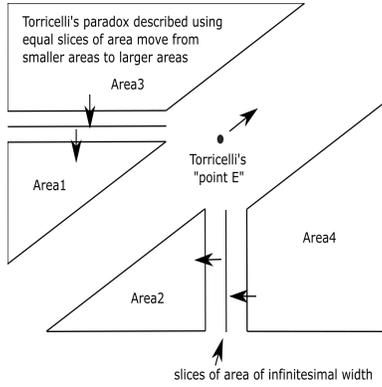


Figure 12: Torricelli's parallelogram with area being removed from Area1 to Area3 and from Area2 to Area4 via slices of area

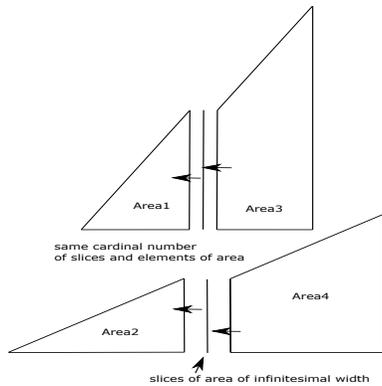


Figure 13: Torricelli's parallelogram with area being removed from Area1 to Area3 and from Area2 to Area4 via vertical slices of area

Let me point out in the graphic there are three elements of area shown in the column (and not six elements of length) to conceptually represent the cardinality. We can also see that the elements of area in the top triangle have the same "height" as the "width" of the elements in the bottom triangle and vice versa. We can also see that the top and the bottoms have the same cardinal number of slices (and we can assume number of elements). If we are not averse to counting the number of elements along the bottom of each triangle, we can see that they too have the

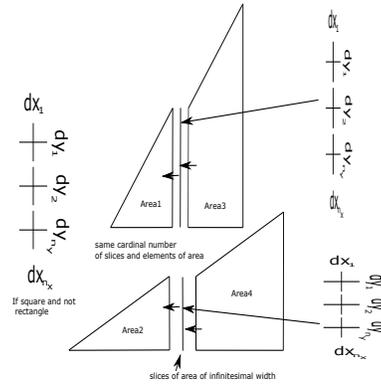


Figure 14: Torricelli's parallelogram with area being removed from Area1 to Area3 and from Area2 to Area4 via columns of vertical infinitesimal elements of equal area

same cardinal number.

It will be helpful to represent these columns of area with Figure 15 where we have chosen the real number values to represent the lengths of the sides and bottoms of the top and bottom triangle. Note that we can write that each column of area in the top is equal to each column of area in the bottom,

$$(n_{y_{top}} dy_{top}) dx_{top} = (n_{y_{bot}} dy_{bot}) dx_{bot}. \quad (66)$$

The height of the labeled top column is

$$n_{y_{top}} dy_{top} = 1 \quad (67)$$

and the height of the labeled bottom column is

$$n_{y_{bot}} dy_{bot} = \frac{1}{2}. \quad (68)$$

The bottom column is twice as "thick" as the top column and the bottom column is half the height of the top column since they possess equal infinitesimal areas. *It is important to note that we are measuring the height of a column of elements of area and not the length of a one-dimensional line! It is also important to note that each element is intrinsically curved within itself and not flat (the dx and the dy are not of equal magnitude).*

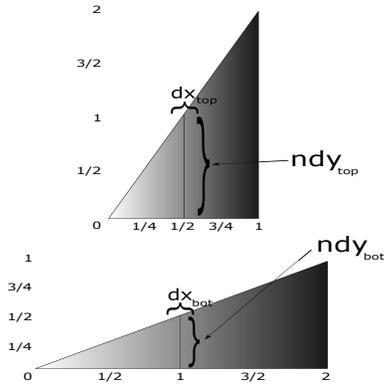


Figure 15: Torricelli's parallelogram composed of columns of elements of area

2.4.1 Comparison of Torricelli ANMC Solution With Leibniz Area Solution

We can now finally go back and consider Leibniz's transcendental law of homogeneity. His example is essentially the same as Torricelli's if we think of both of them as asking "What is a logically and notationally correct method to find an infinitesimal change in area?" Along with Torricelli's example (Figure 16) we can use a common illustration of the product rule [BBE+13] (Figure 17). Each is fundamentally concerned with two strips of area. They are of infinitesimal but non-zero thickness, one along the top and one along the right.

