

Higher-Degree Generalizations of the Method of Moving Points

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Abstract

This report will be presenting a generalization to a previous method to solve Euclidean geometry problems which are parameterizable in one variable, known as the "Method of Moving Points". This method sometimes faces limitations, often unable to directly intersect or parameterize curves with degrees greater than one without tailored geometric analysis. We generalize this method through applying the Veronese map to be able to parameterize higher-degree moving curves, and extend the notion of multiplicity of point-point concurrence to the degree of vanishing of a determinant, to find effective bounds on the degree of higher-degree moving curves. Additionally, through an application of polynomial resultants, we bound the degree of the locus of intersections of higher-degree moving curves. Finally, we present a collection of examples and applications of this theory to solving olympiad geometry problems involving moving circles and factoring their resultant bounds.

1 Introduction

This report will be presenting a generalization to a previous method, known as the "Method of Animation" [1] or the "Method of Moving Points", popularized in the United States math olympiad community by Zack Chroman, to solve Euclidean geometry problems by parameterizing all configurations of the problem in one variable, then writing the condition to be proven as proving a polynomial in this variable is always zero, then bounding the degree of this polynomial and checking the problem holds for a number of cases, one more than the degree of this polynomial.

This technique was later fully rigorized in 2025 by Molnar-Szabo Vilmos [2], giving a working definition of multiplicity, as well as fully working in homogeneous polynomials, and we will be extensively citing results from this publication.

In this report, we will generalize the "Method of Moving Points" to handle multiple intersections of higher-degree curves, such as conics and cubics, as well as demonstrate quick solutions to some Euclidean geometry problems with this method.

Notate complex projective n -space as \mathbb{CP}^n . U_i will be used for a Zariski-open set. We will often be working in the z -centered standard affine chart of \mathbb{CP}^2 , $[x : y : z] \mapsto (x/z, y/z)$, defined on $U_z := \{[x : y : z], z \neq 0\}$, with the line at infinity denoted as $[x : y : 0]$.

Definition 1. Define the circular points at infinity as $[1 : i : 0], [1 : -i : 0]$. All circles are conics passing through these two points, and we will notate these two points as **I** and **J**.

Definition 2. Define a **moving point** of degree d as a regular map from $\mathbb{CP}^1 \rightarrow \mathbb{CP}^2$, $[s : t] \mapsto [F(s, t) : G(s, t) : H(s, t)]$ where F, G, H are homogeneous polynomials, all of degree d . Define the **degree** of a moving point to be d .

We will often interpret a moving point as the parameterization of all possible locations of a point P in the plane.

Example 3. Let P be a point on line ℓ , and let Γ be a circle. Let X be a point on Γ , let line XP intersect Γ again at point Q . We can construct a degree-1 moving point parameterizing P . Let A, B be two points on the line with coordinates $[A_x : A_y : A_z]$ and $[B_x : B_y : B_z]$. Define the (bi)regular map

$$\begin{aligned} \mathcal{P} : \mathbb{CP}^1 &\rightarrow \mathbb{CP}^2 \\ [m : n] &\mapsto mA + nB. \end{aligned}$$

Let P have coordinates $[\mathbf{P}_x : \mathbf{P}_y : \mathbf{P}_z]$, where $\mathbf{P}_i := mA_x + nB_x$. The image of \mathbb{CP}^1 after this map would be line $\ell \in \mathbb{CP}^2$, would give all possible positions for P . Similarly, Q can be given as a degree-2 moving point, as line XP can be given as the line, with coefficients

degree-1 homogeneous polynomials,

$$\begin{bmatrix} \mathbf{P}_y Q_z - \mathbf{P}_z Q_y \\ \mathbf{P}_z Q_x - \mathbf{P}_x Q_z \\ \mathbf{P}_x Q_y - \mathbf{P}_y Q_x \end{bmatrix} \cdot [x : y : z] = 0$$

and its intersection with the circle can be given as effectively the construction of the well-known homogeneous degree-2 polynomial parameterization of a circle, by its isomorphism to \mathbb{CP}^1 .

We will cite Theorem 8.1 and Theorem 9.2 from Vilmos's paper, [2], reproduced below, with the proofs omitted.

Theorem 4 (Locus-degree Theorem). H is a degree n polynomial moving point. Then the locus of H is a degree k algebraic curve, where $k \mid n$, and for every smooth point P on the curve (i.e. tangent space is nondegenerate), then P will have $\frac{n}{k}$ preimages in \mathbb{CP}^1 . For every order d singularity B , it will have $\frac{n}{k} \cdot d$ preimages in \mathbb{CP}^1 .

Define the degree of a moving line as the degree of its equation's coefficients as homogeneous polynomials.

Theorem 5 (Maszo-theorem). For two moving points A, B moving on a common conic, then $\deg \overline{AB} = \frac{\deg A + \deg B}{2}$.

We will also cite its definitions of "multiplicity" of certain conditions.

