Quantization of free electromagnetic field without infinity in terms of finite series or integral in finite interval

Chol Jong [1]

Abstract We present a new approach to the quantization of free electromagnetic fields without the infinity problem. Our work shows that from the physical viewpoint, the energy of a free electromagnetic field in a finite volume should be represented necessarily by a finite series, since fields consist of a finite number of photon. From the mathematical viewpoint, it is explained that the quantization of electromagnetic field reduces the infinite Fourier series to a finite series and preserve the relativistic invariance. It is demonstrated that the cut-off of series for the energy of photons is uniquely and objectively determined based on the assumption about photon. Based on this perspective, the interaction between free electromagnetic field and matter is described in terms of an integral in a finite interval which does not comprise zero and infinity. Our methodology always gives finite results and thus does not need renormalization. Ultimately, it is demonstrated that it is possible to construct a new quantization theory of electromagnetic field without infinity and renormalization thereof.

Keywords Quantum field theory · Infinity · Particle number operator · Fourier series · Zero-point energy

1 Introduction

The canonical quantization of free electromagnetic field is carried out by substituting the particle number operators for expansion coefficients of the vector potential of classical electromagnetic field. As a result, we obtain as the Hamiltonian for quantized energy

$$\hat{H} = \sum_{k,\sigma} \hbar\omega_k \hat{a}^\dagger_{k\sigma}\hat{a}_{k\sigma} + \frac{1}{2}\sum_{k,\sigma}\hbar\omega_k.$$

Then the infinity problem arises. Evidently, in any case, the quantized energy of free electromagnetic field in a confined space is divergent because it is composed of infinitely many numbers of finite quantities. A proposal to solve the infinity problem of the zero-point energy is to represent, by modifying it, as

$$W = \frac{1}{2}\sum_{k,\sigma}\hbar\omega_k \exp\left(-\lambda\frac{\omega_k}{c}\right).$$

Since a factitious parameter λ has been introduced here, the solution is not perfect [1]. We still have not found a satisfactory method for solving in a general way the infinity problem arising from the quantization of free electromagnetic field. The infinity problem arising when dealing with the interaction between quantized electromagnetic field and matter is solved based on renormalization theory.

Originally, renormalization theory came from the research on quantum scattering theory. John Ward, Yang, Mills and Salam's innovative studies to resolve the problem of overlapping divergences of the scattering matrix [2–4] contributed to the early development of quantum scattering theory. In later time, renormalization theory was furthermore elaborated by Stueckelberg and Green [5], Bogoliubov and Parasiuk [6], Wolfhart Zimmermann. On the other hand, Gelfand, Yaglom and Cameron contributed to making the Feynman history integral rigorous [7, 8].

The recent researches show a wide spectrum of renormalization technique that covers the studies such as renormalization group flow [9], renormalization group function and equation [10], renormalized perturbation theory [11], renormalization theory on the perturbative Feynman graph

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expansion [12], and on the other hand, connections between these different techniques of renormalization [13]. Renormalization theory recognized as a successful theory continues to extend its coverage to problems relevant to Lorentz and gauge symmetry and the requirement for cosmological spacetime [14–17].

Up to date, renormalization theory has played a key role in solving the infinity problem of the scattering matrix. Nevertheless, the objective reality manifestly indicates that renormalization theory still has not been framed as a consistent theory, since it has not provided any promising avenue towards the final solution of the infinity problem [18]. The facts imply that it is necessary to explore a new theory without the infinity problem, in parallel with the development of renormalization [17,19,20].

The present situation leads us to a practical view that it is desirable and feasible to formulate a consistent theory on the quantization of electromagnetic field without dealing with the infinity problem. In this regard, it is remarkable that there are attempts to construct a new formulations of scattering theory without infinity [17,19].

Our aim is to construct a new approach without infinity. It is unreasonable to apply renormalization to solve the infinity problem caused by the quantization of free electromagnetic field. In fact, the cause of infinity lies in not the matter interacting with an electromagnetic field but the quantization of the electromagnetic field. Therefore, the quantization must not depend on mass or charge and others pertaining to matter. With such an understanding, we present a new method of the quantization of free electromagnetic field without infinity by using a cut-off method which is physically well-grounded.

The remaining paper is organized as follows. In Sect. 2, the quantization of free electromagnetic field in terms of finite series is dealt with, which solves the infinity problem in a natural way. In Sect. 3, the interaction between electromagnetic field and matter is described based on the representation of free electromagnetic field in terms of the integral in a finite interval, from which the infinity problem does not arise. In Sect. 4, the discussion is given. The paper is concluded in Sect. 5.

2 Quantization of free electromagnetic field in terms of finite series

2.1 Background

To examine the infinity problem rigorously and systematically, it is necessary to revisit general knowledge of the quantization of free electromagnetic field. The description below refers primarily to Greiner's one [1]. By the Lorentz condition

$$\mathrm{div}\mathbf{A} - \varepsilon_0\mu_0\frac{\partial\varphi}{\partial t} = 0,$$

the wave equations for the vector potential $\mathbf{A}$ and the scarlar potential φ are represented as

$$\triangle\mathbf{A}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{r},t) = 0, \tag{1}$$

$$\triangle\varphi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{2}$$

respectively. By introducing the following quantities: $x_4 = ict, A_4 = i\frac{\varphi}{c}$, Eqs. (1) and (2) are expressed in the four-dimensional form as

$$\sum_{\nu,\,\mu=1}^{4}\frac{\partial^2 A_\nu}{\partial x_\mu^2} = 0, \tag{3}$$

where $x_1 = x,\ x_2 = y,\ x_3 = z,\ A_1 = A_x,\ A_2 = A_y,\ A_3 = A_z$. Therefore, Eq. (3) satisfies the requirement for the relativistic invariance.

