Geometric Law of Quantum Mechanics and the Imaginary Unit

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Abstract

Mathematically, the imaginary unit i has long been regarded as an algebraic generator extending the real numbers to the complex field.
Yet a century after the birth of quantum mechanics (1925–2025), the answer to its oldest question is now clear:
quantum mechanics is not merely a mathematical structure—it is a physical reality, and all quantities appearing in its non-commutative relations are physical.

Therefore, i is not a mere algebraic extension but the universal operator demanded by the geometric completeness of non-commutative reality between position and momentum.
The imaginary unit is a physical reality.
The canonical relation $[x, p] = i\hbar$ is not merely algebraic—it is a **geometric law**, showing that the complex structure of quantum mechanics arises inevitably from physical reality itself.
It stands without contradiction, without complexity. This is nature.

In the (x, p) plane, i represents a 90° rotation and $\hbar$ quantizes the curvature of that rotation.
Complex numbers thus arise as the natural geometric language of non-commutative physics, providing a geometric resolution to the century-old question of physical completeness first raised by Einstein, Podolsky and Rosen (1935).

1. Origin and Necessity: why i is not imaginary

The position–momentum commutator of quantum mechanics has, since the theory's inception, never shown any deviation or approximation that would cast doubt on its correctness.
We therefore begin from the established fact that

$$[x, p] = i\hbar$$

is an exact physical law, not a provisional rule.

Quantum mechanics has employed complex numbers for a hundred years without understanding their necessity.
This canonical relation is usually regarded as algebraic, yet it is in essence a geometric law.
It defines a conjugate pair—position and momentum—whose operations do not commute because their order implies rotation.
Non-commutativity is therefore geometric in nature, and its most direct expression is rotation itself.
The same geometric rotation governs the phase of a quantum state through time, making i simultaneously the direction indicator and generator of rotation for every physical evolution.

Heisenberg's relation $[x,p] = i\hbar$ breaks the planar symmetry of the position–momentum space.
To restore closure of that geometry, a 90° rotation is required—the rotation embodied by i.
Thus i is the minimal geometric completion demanded by non-commutativity.

Earlier approaches assumed i a priori as an algebraic device; here i emerges as a physical consequence of the non-commuting observables x and p.
This reverses the causal order: i is not a tool describing non-commutativity—it is its inevitable outcome.
In this sense, i is the closure element of a broken symmetry, the rotation that re-establishes geometric consistency.
The imaginary unit is a physical reality.

Weyl (1928) recognized $[x,p] = i\hbar$ as a Lie algebra with structure constant i [3];
Dirac (1930) systematized it as the foundational postulate of quantum mechanics [4],
while von Neumann (1931) rigorously proved the uniqueness of its Hilbert-space representation [5].
While those works revealed its algebraic necessity, the present reinterpretation exposes its **geometric inevitability**:
the commutator defines a curved rotational manifold whose minimal closure is the action of i.

2. Quantum Law and the Imaginary Unit

The primordial symmetry of nature does not geometrically distinguish between position x and momentum p.
The non-commutative relation of quantum mechanics, $[x,p] = i\hbar$, signifies that this symmetry is spontaneously broken.
The magnitude of the area generated by the two operators measures the degree of this breaking, while its orientation is given by the imaginary unit i.
Thus, $i\hbar$ is the geometric remnant of spontaneous symmetry breaking, defining the intrinsic non-commutative structure of quantum mechanics.

$$[x,p] = i\hbar$$

is invariant under any rotation of the coordinate system in the (x,p) plane.
Its left side represents the oriented area generated by the two operators x and p; its right side gives that area's magnitude $\hbar$ and its orientation i.
Therefore, i is not a mere symbol but the **orientation indicator** of phase-space rotation.

The invariance of $[x,p]$ under rotation means the entire relation is geometrically closed.
Hence i serves as the **rotation generator** of the non-commutative plane, and the equality holds for all rotated frames.
In the geometric language of quantum mechanics, the imaginary unit is both the **direction of curvature** and the **operator that produces it.**

We thus identify

$$[x,p] = i\hbar$$

as a **geometric law of nature**:
the minimal oriented area element of the position–momentum space has magnitude $\hbar$ and orientation i.
This re-establishes quantum mechanics as a theory of intrinsic geometry, not merely algebra.

3. Implications: geometric completeness and universality

Einstein, Podolsky and Rosen (1935) argued that quantum mechanics was incomplete because it lacked local realism.
Reinterpreting i as the intrinsic geometric rotation of the (x, p) plane transforms this question: quantum mechanics may be nonlocal in measurement, yet it is **geometrically closed** through i; completeness is achieved not through locality but through curvature.

The analogy extends to general relativity: in Einstein's theory, spacetime curvature produces gravity;
in quantum theory, phase-space curvature—quantized by $\hbar$—produces dynamics through i.
Both describe motion as geometry—one real, the other complex.
This parallel provides a conceptual foundation for unifying quantum mechanics and gravity, and for understanding any system governed by non-commuting structures, including learning architectures evolving on curved informational manifolds.

Conclusion

This work establishes a fundamental principle at the foundation of quantum mechanics: the non-commutative law $[x, p] = i\hbar$ is not an algebraic relation but a geometric necessity of nature.
We have confirmed that the imaginary unit i is not imaginary at all: it is a **physical quantity**.
Through this recognition, elementary mathematics obtains a firm foundation from quantum mechanics itself.
The imaginary unit, once regarded as an abstract construct, is newly discovered as the physical indicator of rotational symmetry in nature.
Mathematicians created numbers; physicists discovered them.

The real number measures translation; the imaginary number measures rotation.
Therefore, a complex number $(a + ib)$ is the minimal representation describing the combined state of translation and rotation in nature.
The non-commutative commutator of quantum mechanics is thus an oriented curvature relation.

On the centennial of quantum mechanics (1925–2025), the geometric essence of its foundation and the physical existence of i have revealed themselves.
Quantum mechanics, mathematics, and geometry now form a single consistent structure; elementary mathematics leaves the realm of abstraction and gains a new footing on the physical reality of nature.

$[x, p] = i\hbar$ is a geometric law: the left side is the oriented area made by x and p, the right side

gives its direction i and magnitude $\hbar$.
From this, we define the positive orientation of a two-dimensional plane—the direction of the cross product of two vectors—as the imaginary unit itself.

Just as Newton's laws define classical mechanics, the non-commutative geometric relation $[\boldsymbol{x}, \boldsymbol{p}] = \boldsymbol{i\hbar}$ defines nature in the quantum domain.

As a result, the commutation relation of quantum mechanics is proven self-consistent, and by revealing the physical foundation of the imaginary unit we place mathematics itself on a firmer ground.
Our discussion is closed.

Epilogue

For a hundred years it was used, yet not understood.
The imaginary unit is no longer imaginary—it is the geometry through which nature speaks.
It is without contradiction or complexity.
This is nature.

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Figure 1 | Quantum geometry of non-commutativity.
The quantum law $[x, p] = i\hbar$ defines a complete geometric structure that embraces the imaginary unit by orientation (i), the quantum curvature by area (ħ), and rotational symmetry (π).
The conjugate coordinates x and p form an orthogonal plane.
The red quarter-circle arrow (i) indicates the 90° rotation linking x and p, while the shaded cell denotes the minimal oriented area of magnitude $\hbar$.
Together, i, $\hbar$, and π constitute the fundamental quantum identity of physics—expressing the unity of orientation, curvature, and symmetry in quantum geometry.

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