1. The point-point multiplicity of the coincidence of two moving points A, B is the order of the zero of $A - B$ at the intersection point, calculated by taking a chart of \mathbb{CP}^1 containing the intersection point.
2. The point-line multiplicity of the coincidence of a moving point and a moving line is given by the order of the zero of their dot product at the coincidence, calculated by taking a chart of \mathbb{CP}^1 containing their intersection point.

Finally, we will cite a lemma from Vilmos which allows us to work in (holomorphic) affine charts of \mathbb{CP}^1 to count multiplicity much easier.

Lemma 6 (point-point multiplicity is preserved, lemma 4.1 in Vilmos). Let $p, q : \mathbb{CP}^1 \rightarrow M$ be two holomorphic maps and $\phi : M \rightarrow N$ a holomorphic map, where M and N are m and n dimensional complex manifolds. Then for every $t_0 \in U \in \mathbb{CP}^1$ (where U is an affine chart) where $p(t_0) = q(t_0)$, the point-point multiplicity of $\phi \circ p$ and $\phi \circ q$'s concurrence at t_0 is at least the point-point multiplicity of p and q 's concurrence at t_0 . Furthermore if ϕ has a Jacobian matrix of rank m at $p(t_0)$, then the two multiplicities are equal.

2 Moving Curves

2.1 Resultants and the Veronese embedding

Definition 7. Given two polynomials in x with coefficients $a_m \dots a_0$ and $b_m \dots b_0$, define the **resultant** of this system as this determinant

$$\begin{bmatrix} a_m & a_{m-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m & a_{m-1} & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n & b_{n-1} & \cdots & b_0 \end{bmatrix}$$

Given two homogeneous polynomials \mathcal{P}, \mathcal{Q} in multiple variables $x, y, z, w \dots$, we can treat $y, z, w \dots$ as coefficients of a polynomial in x . Then, the resultant of these two polynomials, taken in x , is going to be an expression in $y, z, w \dots$ (no x) which will be equal to zero exactly when these two polynomials \mathcal{P}, \mathcal{Q} are both zero.

In other words, the zeroes of the resultant will be a projection of all intersection points of the zero sets of \mathcal{P} and \mathcal{Q} , through $(1 : 0 : 0 \dots)$ onto the line $x = 0$.

First, let's generalize the method of moving points to "moving curves". Let our parametrization variable be t in $(x : y) \in \mathbb{C}\mathbb{P}^1$ (sometimes, for ease of algebra we will take the chart sending it to $x/y \in \mathbb{C}$, and the degree of a point will be the maximal degree of each coefficient in t , but in reality it should be homogeneous in x, y).

Definition 8 (Degree of a moving curve). The **moving degree** d of a moving curve is the degree of its coefficients, as a polynomial in $[s : t]$.

As a convention to avoid confusion, in this section, we will use capital letters to denote the algebraic degree of a curve, and we will use lowercase letters to denote the moving degree of a curve.

Definition 9 (Veronese map). Define the degree-2 Veronese map as such [5]:

$$\mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^5$$

$$[x : y : z] \mapsto [x^2 : xy : xz : y^2 : yz : z^2].$$

Analogously, define the degree- N Veronese map for higher degree monomials, sending

$$\mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^{\binom{N+2}{2}-1}$$

$$[x : y : z] \mapsto [x^N : x^{N-1}y : x^{N-1}z : x^{N-2}y^2 : x^{N-2}yz : \dots : y^N : y^{N-1}z : \dots : yz^{N-1} : z^N].$$

The image of the projective plane after this map is the Veronese surface, lying in a subvariety of $\mathbb{C}\mathbb{P}^{\binom{N+2}{2}-1}$.

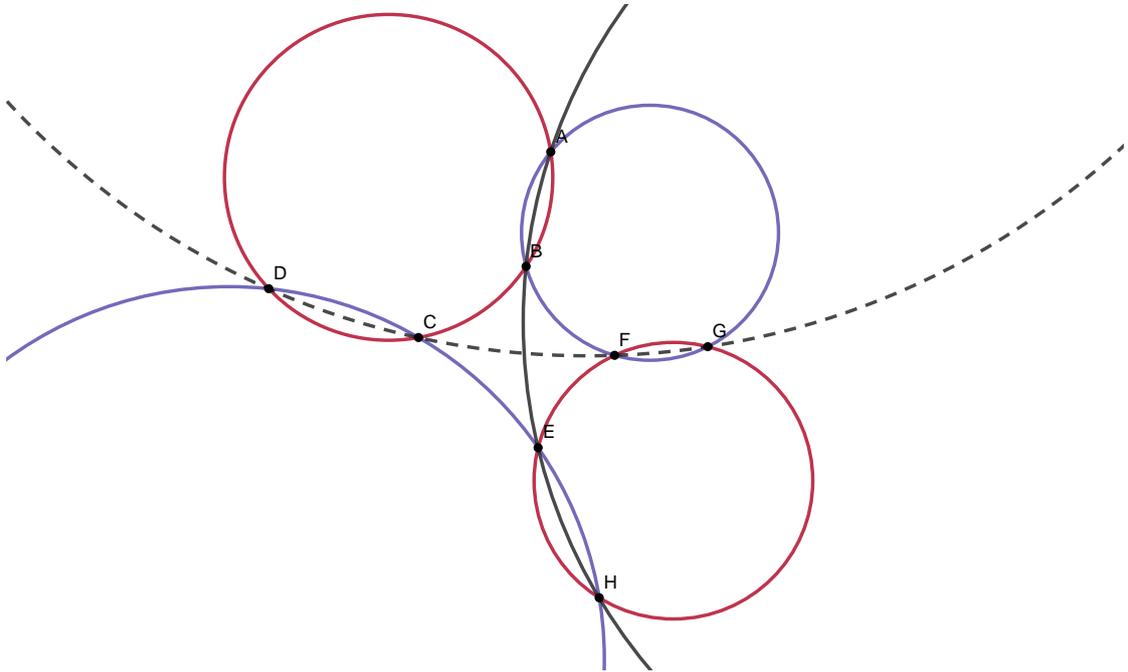
The image of any degree- N plane curve, after the degree N Veronese map, will lie in the intersection of a hyperplane with dimension $\binom{N+2}{2} - 2$ with the Veronese surface.