Meanwhile, the Lorentz condition is represented in the four-dimensional form as

$$\sum_{\mu=1}^{4}\frac{\partial A_\mu}{\partial x_\mu} = 0. \tag{4}$$

Thus, the Lorentz condition also satisfies the requirement for the relativistic invariance.

Let us consider whether a superposition of normal modes

$$A_\nu = \sum_\lambda a_\lambda A_{\nu,\lambda}, \tag{5}$$

satisfies the requirement for the relativistic invariance. Since every normal mode is relativistically invariant, i.e.,

$$\sum_{\nu,\,\mu=1}^{4}\frac{\partial^2 A_{\nu,\lambda}}{\partial x_\mu^2} = 0,$$

from the left side of the above equation, we have

$$\sum_\lambda a_\lambda \sum_{\nu,\,\mu=1}^{4}\frac{\partial^2 A_{\nu,\lambda}}{\partial x_\mu^2} = \sum_{\nu,\,\mu=1}^{4}\frac{\partial^2}{\partial x_\mu^2}\sum_\lambda a_\lambda A_{\nu,\lambda} = \sum_{\nu,\,\mu=1}^{4}\frac{\partial^2 A_\nu}{\partial x_\mu^2}. \tag{6}$$

Thus, we get

$$\sum_{\nu,\,\mu=1}^{4}\frac{\partial^2 A_\nu}{\partial x_\mu^2} = 0. \tag{7}$$

From this, it follows that an arbitrary superposition of normal modes satisfies the relativistic requirement. Importantly, this shows that the number of superposed normal modes is irrelevant to the relativistic invariance.

The normal modes that are solutions of Eq. (1) are determined so that the vector potential can satisfy the following boundary conditions:

$$\begin{aligned}&\mathbf{A}(L, y, z, t) = \mathbf{A}(0, y, z, t),\\ &\mathbf{A}(x, L, z, t) = \mathbf{A}(x, 0, z, t),\\ &\mathbf{A}(x, y, L, t) = \mathbf{A}(x, y, 0, t).\end{aligned} \tag{8}$$

A normal solution of Eq. (1) takes the form of

$$\mathbf{A}(x, y, z, t) = \mathbf{A}(x, y, z)e^{i\omega t} \tag{9}$$

or

$$\mathbf{A}(x, y, z, t) = \mathbf{A}(x, y, z)e^{-i\omega t}. \tag{10}$$

Substituting Eq. (9) or (10) into Eq. (1), we obtain

$$\left(\triangle + \frac{\omega^2}{c^2}\right)\mathbf{A}(x, y, z) = 0. \tag{11}$$

A normal solution of Eq. (16) is given by

$$\mathbf{A}_{k\sigma}(x, y, z) = N_{k\sigma}\varepsilon_{k\sigma}e^{i\mathbf{kx}}$$

or

$$\mathbf{A}_{k\sigma}(x, y, z) = N_{k\sigma}\varepsilon_{k\sigma}e^{-i\mathbf{kx}},$$

where $\mathbf{k}$ is the wave vector and $\varepsilon_{k\sigma}$ the polarization vector vertical to the wave vector. Therefore, for a single mode, the general solution of Eq. (1) is represented as

$$\mathbf{A}_{k\sigma}(\mathbf{x}, t) = N_{k\sigma}\varepsilon_{k\sigma}\left[a_{k\sigma}(t)e^{i\mathbf{kx}} + a^*_{k\sigma}(t)e^{-i\mathbf{kx}}\right],$$

where

$$a_{k\sigma}(t) = a^{(1)}_{k\sigma}(0)e^{-ik\omega_k t} + a^{(2)}_{k\sigma}(0)e^{ik\omega_k t}.$$

A superposition of all modes of oscillation is written as

$$\begin{aligned}\mathbf{A}(\mathbf{x}, t) &= \sum_k \sum_\sigma \left[a_{k\sigma}(t)\mathbf{A}_{k\sigma}(\mathbf{x}) + a^*_{k\sigma}(t)\mathbf{A}^*_{k\sigma}(\mathbf{x})\right]\\ &= \sum_k \sum_\sigma N_k\varepsilon_{k\sigma}\left[a_{k\sigma}(t)e^{i\mathbf{kx}} + a^*_{k\sigma}(t)e^{-i\mathbf{kx}}\right].\end{aligned} \tag{12}$$

Finally, we represent the vector potential as

$$\begin{aligned}\mathbf{A}(\mathbf{x}, t) = \sum_{\mathbf{k},\sigma} N_k\varepsilon_{k\sigma}&\left[a^{(1)}_{k\sigma}(0)e^{i(\mathbf{kx}-\omega_k t)} + a^{(1)*}_{k\sigma}(0)e^{-i(\mathbf{kx}-\omega_k t)}\right.\\ &\left.+a^{(2)}_{k\sigma}(0)e^{i(\mathbf{kx}+\omega_k t)} + a^{(2)*}_{k\sigma}(0)e^{-i(\mathbf{kx}+\omega_k t)}\right].\end{aligned} \tag{13}$$

This is a Fourier expansion of $\mathbf{A}(\mathbf{x}, t)$.

To expand a function into a Fourier series, the given function should be a periodic function or a function defined in a finite interval. Meanwhile, an aperiodic function defined in a finite interval should be represented by a Fourier integral. According to Eqs. (1) and (8), the vector potential $\mathbf{A}(\mathbf{x}, t)$ is a periodic function defined in a definite volume L^3. In fact, the volume is determined uniquely by taking into consideration the periodicity of an electromagnetic field or the space which the electromagnetic field occupies. Therefore, L is not factitious but objective.