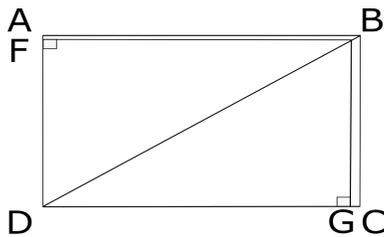


Figure 16: Torricelli's parallelogram setup to examine top and right slices of area

Essentially, for Leibniz's example, let

$$XY = \text{area} \quad (69)$$

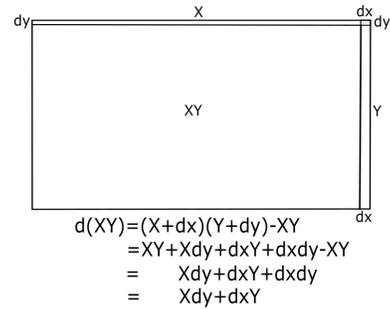


Figure 17: Leibniz's transcendental law of homogeneity from the product rule. Setup as the change in area of a rectangle to examine the top and right slices of area

so that

$$d(XY) = \text{infinitesimal change in area.} \quad (70)$$

Leibniz adds infinitesimals dx and dy to this notation for X and Y so that

$$d(XY) = (X + dx)(Y + dy) - XY \quad (71)$$

denotes a change in area of XY . From treating his terms algebraically he ends up with

$$d(XY) = XY + Xdy + dxY + dx dy - XY \quad (72)$$

which gives

$$d(XY) = Xdy + dxY + dx dy. \quad (73)$$

Leibniz drops the last term ($dx dy$) to make the equation

$$d(XY) = Xdy + dxY = Xdy + Ydx. \quad (74)$$

One author's paraphrasing of Leibniz's justification [Bos74] is

A quantity which is infinitely small with respect to another quantity can be neglected if compared with that quantity. Thus all terms in an equation except those of the highest order of infinity, or the lowest order of infinite smallness, can be discarded.

In other words for our example, $dx dy$ is of a “lower order” than $X dy$ and $Y dx$. This is an important conceptual distinction from CPNAHI in that Leibniz does not seem that he views $X dy$ as a row of elements of area nor does he view the extraneous $dx dy$ as an element of area.

With an eye on not making this introductory paper longer than needed to achieve it’s goals, let us take Equation 66 and claim that we can equate these to the top and right slices of area in Leibniz’s example,

$$n_{y_{top}} dy_{top} = X, \quad (75)$$

$$n_{y_{bot}} dy_{bot} = Y, \quad (76)$$

$$dx_{top} = dx, \quad (77)$$

and

$$dy_{bot} = dy. \quad (78)$$

Let us note here for now that there is no extraneous $dx dy$ for our example. I leave the proof of this to another paper.

3 Conclusion

With CPNAHI and the ANMC we have demonstrated the concept that a relative number of elements and their relative magnitudes are required to quantitatively describe finite measures using infinitesimals. Without these two sides of the same coin of measurement and that we can have not only different sizes of infinity but different sizes of infinitesimals, they may remain paradoxical.

4 Primitive Notion and Postulates

4.1 CPNAHI Primitive Notion

Let a homogeneous infinitesimal (HI) be a primitive notion.

4.1.1 CPNAHI Postulates

1. Postulate of Homogeneity: Homogeneous Infinitesimals (HIs) can have the property of direction with magnitude which gives length for one direction, area for two, volume for three etc. Only HIs of length can sum to create lines. Only HIs of area can sum to create area. Only HIs of volume can sum to create volume. etc..⁸.
2. HIs conform to the boundaries of any shape.
3. HIs can be adjacent or non-adjacent to other HIs.
4. A set of HIs can be a closed set.
5. A lineal line is defined as a closed set of adjacent HIs (path) with the property of length. These HIs have one direction.
6. An areal line is defined as a closed set of adjacent HIs (path) with the property of area. These HIs possess two orthogonal directions.
7. A voluminal line is defined as a closed set of adjacent HIs (path) with the property of volume. These HIs possess three orthogonal directions.
8. Higher directional lines possess higher orthogonal directions.
9. The cardinality of these sets is infinite.
10. The cardinality of these sets can be relatively less than, equal to or greater than the cardinality of another set and is called *relative cardinality*(n or RC).
11. Postulate of HI proportionality: RC, HI magnitude and sums each follow Eudoxus’ theory of proportion.
12. The magnitudes of a HI can be relatively less than, equal to or the same as another HI.
13. The magnitude of a HI can be null.

⁸This is also in accordance with Eudoxus’ theory of proportions which I view as equivalent to not being possible to sum heterogeneous infinitesimals. In simpler words, “stacked” two-dimensional planes cannot integrate into a volume.

14. If the HI within a line is of the same magnitude as the corresponding adjacent HI, then that HI is intrinsically flat relative to the corresponding HI.
15. If the HI within a line is of a magnitude other than equal to or null as the corresponding adjacent HI, then that HI is intrinsically curved relative to the corresponding HI.
16. A HI that is of null magnitude in the same direction as a path is defined as a point. A lineal HI point has the property of 0 dimensions. An areal HI point has the property of length orthogonal to the path. A voluminal HI point has the property of area orthogonal to the path.
17. Adjacent points within adjacent areal lines are said to create an arc (i.e. the circumference of a circle).
18. Adjacent points within adjacent voluminal lines are said to create a surface (i.e. the surface of a sphere).

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