Further, after the Veronese map, a degree- d moving curve corresponds to a hyperplane in $\mathbb{C}\mathbb{P}^{\binom{N+2}{2}-2}$ moving with degree d , given by the matrix transpose of its coefficients.

It is often useful to dualize everything in the $\mathbb{C}\mathbb{P}^{\binom{N+2}{2}-1}$, swapping the hyperplanes (degree- N curves) with the points.

As an example of this technique, we give a clean solution to a classical theorem.

Example 10 (Bundle theorem). With points defined in the diagram, prove if the quadrilateral $ABEH$ is cyclic, then quadrilateral $DCFG$ is cyclic.



Proof. After applying the degree-2 Veronese map, circles become dim-4 hyperplanes passing through the line containing the images of \mathbf{I}, \mathbf{J} . Choose a fixed dim-3 linear subspace \mathcal{P} in $\mathbb{C}\mathbb{P}^5$ not including the line containing the images of \mathbf{I}, \mathbf{J} as a subvariety, and map each circle uniquely to the intersection of its corresponding dim-4 hyperplane with \mathcal{P} . Thus, by projection down from $\mathbb{C}\mathbb{P}^5$ to $\mathbb{C}\mathbb{P}^3$, the set of circles in $\mathbb{C}\mathbb{P}^2$ are all planes in $\mathbb{C}\mathbb{P}^3$. Take the projective dual of all planes in this $\mathbb{C}\mathbb{P}^3$ to get a bijection of circles in $\mathbb{C}\mathbb{P}^2$ to the set of points in $\mathbb{C}\mathbb{P}^{3*} \cong \mathbb{C}\mathbb{P}^3$. Since all operations we have taken are given by linear homogeneous polynomials, a pencil of coaxial circles, all linear combinations of each other, will map to the line connecting two points in $\mathbb{C}\mathbb{P}^3$. Intersection of two lines corresponds to two pencils of circles having a common circle, implying the two pairs of real fixed points of these pencils form a cyclic quadrilateral.

Then, our problem is translated to constructing four points $c_{12}, c_{23}, c_{34}, c_{14}$ in $\mathbb{C}\mathbb{P}^3$ such that the line connecting c_{23}, c_{12} and the line connecting c_{34}, c_{14} intersect, then the line

connecting c_{23}, c_{34} and the line connecting c_{12}, c_{14} also intersect. However, this is true since the four points must all be coplanar. \square

Theorem 11 (Interpolation formula). For $\binom{N+2}{2} - 1$ points P_i in the plane, the points on a degree- N curve passing through all of the points will be given by the preimages of all $Q \in \text{Veronese surface} \in \mathbb{CP}^{\binom{N+2}{2}-1}$ satisfying

$$\det \begin{bmatrix} \text{coordinates of } P_1 \text{ after the Veronese} \\ \text{coordinates of } P_2 \text{ after the Veronese} \\ \vdots \\ \text{coordinates of } P_{\binom{N+2}{2}-1} \text{ after the Veronese} \\ Q_1, Q_2, Q_3, \dots, Q_{\binom{N+2}{2}-2}, Q_{\binom{N+2}{2}-1} \end{bmatrix} = 0.$$

Currently this is only well-defined if P_i all lie on a codimension-1 hyperplane, and thus not everywhere-regular - we will address this shortly.

Proof. All points $Q = [Q_1 : Q_2 : \dots : Q_{\binom{N+2}{2}-1}]$ satisfying this determinant condition will be coplanar on a codimension-1 hyperplane with all of P_i , and the intersection of this hyperplane with the Veronese surface gives the image of our desired curve after the Veronese map. \square

The components of this determinant give a rational map from $\mathbb{CP}^1 \dashrightarrow \mathbb{CP}^{\binom{N+2}{2}-1}$, with each of the $\binom{\binom{N+2}{2}-2}{2} \times \binom{\binom{N+2}{2}-2}{2}$ minors' determinants corresponding to one coordinate in $\mathbb{CP}^{\binom{N+2}{2}-1}$.