In terms of the vector potential, the energy of electromagnetic field is represented as

$$\begin{aligned}H_c &= \frac{1}{8\pi}\int_{L^3} \mathrm{d}^3x\left(\mathbf{E}^2 + \mathbf{H}^2\right)\\ &= \frac{1}{8\pi}\int_{L^3} \mathrm{d}^3x\left[\frac{1}{c^2}\frac{\partial\mathbf{A}}{\partial t}\cdot\frac{\mathbf{A}^*}{\partial t} + (\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{A}^*)\right],\end{aligned} \tag{14}$$

where the subscript c means "classical". Substituting Eq. (13) into Eq. (14), we obtain

$$H_c = \sum_{k,\sigma}\frac{\omega_k^2}{4\pi c^2}N_k^2L^3(a_{k\sigma}a^*_{k\sigma} + a^*_{k\sigma}a_{k\sigma}), \tag{15}$$

where the orthogonality conditions

$$\left.\begin{aligned}&\textstyle\int_{L^3}\mathrm{d}^3x e^{i\mathbf{kx}}e^{-i\mathbf{kx}} = L^3\delta_{\mathbf{kk}'},\\ &\varepsilon_{k\sigma}\cdot\varepsilon_{k\sigma'} = \delta_{\mathbf{kk}'},\\ &\varepsilon_{k\sigma}\cdot\varepsilon_{-k\sigma'} = -(-1)^\sigma\delta_{\sigma\sigma'}\end{aligned}\right\} \tag{16}$$

were taken into consideration. Obviously, according to Eq. (14), H_c is finite. Assuming the normalization constant

$$N_k = \sqrt{\frac{2\pi\hbar c^2}{L^3\omega_k}}$$

makes a further step toward quantization, and thus the energy of electromagnetic field is represented by the sum of the energies of harmonic oscillators as

$$H_c = \frac{1}{2}\sum_{k,\sigma}\hbar\omega_k(a_{k\sigma}a^*_{k\sigma} + a^*_{k\sigma}a_{k\sigma}) \tag{17}$$

$$= \sum_{k,\sigma}\hbar\omega_k a_{k\sigma}a^*_{k\sigma}. \tag{18}$$

On a classical basis, both expressions (17) and (18) are equivalent.

Next, in order to quantize the field, it is assumed that $a_{k\sigma}$ and $a^*_{k\sigma}$ correspond to the photon annihilation and creation operators. These operators satisfy the following commutation relations.

$$\left.\begin{aligned}&\left[\hat{a}_{k\sigma}, \hat{a}^\dagger_{k'\sigma'}\right]_- = \delta_{kk'}\delta_{\sigma\sigma'}\\ &[\hat{a}_{k\sigma}, \hat{a}_{k'\sigma'}]_- = 0\\ &\left[\hat{a}^\dagger_{k\sigma}, \hat{a}^\dagger_{k'\sigma'}\right]_- = 0\end{aligned}\right\}, \tag{19}$$

where the creation operator $\hat{a}^\dagger_k$ and the annihilation operator $\hat{a}_k$ are defined as

$$\hat{a}_k \mid n_1, n_2, \cdots, n_k, \cdots\rangle = \sqrt{n_k} \mid n_1, n_2, \cdots, n_k - 1, \cdots\rangle,$$
$$\hat{a}^\dagger_k \mid n_1, n_2, \cdots, n_k, \cdots\rangle = \sqrt{n_k + 1} \mid n_1, n_2, \cdots, n_k + 1, \cdots\rangle,$$

respectively.

With the help of Eq. (19), Eqs. (17) and (18) are translated into the following operators:

$$\hat{H} = \frac{1}{2}\sum_{k,\sigma}\hbar\omega_k(\hat{a}^\dagger_{k\sigma}\hat{a}_{k\sigma} + \hat{a}_{k\sigma}\hat{a}^\dagger_{k\sigma})$$
$$= \sum_{k,\sigma}\hbar\omega_k\hat{a}^\dagger_{k\sigma}\hat{a}_{k\sigma} + \frac{1}{2}\sum_{k,\sigma}\hbar\omega_k. \quad (20)$$

and

$$\hat{H} = \sum_{k,\sigma}\hbar\omega_k\hat{a}^\dagger_{k\sigma}\hat{a}_{k\sigma}, \quad (21)$$

respectively. The second term of Eq. (20), i.e., the zero-point energy

$$E_z = \frac{1}{2}\sum_{k,\sigma}\hbar\omega_k \quad (22)$$

is infinite, since it is related to infinitely many field oscillators. This result is contradictory to the fact that the measurement-based quantity, Eq. (18) is always finite.

This is the infinity problems that arise from the quantization of free electromagnetic field. The above description shows that the quantization of any free electromagnetic field unexceptionally leads to the infinity problem. Physical quantities such as energy and momentum in a finite volume should be finite in any case. In fact, Eq. (14) is equivalent to Eq. (17) and (18), and therefore Eqs. (20) and (21) should give finite results too. On the other hand, for the Fourier series Eq. (13) to be convergent, the condition

$$\lim_{\mathbf{k}\to\infty} N_k\varepsilon_{k\sigma}\left[a^{(1)}_{k\sigma}(0)e^{i(\mathbf{kx}-\omega_k t)} + a^{(1)*}_{k\sigma}(0)e^{-i(\mathbf{kx}-\omega_k t)}\right.$$
$$\left.+a^{(2)}_{k\sigma}(0)e^{i(\mathbf{kx}+\omega_k t)} + a^{(2)*}_{k\sigma}(0)e^{-i(\mathbf{kx}+\omega_k t)}\right] = 0$$

should be satisfied. But on account of Eq. (22), this condition is not satisfied. This fact shows that the adopted scheme of quantization is incorrect. Evidently, Eq. (15) is not precise in the quantum sense but accurate in the classical sense. However, Eqs. (20) and (21) which are the outcome of quantum translation of Eq. (14) are neither precise nor accurate in the quantum sense as well as in the classical sense. This shows that the very quantum translation is problematic. Therefore, it is inevitable to examine the quantum translation, i.e., canonical quantization.

2.2 Quantum translation of classical relation in terms of finite series

The purpose of this subsection is to show why an electromagnetic field should be represented by a finite series instead of an infinite series. Let us consider the problem of the Fourier series expansion of electromagnetic field. It is a common knowledge that a periodic function or a bounded-region function can be expanded by a harmonic function of a fundamental frequency and its multiple frequencies. This is because this system of functions is orthogonal and complete. The case of electromagnetic field is the same as well. Of course, from the classical viewpoint, the vector potential can be represented by the superposition of infinitely many numbers of harmonic oscillations. But we should necessarily take into consideration the fact that in the case of quantum oscillation, every harmonic oscillation must possess a definite quantized energy. Therefore, we can easily see that since an infinite sum of definite terms is not finite, the Fourier series for the quantization of electromagnetic field should be expressed as a series of finite terms. In this case, the relation of photon $E = \hbar\omega$ helps to calculate the number of photons, and thus the Hamilton function (15) can be represented by use of the number of photons.

For the purpose of quantization, we rewrite Eq. (15) as

$$H_c = \sum_{k,\sigma}\frac{\omega_k^2}{4\pi c^2}N_k^2L^3(a_{k\sigma}a^*_{k\sigma} + a^*_{k\sigma}a_{k\sigma})$$
$$= \sum_{k,\sigma}\hbar\omega_k\frac{\omega_k}{4\pi c^2\hbar}N_k^2L^3(a_{k\sigma}a^*_{k\sigma} + a^*_{k\sigma}a_{k\sigma}).$$

At this stage, we do not insert the particle number operators for $a^*_{k\sigma}, a_{k\sigma}$, and consider $a_{k\sigma}a^*_{k\sigma}$ to be identical to $a^*_{k\sigma}a_{k\sigma}$. Then we have

$$H_c = \sum_{k,\sigma}\hbar\omega_k\left(\frac{\omega_k N_k^2 L^3|a_{k\sigma}(t)|^2}{2\pi c^2\hbar}\right). \quad (23)$$

Here, the expression

$$\tilde{n}_k = \frac{\omega_k N_k^2 L^3|a_{k\sigma}(t)|^2}{2\pi c^2\hbar} \quad (24)$$

should give the number of photons corresponding to every mode of oscillation. Then the number of photons n_k should be a definite integer.

If $\tilde{n}_k$ is smaller than 1, the responding mode obviously cannot enter the spectrum in the quantum sense even if it is possible purely in the mathematical sense. This tells us that the Fourier series expansion of electromagnetic field is an approximation in the quantum sense. Therefore, the Fourier series of free electromagnetic field at the quantum level should include only a finite number of the modes. This is the content of classical possibility and quantum impossibility concerning the Fourier series expansion of electromagnetic field.

To obtain the result satisfying the requirement of quantization, we should take an integer that is nearest to Eq. (24). That is, the number of photons is written by

$$n_k = \mathrm{int}(\tilde{n}_k) = \mathrm{int}\left(\frac{\omega_k N_k^2 L^3|a_{k\sigma}(t)|^2}{2\pi c^2\hbar}\right), \quad (25)$$

where $\mathrm{int}(x)$ stands for the operation taking an integer nearest to x.

In view of Eq. (25), the quantized Hamilton function is represented as

$$H_q = \sum_{k,\sigma} \hbar\omega_k \mathrm{int}\left(\frac{\omega_k N_k^2 L^3 |a_{k\sigma}(t)|^2}{2\pi c^2 \hbar}\right) = \sum_{k,\sigma} \hbar\omega_k n_k. \qquad (26)$$

where the subscript q refers to "quantum". In fact, Eq. (26) also is a natural and direct consequence which issues from the idea for photon. The energy of an electromagnetic field is the sum of the energies that photons in all normal modes have. According to the second quantization, Eq. (26) is transformed into

$$\hat{H} = \sum_{k,\sigma} \hbar\omega_k \hat{n}_k = \sum_{k,\sigma} \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k. \qquad (27)$$

Eq. (27) is the direct description of the fact that light consists of photons as well.

Let us consider the fact that by quantization, the series expansion of electromagnetic field gives necessarily a finite number of terms. We discuss this matter with the energy of a photon:

$$\varepsilon = \hbar\omega = \frac{hc}{\lambda}. \qquad (28)$$

Since the electromagnetic wave is in a box defined by L, in view of the periodicity condition Eq. (8), the minimum energy of a photon should be

$$\varepsilon_{\min} = \frac{hc}{L}. \qquad (29)$$

Hence, energy of a photon for every mode is given as

$$\varepsilon_i = \frac{hc}{L} i,$$

where i is an integer. Therefore, every mode has a definite energy that is not infinitesimal. Then the energy of free electromagnetic field calculated based on classical theory, H_c, should approximate to that translated based on quantum theory:

$$H_q = \sum_{i=1} \frac{hc}{L} i \cdot n_i \approx H_c. \qquad (30)$$

where n_i is the number of photons for a given oscillation mode. Then H_q should be taken so that it is nearest to H_c.