It is important to recognize that this map is defined on the open subset of \mathbb{CP}^1 of when the previous determinant is not equal to the zero polynomial, as then all components of the determinant will be identically zero and our map will not be well-defined there.

Theorem 12. If P_1, P_2, P_3, \dots are moving points, with there existing no value $(s : t) \in \mathbb{CP}^1$ such that the previous determinant is the zero polynomial, then the degree- N moving curve through all of them has moving degree

$$N \cdot (\text{sum of degrees of the moving points}).$$

Proof. After the degree- N Veronese map is applied, the coordinates of a moving point P_i moving with degrees $\deg P_i$ will be polynomials of degrees $\deg P_i \cdot N$. The row in the above matrix corresponding to P_i will thus have homogeneous terms of degree $\deg P_i \cdot N$. Then, each component of our determinant will have degree $\sum(\deg P_i \cdot N) = N \cdot \sum(\deg P_i)$. Since this determinant is never the zero polynomial, the components in each coordinate will never be zero at the same $(s : t) \in \mathbb{CP}^1$ so our rational map is everywhere-defined and thus regular. \square

Suppose there is a point $(r_s : r_t) \in \mathbb{CP}^1$ such that this determinant is the zero polynomial. Let U_i be a Zariski-open holomorphic chart of \mathbb{CP}^1 containing a neighborhood

of $(r_s : r_t)$ (in the analytic topology in a standard affine chart of \mathbb{CP}^1 containing it), sending $(r_s : r_t) \mapsto r \in \mathbb{C}$. Restricted to U_i , we get a well-defined multiplicity for each coefficient of the determinant vanishing, by the number of iterated derivatives of it which vanish at r . (This is the same among all holomorphic charts, by lemma 4.1 in Vilmos). Define the **multiplicity of vanishing** at r analogously as the **minimal** multiplicity over all coefficients of the determinant.

Note when the determinant vanishes at r with multiplicity n , then in U_i , the determinant can be given by the polynomial $F_1(z)(z-r)^n \cdot Q_1 + F_2(z)(z-r)^n \cdot Q_2 + \dots = 0$ for some polynomials $F_i(z)$, with at least one F_i being nonzero at r .

If the determinant is the zero polynomial ever, then we get degree-reduction lemmas, via the procedure to extend a rational map from $\mathbb{CP}^1 \dashrightarrow \mathbb{CP}^n$ to a regular map.

Theorem 13 (Degree-bound on interpolated curve). For $\binom{N+2}{2} - 1$ moving points on the plane, then the moving degree of a degree- N curve passing through all of them will be maximally bounded by

$$N \cdot (\text{sum of degrees of the moving points})$$

- (number of times the previous determinant is zero, including multiplicity).

Proof. We defined a rational map from $\mathbb{CP}^1 \mapsto \mathbb{CP}^{\binom{N+2}{2}-1}$. Since our rational map is given in each coordinate by polynomials homogeneous in $[s : t]$ (call them $F_1(s : t)$), the only time this map is not defined is for $[r_s : r_t]$ when all of the homogeneous polynomials F_i are zero, with determinant multiplicity m as defined before. When this happens, we can write our map (globally) as $[s : t] \mapsto [F_1(s, t)(r_s t - r_t s)^m + F_2(s, t)(r_s t - r_t s)^m + \dots] = 0$ such that there exists $F_i \neq 0$ at $[r_s : r_t]$. However, this rational map is identical to the regular map $[s : t] \mapsto [F_1(s, t) + F_2(s, t) + \dots] = 0$, dropping the expected degree by m . Repeating this procedure over every time the rational map is undefined, we get our desired result. \square

When n moving points P_i concur, we get more reduction. The amount of reduction will be much easier to calculate if we work in an affine chart, as we will look at derivatives near the concurrence. Let the points be parameterized by t near the concurrence.

Then we get the multiplicity of vanishing of the determinant is given by its corank, plus additional terms from first and higher order derivatives vanishing simultaneously as well. However, the corank of the higher terms is generally hard to fully count nicely, and usually in geometry problems this exact reduction is not needed to solve them.

2.2 Geometric degree-reduction lemmas

For this section, we will give a few lemmas for commonly-seen situations when degree of an interpolated moving curve drops, and maximally bound it in each case, helping

to connect the geometric picture with the algebraic picture. Assume all moving points defined here are non-cuspidal; that is, they always have a well-defined first derivative in an appropriate affine chart.

Lemma 14 (Concurrence degree-reduction Lemma). When n_0 moving points P_i concur, the degree of the interpolated curve will drop by at least $n_0 - 1$ (summed over all concurrences).