From Eq. (30), it is obvious that there exists a definite $i_{\max}$ which satisfies the following condition:

$$i_{\max} = \mathcal{N} : \min\{|H_q - H_c|\} = \min\left\{\left|\sum_{i=1}^{\mathcal{N}} \frac{hc}{L} i \cdot n_i - H_c\right|\right\}. \qquad (31)$$

According to Eq. (31), $i_{\max}$ is not factitious and is determined uniquely. The existence of $i_{\max}$ indicates that in the case of quantization, Eq. (26) always should be a finite series. In fact, if ω_k is not restricted, the energy of a single mode $\hbar\omega_k$, as an extreme case, may be greater than H_c, which is impossible. Given $i_{\max}$, the maximum energy of a photon in an electromagnetic field should be

$$\varepsilon_{\max} = \frac{hc}{L} i_{\max}. \qquad (32)$$

Thus, the number of oscillation modes is always finite and Eq. (30) should be represented as a finite series:

$$H_q = \sum_{i=1}^{i_{\max}} \frac{hc}{L} i \cdot n_i.$$

In view of the relation between angular frequency and wavelength $\omega = \frac{2\pi c}{\lambda}$, we write the minimum angular frequency as

$$\omega_{\min} = \frac{2\pi c}{L}, \qquad (33)$$

and possible angular frequencies as

$$\omega_i = \frac{2\pi c}{L} \cdot i = \omega_{\min} \cdot i, \qquad (34)$$

and the maximum angular frequency as

$$\omega_{\max} = \frac{2\pi c}{L} i_{\max} = \omega_{\min} \cdot i_{\max}. \qquad (35)$$

From the relation between wave vector and angular frequency:

$$k_i = \frac{\omega_i}{c} = \frac{2\pi}{L} \cdot i,$$

it follows that the wave vector k_i in the sum of Eq. (26) begins from $k_{\min} = \frac{2\pi}{L}$ and cannot exceed at most $k_{\max} = \frac{2\pi}{L} i_{\max}$.

It turns out that $k_{\min}$ and $k_{\max}$ determined as a result of a kind of cutoff are not factitious but objective. This tells us that the energy of electromagnetic field should be represented by taking a finite number of terms from a Fourier series. Namely,

$$H_q = \sum_{k=k_{\min},\sigma}^{k_{\max}} \hbar\omega_k n_k, \qquad (36)$$

Thus, the Hamilton operator should be represented as

$$\hat{H} = \sum_{k=k_{\min},\sigma}^{k_{\max}} \hbar\omega_k \hat{n}_k = \sum_{k=k_{\min},\sigma}^{k_{\max}} \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k. \qquad (37)$$

From this, it follows that the number of modes of free electromagnetic field is finite and although there are zero-point oscillations, in other words, even if we adopt Eq. (20), its number is finite. In this case, the Hamilton operator is represented as

$$\hat{H} = \sum_{k=k_{\min},\sigma}^{k_{\max}} \hbar\omega_k \left(\hat{n}_k + \frac{1}{2}\right) = \sum_{k=k_{\min},\sigma}^{k_{\max}} \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right).$$

Thus, the zero-point energy is

$$E_z = \sum_{k=k_{\min}}^{k_{\max}} \frac{\hbar\omega_k}{2},$$

which is finite.

It should be emphasized that it is not necessarily that the number of normal modes of electromagnetic field should be infinite in order to satisfy the relativistic requirement. As showed in Eq. (5)–(7), every mode of oscillation is invariant with respect to the Lorentz transformation which is a linear transformation. Accordingly, the relativistic invariance of the vector potential $\mathbf{A}(\mathbf{x}, t)$ as a superposition of normal modes does not depend on the number of superposed normal modes. As the simplest example, a monochromatic electromagnetic field has to be represented as a single mode, and evidently in this case, the field is invariant with respect to the Lorentz transformation.

2.3 Qualitative explanation of Casimir effect

The Casimir effect is the phenomenon showing that there is attraction between two separated metal plates in electromagnetic fields. The effect usually is explained based on the zero-point energy of free electromagnetic field. To avoid the infinity problem of the zero-point oscillation energy, some researchers rely on the view that a good conductor is a bad conductor for the light of short wavelength [1] and adopt a modification of Eq. (22):

$$E_z = \frac{1}{2}\sum_{k,\sigma} \hbar\omega_k \exp\left(-\lambda\frac{\omega_k}{c}\right), \tag{38}$$

where λ is a parameter which makes the series converge. For the final result, one makes $\lambda \to 0$. In this way, one can obtain a finite result of the zero-point energy. In all respects, it is factitious to multiply every term by the exponential function, $\exp\left(-\lambda\frac{\omega_k}{c}\right)$.

First, this approach is not consistent because an objective free electromagnetic field irrelevant to interaction is described by borrowing the conception of interaction. Next, it makes the conception of photon meaningless. To explain the Casimir effect, one modifies an infinite zero-point energy to take the form of Eq. (38). The purpose is that even if the number of the modes of oscillation is infinity, the portion which every mode of oscillation contributes to the zero-point energy could be adjusted so as for the sum to be a finite quantity. Under such a modification, the number of the modes remains also infinite. This means that the number of photons is infinite and thus a photon can possess an infinitely low energy irrespective of frequencies of harmonic oscillation. In this manner, Eq. (38) for obtaining a finite energy from an infinite one leads to the violation of the original concept of photon.

Let us explain the Casimir effect, based on the knowledge of Sect. 2.2. If we make a confined region in the electromagnetic field using two plates, the number of normal modes of electromagnetic field gets smaller than that in the semiinfinite space. This is because modes which do not satisfy the periodicity condition given by two plates are not allowed. The force exerted on a plate is determined by the difference between the light pressure between two metal plates and the light pressure in the semiinfinite space.