Proof. If n_0 moving points concur, then by row-reduction, after the Veronese map, we can construct a matrix with $n_0 - 1$ rows which all vanish simultaneously at the time of concurrence. This gives the determinant a multiplicity of vanishing with at least $n_0 - 1$ at this root.

Remark 15. Note how this immediately gives the formula for the degree of a moving line through moving points A and B as $\deg A + \deg B - \{\#(A = B)\}$. By duality in $\mathbb{C}\mathbb{P}^2$, the degree of the point of intersection of two moving lines ℓ_A and ℓ_B is given by $\deg \ell_A + \deg \ell_B - \{\#(\ell_A = \ell_B)\}$.

□

Lemma 16 (Collinearity degree-reduction Lemma). For a conic through five points, whenever four points are collinear, the moving degree of the conic drops by at least 1, summed over all occurrences of collinearity.

Proof. The image of a line after the Veronese map is a conic in $\mathbb{C}\mathbb{P}^5$ lying on some dim-2 plane. Thus, when four points are collinear, after the Veronese map, they will go to four points lying on a common dimension-2 plane. Thus the interpolating conic's matrix will have corank at least 1, so the determinant will vanish with multiplicity at least 1. □

Lemma 17 (5-collinear degree-reduction Lemma). For a conic through five points, whenever all five points are collinear, the degree drops by at least 2.

Proof. Similar to the previous proof, this gives five points in $\mathbb{C}\mathbb{P}^5$ lying on a dim-2 plane, and thus a corank of at least 2, since for five points lying on a common dim-2 plane, at least two must be linear combinations of the other three, as three non-collinear points determine a plane. □

This clearly generalizes in a similar way to concurrence degree-reduction for higher-degree curves.

For cubics, we have an interesting example of degree-reduction:

Lemma 18 (Cayley-Bacharach degree reduction). For nine points, whenever they lie on the intersection of two non-degenerate cubics, the moving degree of the moving cubic interpolated through them will drop by at least 1.

Proof. Consider the degree-3 Veronese map into $\mathbb{C}\mathbb{P}^9$. The image of the nine points in the intersection of two cubics will be given by the points of intersections of two dim-8 hyperplanes, which has a common dim-7 linear subspace, with the degree-3 Veronese surface. Since any degree Veronese surface contains no lines [5], the nine moving points lie on a dim-7 linear subspace, and thus the corank of the previously constructed matrix is at least 1. \square

Theorem 19 (Intersection locus degree formula). Two curves of degrees A and B in $\mathbb{C}\mathbb{P}^2$ moving with degrees m and n , never sharing a common irreducible component, will intersect in a locus of maximally degree $An + Bm$.

Proof. In a standard affine chart of $\mathbb{C}\mathbb{P}^1$, the locus of intersections will be given by the zero set of a resultant in t of the two curves. By the determinant definition of resultant, we have m terms of degree A multiplied by n terms of degree B , so the total degree of the resultant is $An + Bm$ in t . In the other chart of $\mathbb{C}\mathbb{P}^1$, since we have homogeneity of the polynomials defining our moving curves, we also have degree $An + Bm$. \square

If they do share a common irreducible component at some point, then the resultant must contain the whole irreducible component, and thus has to have the irreducible component as a factor. This allows for further degree-reduction sometimes, and will be used in the examples.

Remark. Note how this is compatible with the degree-formula for the intersection of two moving lines ℓ_A and ℓ_B as $\deg \ell_A + \deg \ell_B - \{\#(\ell_A = \ell_B)\}$, as when $\ell_A = \ell_B$, we have a line in the resultant, dropping its degree by 1. Thus the degree of the resultant here exactly corresponds to the degree of the moving point formed by intersecting the two lines.

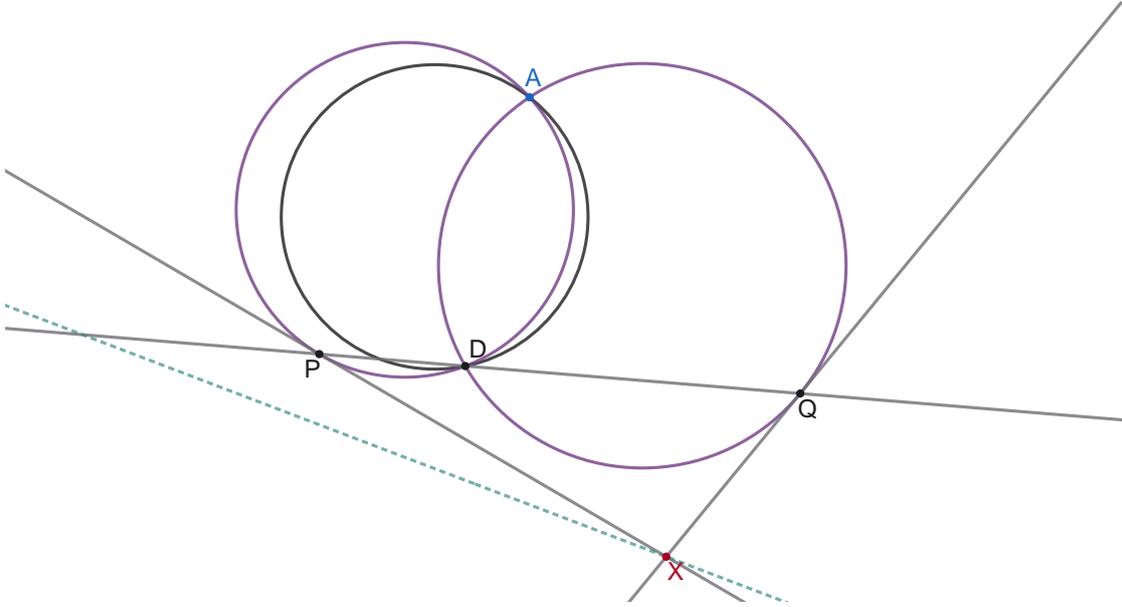
Lemma 20 (Condition for passing through a point). A degree N curve moving with degree d will always pass through a point P if it passes through P for $d + 1$ cases.