The possible modes of wavelengths between the two metal plates are given as

$$\lambda_n = \frac{L}{n}, \tag{39}$$

where n is integers starting from 1. Accordingly, the angular frequencies are represented as

$$\omega_n = \frac{2\pi c}{L}n. \tag{40}$$

According to Eqs. (39) and (40), the number of normal modes constituting an electromagnetic field between the two metal plates is finite and in addition gets less than that in Let us explain the Casimir effect based on the knowledge of Sect. 2.2. If we make a confined region in the electromagnetic field using two plates, the number of normal modes of electromagnetic field gets smaller than that in the semiinfinite space. This is because there are modes which does not satisfy the periodicity condition given by two plates. The force exerted on a plate is determined by the difference between the light pressure between two metal plates and the light pressure in the semiinfinite space. The decrease in the number of normal modes gives rise to the decrease of energy of electromagnetic field between two metal plates. The decrease of the energy in the space between two metal plates means the decrease of the momentum of electromagnetic field. As a result, the force proportional to the difference between the two momenta of electromagnetic fields in and out of the space between the two plates directs from the outside to the inside and as a result it seems as if the two metal plates pulled each other. This is the explanation of the Casimir effect from the point of view of our work.

3 Interaction between electromagnetic field and matter

3.1 Background

It is necessary to examine how the infinity problem arises in the case of addressing the interaction between electromagnetic field and matter. The description below refers to Greiner's one [1]. The Hamilton operator for the interaction between electron and electromagnetic field is represented as

$$\hat{H}_{\text{int}} = -\frac{e}{mc}\hat{\mathbf{p}}\cdot\hat{\mathbf{A}} + \frac{e^2}{2mc^2}\hat{\mathbf{A}}^2. \tag{41}$$

We use the following notations:

$$\hat{H}'_{\rm int} = -\frac{e}{mc}\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}$$

and

$$\hat{H}''_{\rm int} = \frac{e^2}{2mc^2}\hat{\mathbf{A}}^2.$$

Then the first order perturbation is given as

$$E_p^{(1)} = \left\langle \mathbf{p}|\hat{H}_{\rm int}|\mathbf{p}\right\rangle = \left\langle \mathbf{p}|\hat{H}''_{\rm int}|\mathbf{p}\right\rangle = \frac{e^2}{2mc^2}\left\langle \mathbf{p}|\hat{\mathbf{A}}^2|\mathbf{p}\right\rangle. \tag{42}$$

Here, we take into account that since in the first approximation, $\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}$ does not involve $\hat{a}^\dagger\hat{a}$ or $\hat{a}\hat{a}^\dagger$ describing the emission and absorption of light, it does not contribute to perturbation. But $\hat{\mathbf{A}}^2$ is able to contribute to the first approximation because it describes the simultaneous emission and absorption. With the help of Eq. (42), we calculate the first approximation to get

$$\begin{aligned} E_p^{(1)} &= \frac{e^2}{2mc^2}\sum_{k\sigma}\sum_{k'\sigma'}\left(\frac{2\pi\hbar c^2}{L^3}\right)\frac{\varepsilon_{k\sigma}\cdot\varepsilon_{k'\sigma'}}{\sqrt{\omega_k\omega_{k'}}} \\ &\times\int\frac{1}{\sqrt{L^3}}e^{-iqx}e^{i(k-k')x}\frac{1}{\sqrt{L^3}}e^{iqx}\mathrm{d}^3x\delta_{\sigma\sigma'} \\ &= \frac{e^2}{2mc^2}\sum_{k\sigma}\left(\frac{2\pi\hbar c^2}{L^3}\right)\frac{\varepsilon_{k\sigma}\cdot\varepsilon_{k'\sigma'}}{\omega_k}. \end{aligned}$$

Transforming the sum into an integral, we rewrite the above expression as

$$\begin{aligned} E_p^{(1)} &= \frac{e^2}{2mc^2}\frac{L^3}{(2\pi)^3}2\int\frac{k^2\mathrm{d}k\mathrm{d}\Omega_{\rm k}}{\omega_k}\left(\frac{2\pi\hbar c^2}{L^3}\right) \\ &= \frac{e^2\hbar}{\pi mc}\int_0^\infty k\mathrm{d}k = \infty. \end{aligned} \tag{43}$$

Obviously, the contribution to the energy of the free electron is infinite irrespective of the electron momentum $\mathbf{p}$. This relation is identical for all electrons. Thus, we are faced with the first case of the infinity problem $\int_0^\infty k\mathrm{d}k = \infty$.

Next, let us consider the second approximation of $\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}$. The second-order perturbation is represented as

$$E_p^{(2)} = \sum_I\frac{\left\langle p|\hat{H}'_{\rm int}|I\right\rangle\left\langle I|\hat{H}'_{\rm int}|p\right\rangle}{E_p - E_I},$$

where $\hat{\mathbf{A}}$ is linear in the creation and annihilation operator for photon, so the intermediate state $|I\rangle$ must contain only a photon. Consequently, the state function for electron is represented as

$$|I\rangle = b_q^\dagger|0\rangle_p\hat{a}_{k\sigma}^\dagger|0\rangle_{\rm rad},$$

$$|p\rangle = b_q^\dagger|0\rangle_p,$$

respectively. In addition, we have

$$E_p^{(0)} = \frac{(\hbar k)^2}{2m}$$

and

$$E_I = \frac{(\hbar q)^2}{2m} + \hbar\omega_k.$$

The second-order perturbation is computed to be

$$\begin{aligned} E_p^{(2)} &= \sum_I\frac{\left\langle p|\hat{H}'_{\rm int}|I\right\rangle\left\langle I|\hat{H}'_{\rm int}|p\right\rangle}{E_p - E_I} = \frac{e^2}{m^2c^2}\sum_I\frac{\left|\left\langle p|\hat{H}'_{\rm int}|I\right\rangle\right|^2}{E_p - E_I} \\ &= \frac{(\hbar k)^2}{2m}\left(-\frac{e^2}{mc^2\pi}\int_{-1}^{1}\mathrm{d}(\cos\theta)(1-\cos\theta)\times\right. \\ &\left.\int_0^\infty\frac{\mathrm{d}k}{1-\frac{v}{c}\cos\theta+\frac{\hbar kc}{2mc^2}}\right). \end{aligned} \tag{44}$$

Evidently, the last integral with respect to k diverges logarithmically, since for large k, it behaves like $\int\frac{\mathrm{d}k}{k}$. It is obvious that the self-energy $E_p^{(2)}$ is divergent. As a result, we encounter the second case of the infinity problem, $\int\frac{\mathrm{d}k}{k} = \infty$. Introducing the notation:

$$D = \frac{e^2}{mc^2\pi}\int_{-1}^{1}\mathrm{d}(\cos\theta)(1-\cos\theta)\int_0^\infty\frac{\mathrm{d}k}{1-\frac{v}{c}\cos\theta+\frac{\hbar kc}{2mc^2}}, \tag{45}$$

we have

$$E_p = E_p^{(0)} + E_p^{(2)} = \frac{(\hbar k)}{2m}(1-D).$$

Making the nonrelativistic approximation $\frac{v}{c} \ll 1$ and $\frac{\hbar\omega}{mc^2} \ll 1$ in D yields

$$D = \frac{4}{3}\frac{e^2}{mc^2\pi}\int_0^\infty\mathrm{d}k. \tag{46}$$

Thus, the energy of an electron is represented as

$$E_p = \frac{(\hbar k)^2}{2m}\left(1-\frac{4}{3}\frac{e^2}{mc^2\pi}\int_0^\infty\mathrm{d}k\right). \tag{47}$$

Then we are faced with the third case of the infinity problem, $\int\mathrm{d}k = \infty$. Relation $E_p^{(0)} = \frac{(\hbar k)^2}{2m}$ represents the bare mass of an electron. From Eq. (47), the experimental mass of an electron should be represented as

$$\frac{1}{m_{\rm exp}} = \frac{1}{m}(1-D). \tag{48}$$

Thus, the relation between the fictitious mass m and the experimental mass m_{exp} of an electron is represented as

$$m_{\mathrm{exp}} = \frac{m}{1 - D}. \tag{49}$$

This should be finite because in experiments, the mass of an electron is determined to be finite. Since D is infinite, we are forced to consider that the mass of an electron, m, is fictitious, and thus change it so as for the experimental mass to be the real mass. In this manner, the self-energy which diverges is interpreted reasonably using the method of renormalization. As a whole, the kinds of infinity concerning the application of the perturbation theory are categorized as

$$\int_0^\infty \frac{1}{k}\mathrm{d}k, \qquad \int_0^\infty k\mathrm{d}k, \qquad \int_0^\infty \mathrm{d}k. \tag{50}$$

Namely, it involves the logarithmic, squared and linear divergence.

3.2 Representation of perturbation in terms of integral in a finite interval

It is clear that the divergent integrals such as Eq. (50) are not consistent with the description of subsection 2.2. As in the quantization of free electromagnetic field, it is necessary to take the interval of integration not as the interval from zero to infinity but as a finite interval from k_{min} to k_{max} in view of the quantum condition. This is the correct key for solving the infinity problem. Accordingly, the divergent integrals in Eq. (50) are rebuilt as the integrals in the finite interval:

$$\int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} \frac{1}{k}\mathrm{d}k, \qquad \int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} k\mathrm{d}k, \qquad \int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} \mathrm{d}k,$$

which obviously become finite. Thus, such a treatment makes the problem of mass renormalization useless. It is possible to newly interpret the meaning of Eq. (47). With the help of the integral in a finite interval, Eq. (47) is recast as

$$\begin{aligned} E_p &= \frac{(\hbar k)^2}{2m}\left(1 - \frac{4}{3}\frac{e^2}{mc^2\pi}\int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} \mathrm{d}k\right) \\ &= \frac{(\hbar k)^2}{2m}\left[1 - \frac{4}{3}\frac{e^2}{mc^2\pi}(k_{\mathrm{max}} - k_{\mathrm{min}})\right] = \frac{(\hbar k')^2}{2m}, \end{aligned} \tag{51}$$

where

$$k' = k\sqrt{1 - \frac{4}{3}\frac{e^2}{mc^2\pi}(k_{\mathrm{max}} - k_{\mathrm{min}})} \ .$$

From this, the effect of the perturbation should be interpreted as giving rise to not the variation in mass but the change of wave vector. Ultimately, we come to understand the cause of infinity and the advantage of our method over renormalization.

4 Discussion

In this work, we have shown that a simple physical and mathematical viewpoint can lead to the perfect solution of the infinity problem of electromagnetic field without relying on renormalization theory. According to the conventional quantum field theory, it seems to be inevitable that the quantization of free electromagnetic field which really possesses a finite energy in a finite volume leads to unreasonable results such as the infinite energy due to the zero-point energy originating from infinitely many spectrum terms. We began with the classical Hamilton function to quantize electromagnetic fields. The quantization, in essence, is a quantum translation of the classical representation. Surprisingly, the objectively finite quantities in the classical sense are translated into infinite ones in the quantum sense. It is a conventional view that quantum theory can give more exact description of physical phenomena than classical theory can do, and can solve a lot of problems that are impossible to be solved relying on classical theory. However, we are faced with such a serious situation that looks like unreasonable quantum results versus reasonable classical results. Since the Maxwell equation is correct at the classical level, the classical representation of free electromagnetic field is absolutely correct, if slight deviations relevant to the quantum realm are ignored. This situation shows an unreasonable aspect of the quantization of electromagnetic field. As the biggest challenge to quantum field theory, the infinity problem shows that the theory still has been unfinished, and emphasizes the necessity of encouraging significant innovations.