Proof. The condition for a moving hyperplane to pass through a fixed point is a linear equation in the coordinates of the hyperplane, so there can be at most d solutions unless the hyperplane always contains the fixed point. \square

Now we will solve some olympiad geometry problems with this technique, to demonstrate its utility and clarify the theory.

3 Applications and examples

Example 21 (2023 ELMO Shortlist G4). Let D be a point on segment PQ . Let ω be a fixed circle passing through D , and let A be a variable point on ω . Let X be the intersection of the tangent to the circumcircle of $\triangle ADP$ at P and the tangent to the circumcircle of $\triangle ADQ$ at Q . Show that as A varies, X lies on a fixed line.



Solution. Let \mathbf{I}, \mathbf{J} be the circular points at infinity. Parameterize $A \in \omega \in \mathbb{CP}^2$ as a degree-2 moving point on the circle ω . By the [Locus-degree Theorem](#), A will cover the circle once.

Claim. Circles (ADP) and (ADQ) move with degree 1.

Note there are three positions for A where $A = \mathbf{I}, \mathbf{J}, D$, since A covers the circle once. By the generalized degree formula, we have the circle (ADP) maximally moves with degree

$$2(\deg \mathbf{I} + \deg \mathbf{J} + \deg D + \deg P + \deg A) - [\# \text{ of } A = \mathbf{I}, \mathbf{J}, D] = 2(0 + 0 + 0 + 0 + 2) - 3 = 1,$$

and similarly for (ADQ) .

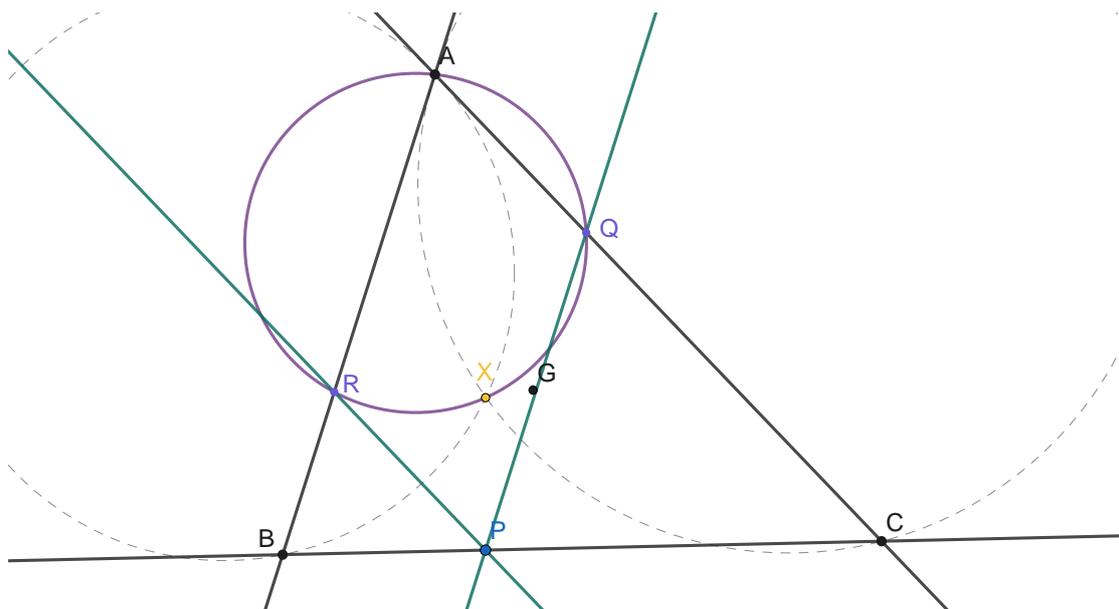
Let $\mathcal{C}_1(x : y : z) := (ADP)$. The tangent at P is given by the line

$$x \frac{\partial \mathcal{C}_1}{\partial x}(P) + y \frac{\partial \mathcal{C}_1}{\partial y}(P) + z \frac{\partial \mathcal{C}_1}{\partial z}(P) = 0.$$

Since P is a fixed point, and $\frac{\partial \mathcal{C}_1}{\partial x}$ varies linearly as the coefficients of \mathcal{C}_1 vary linearly, the tangent at P also moves with degree-1. Similarly, the tangent at Q moves with degree-1.

The two tangents are identical when $A = PQ \cap \omega$, implying their intersection X moves with degree $1 + 1 - 1 = 1$, so X moves on a fixed line by the [Locus-degree Theorem](#). \square

Example 22 (2008 USA IMO Team Selection Test P7). Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC . Points Q and R lie on sides AC and AB respectively, such that $\overline{PQ} \parallel \overline{AB}$ and $\overline{PR} \parallel \overline{AC}$. Prove that, as P varies along segment BC , the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.



Solution. First, we want to prove X exists.

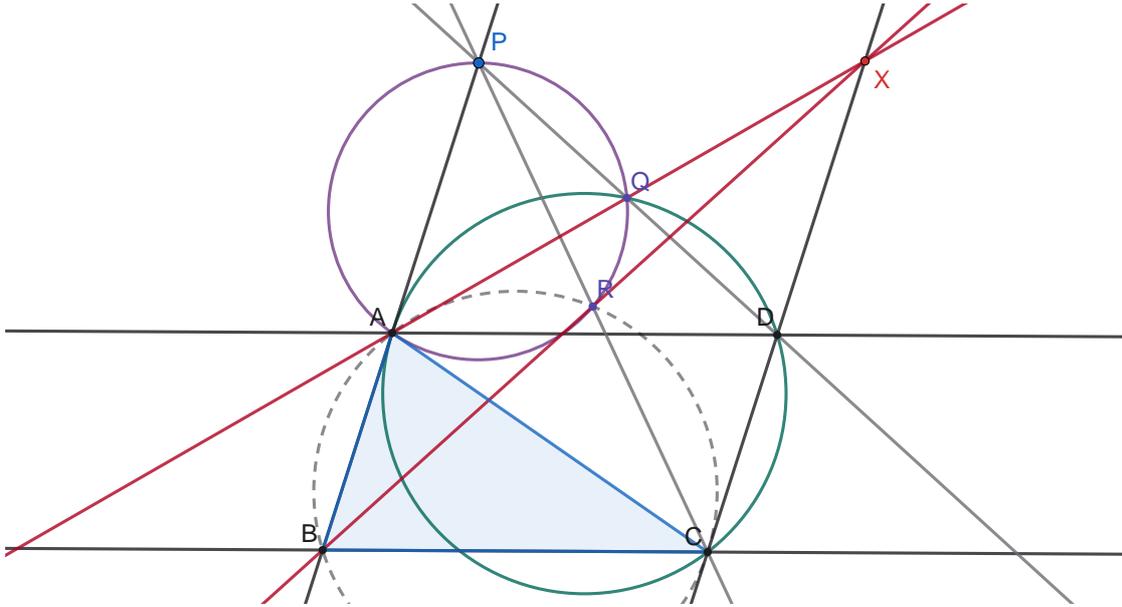
Move P , then points Q, R will move with degree 1 each, by projecting through the point at infinity along AB and AC respectively, so circle (AQR) will move maximally with degree $2(\deg Q + \deg R) = 4$. When $P = B$ and $P = C$, we have $Q = A$ and $R = A$ respectively, which decrease our degree bound to maximally 2 by the [Concurrence degree-reduction Lemma](#) applied twice.

When P is at the point at infinity along BC , we have Q, R as the points at infinity along AC and AB respectively. Since \mathbf{I}, \mathbf{J} lie on the line at infinity, we have $Q, R, \mathbf{I}, \mathbf{J}$ collinear here, so we can apply the [Collinearity degree-reduction Lemma](#) to decrease the degree of the circle to degree 1, a tight bound. Since two general values of P give circles (AQR) intersecting at four distinct complex points, (AQR) will move in a pencil of conics, and thus pass through four fixed points. We know it passes through the three fixed points $A, \mathbf{I}, \mathbf{J}$, let the fourth fixed point be X' .

By setting $P = B, P = C$, we know that X' is the intersection of the circle through B tangent to AC at A , and the circle at C tangent to AB at A . In other words, X' is the Miquel point of degenerate quadrilateral $ABAC$, also known as the Dumpty point, known to lie on the A -symmedian, and thus $\angle BAG = \angle CAX$. \square

Next we give an example of the resultant bound for the degree of the intersection of two moving curves.

Example 23 (2021 IMO Shortlist G1). Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B . The circumcircle of ACD meets the segment PD again at Q . The circumcircle of triangle APQ meets the segment PC at R . Prove that lines CD, AQ, BR are concurrent.