The energy relation of photon, $E = \hbar\omega$, clearly is the foundation of the quantization of electromagnetic field. Based on this assumption, the vector potential of a free electromagnetic field is represented necessarily by the superposition of a finite number of normal modes. Our work has demonstrated that the series expansion of the vector potential by means of a finite number of normal modes based on the relation of photon becomes a key to the solutions of the infinity problem. This leads to the obvious conclusion that there is not the infinity problem in the quantization of free electromagnetic field.

Obviously, there is an important difference between the Fourier expansion of free electromagnetic fields and its physical possibility in the quantum sense. Although the Fourier expansion of an electromagnetic field is possible in the purely mathematical sense, in the quantum sense it is not possible, since the composition of normal modes which requires lower energy than that of a photon is not allowed. This is because the electromagnetic field is composed of photons. Therefore, the series expansion of an electromagnetic field based on quantum theory necessarily should be represented by taking a finite number of terms from the Fourier series. In other words, the electromagnetic field in a bounded space always should be composed of a finite number of normal modes with a finite number of photons. All terms of the series expansion correspond to a fundamental frequency Eq. (33) and its integer multiples which should satisfy Eq. (34). This relation is clear from a rudimentary knowledge about the Fourier series. Therefore, quantized electromagnetic fields should be expanded by a finite number of normal modes. In the end, the infinite energy including infinitely many terms of zero-point energies is meaningless. Even if there is the zero-point energy, the number of its modes is finite, so the zero-point energy is finite. Since every term of the series is finite and the number of terms of series too is finite, the energy of an electromagnetic field is never infinite. This view offers the key for solving the infinity problem of free electromagnetic fields.

It is important to examine the rule of quantization in terms of Eq. (17). Purely from the mathematical point of view, $a_{k\sigma}a^*_{k\sigma}$ is identical to $a^*_{k\sigma}a_{k\sigma}$. However, they are considered to be not identical in consideration of the ordering of the amplitudes. Thus, the problem of the zero-point energy arises. It is not possible that an arbitrary confined electromagnetic field always has an infinite zero-point energy. In fact, the infinity problem arises from the application of the operator, $\hat{a}_k\hat{a}^\dagger_k + \hat{a}^\dagger_k\hat{a}_k$. From the aspect of logic, it is not reasonable that the quantum translation starting with classical relations makes the physical world of the zero-point energy emerge and the infinite energy be produced.

On the other hand, the simple consideration of the energy relation of photon gives the number of photons as determined by Eq. (25). Such a physical interpretation does not cause the problem of infinite energy. In this connection, it is necessary to recall how we treat the problem of the radiation and absorption by atom based on the nonstationary perturbation theory. Here, it is considered that the energy of an electromagnetic field in the classical sense:

$$H_c = \left\langle \varepsilon_0 E^2 \right\rangle = 2\varepsilon_0 \mathbf{A}_0^2 \omega^2 \quad (52)$$

should be approximately equal to the quantum energy

$$H_q = n\hbar\omega,$$

namely,

$$H_c = H_q,$$

provided one ignores the deviation due to quantum reason. Therefore, the vector potential and the density of photons should satisfy the following relation

$$2\varepsilon_0 \mathbf{A}_0^2 \omega^2 = n\hbar\omega. \quad (53)$$

From Eq. (53) does not come the infinity problem concerning the quantum treatment of electromagnetic field. Thus, it is concluded that the appearance of the zero-point energy is ascribed purely to the application of the particle number operator and the series expansion, and not due to the problem of physical origin.

In this work, we have shown that it is possible to reconsider the Casimir effect from a new aspect and the effect is in no way related to the zero-point energy. Without considering the zero-point energy, the Casimir effect has been successfully explained.

Our perspective could provide a new foundation capable of resolving the infinity problems of free electromagnetic field in a general way without renormalization. As far as we apply within the framework of the conventional mathematical formalism the perturbation theory to the study on the interaction between electromagnetic field and matter, we cannot avoid the infinity problem. Purely from the mathematical aspect, it is illogic to derive a finite quantity from infinity.

Our formulation makes the infinity problem be solved easily by a finite series and an integral in a finite interval which does not comprise zero and infinity. This shows that it is possible to solve the infinity problem of free electromagnetic field in a simple way. Clearly, there is a possibility of constructing an alternative formulation without renormalization, thus overcoming the infinity problem in principle.

We believe that our formulation is simple but it could contribute to overcoming serious difficulties present in quantum field theory and to renovating the theory.

5 Conclusions

In this work, we have presented an alternative method for quantizing free electromagnetic fields without infinity. An important feature of our formalism is that it is a simple way that does not need any assumptions except for the hypothesis about photon and importantly does not rely on renormalization theory.

Our work has given clear physical and mathematical reasons to represent free electromagnetic fields in terms of a finite series or an integral in a finite interval. Our approach solves the infinity problem without renormalization by replacing the integral of perturbative terms in the infinite interval $[0, \infty)$ with the integral in a finite interval $[k_{\min}, k_{\max}]$. Therefore, this approach is a kind of cut-off one. Our approach clarifies the physical conditions of cut-off in principle and moreover satisfies the relativistic requirement, so is essentially distinguished with the previous ones. Our consideration with sufficient reasons show that the infinity problem at issue does not exist and it is a technical subject associated with method of mathematical treatment. Thus, our methodology keeps the quantization of free electromagnetic fields from infinity.

Our explanation of the Casimir effect shows that our method for quantizing free electromagnetic field is reasonable. In fact, the adopted approach is simple but enables us to solve easily such a big problem that requires one to derive a finite result from an infinite zero-point energy. Based on our work, it is clarified that the problem of zero-point energy has the origin associated purely with mathematical methods and has nothing to do with physics. We believe that our method will offer a new possibility of constructing quantum electrodynamics free from infinity, since the proposed method reflects physical reality reasonably.

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