Solution. Move P on line AB with degree 1. Then Q moves on (ACD) with degree 2, by [Maszo-theorem](#). First we can see (APQ) will be a circle moving with degree-1, as it moves with degree at most $2(\deg P + \deg Q) = 2(3)$, then subtracting 1, twice, when $Q = I$ and $Q = J$. Finally, note geometrically P and Q and A will meet, subtracting 2 from the total degree, but also P and Q have first derivatives scalar multiples of each other at $P = A, Q = A$, subtracting an additional 1 from the degree as the determinant in the [Concurrence degree-reduction Lemma](#) will get an extra zero. Thus, (APQ) will be a circle moving with degree-1.

Claim. R moves on a circle.

Since \overline{DP} is a line moving with degree 1, we know the locus of the two intersection points of \overline{DP} with (APQ) is a curve of degree $2 \cdot 1 + 1 \cdot 1 = 3$ (by the [Intersection locus degree formula](#)), and we also know it is a circular cubic, since (APQ) always passes through the circle points, and DP will eventually pass through the lines DI and DJ . Further, this cubic curve is reducible, since one part of this locus is the line AB , as the known locus component of point P . This implies R moves on a conic, but since line AB does not pass through either circle point, R moves on a circle. By manually verifying the cases $P = B, P = A, P = \text{infinity point along } \overline{BA}$, where $R = A, B, C$, we get R, A, B, C concyclic.

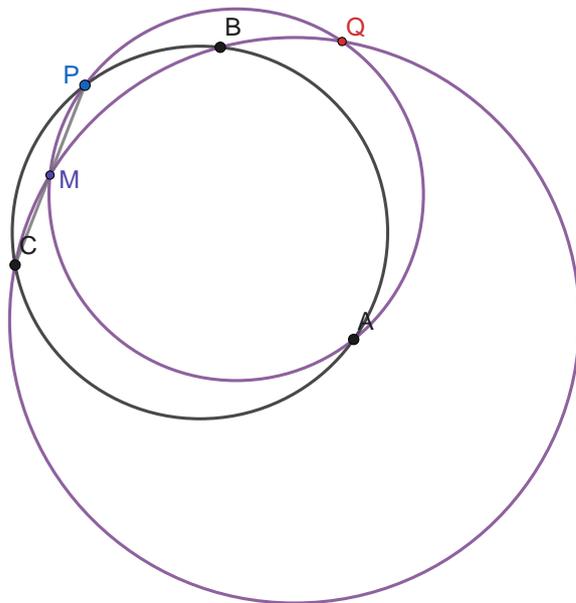
Redefine R as $(ABC) \cap CP$, then after this definition, R moves projectively on fixed conic (ABC) with degree 2 by [Maszo-theorem](#), implying line BR moves with degree 1 by [Maszo-theorem](#) a second time. Since Q moves with degree 2 on fixed conic (ACD) , AQ is a degree 1 line by [Maszo-theorem](#) a third time. The lines BR and AQ are equal when $P = A$, so $X := BR \cap AQ$ moves with degree $1 + 1 - 1 = 1$. Thus, to prove X always lies on line CD , we must manually check at least two cases by the [Condition for passing through a point](#).

When P lies on the intersection of the perpendicular bisector of segment \overline{AD} with AB , we have line PD is tangent to circle (ACD) by symmetry, so $Q = D$. Since

$$\angle BRQ = \angle PRB + \angle PRQ = (180^\circ - \angle CRB) + \angle PAQ = (180^\circ - \angle BAC) + \angle PBC = 180^\circ$$

, we have B, R, Q collinear, so here BR and AQ intersect at D , clearly on line CD . When P is at the point at infinity along AB , circle APQ degenerates into the union of line AC with the line at infinity, and R is point C . Thus in this case, BR and AD intersect at C , also on line CD , so we have verified two cases and the problem is solved. □

Example 24 (2014 IMO Shortlist G4). Consider a fixed circle Γ with three fixed points A, B , and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ , let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC . Prove that as P varies, the point Q lies on a fixed circle.



Move P with degree 2 on (ABC) . Then M will also move with degree 2. (AMP) will then move with degree at most $2 \cdot (\deg P + \deg M) = 2 \cdot (4)$, with the degree reduced by $P = \mathbf{I}, P = \mathbf{J}, P = A, M = \mathbf{I}, M = \mathbf{J}, M = P$, so (AMP) will move with degree at most 2. (CMB) will then move with degree at most 4, dropping when $M = C, M = \mathbf{I}, M = \mathbf{J}$. Thus (CMB) moves with degree 1, a tight bound. The intersection of (CMB) with (AMP) will then be a degree 6 curve, singular at each of the two circle points as they are stationary. One of the points of intersection is M , which moves on a degree 2 circle. When $P = C$, $(CMB) = (AMP)$, giving us a degree 2 irreducible component in the resultant. Since \mathbf{I} and \mathbf{J} are fixed, Q must move on a degree 2 curve, passing through the circle points when $P = \mathbf{I}$ and $P = \mathbf{J}$. Thus Q moves on a circle.

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