Green function in the constant magnetic field

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Abstract

We derive the propagator of the spin 1/2 particle in a constant external magnetic field using the specific mathematical techniques by Schwinger, Wu yang Tsai and others.

1 Introduction

Green's functions are named after the British mathematician George Green, who first developed the concept (Green, 2008) in the 1828. In the modern study of linear partial differential equations, Green's functions are studied largely from the point of view of fundamental solutions instead. Under many-body theory, the term is also used in physics, specifically in quantum field theory, aerodynamics, aeroacoustics, electrodynamics, seismology and statistical field theory, to refer to various types of correlation functions, even those that do not fit the mathematical definition. In quantum field theory, Green's functions take the roles of propagators.

In mathematics, the Green function is the impulse response of an inhomogeneous linear differential operator defined on a domain with specified initial conditions or boundary conditions.

A Green's function, $G(x, x')$, of a linear differential operator $L = L(x)$ is any solution of $LG(x, x') = \delta(x - x')$. We here apply this definition in quantum electrodynamics with the calculation of the Green function of an electron moving in the constant magnetic field.

2 The Green function of electron in a constant magnetic field

We derive the propagator of the spin 1/2 particle in a constant external magnetic field using the specific mathematical techniques by Schwinger, Wu yang Tsai and so on.

The electron propagator is a building stone in the mass operator from which can be computed the energy shift, the power spectrum of the synchrotron radiation, the anomalous magnetic moment and so on. It was intensively discussed for instance by Dittrich et al. (1985). Here we will follows the Dittrich treatment in order to derive the special representation of the Dirac propagator of a particle in a constant external magnetic field.

If we write the Green function equation as

$$G_+(x,x') = \langle x|G_+|x'\rangle, \tag{1}$$

then, from the Green function equation for spin 1/2 particles

$$(\gamma\Pi + m)G_+(x,x') = \delta(x-x') = \langle x|x'\rangle \tag{2}$$

we have

$$G_+ = \frac{1}{\gamma\Pi + m - i\varepsilon}, \tag{3}$$

where

$$\Pi_\mu = p_\mu - eA_\mu, \quad p_\mu = \frac{1}{i}\partial_\mu \tag{4}$$

with simultaneous omitting the charge matrix q in eqs. (3) and (4).

The equivalent form of G_+ in eq. (3) is obviously given by relation

$$G_+ = \frac{\gamma\Pi - m}{(\gamma\Pi)^2 - m^2 - i\varepsilon}, \tag{5}$$

where

$$(\gamma\Pi)^2 = -\Pi^2 - \frac{i}{2}\sigma_{\mu\nu}[\Pi_\mu, \Pi_\nu] \tag{6}$$

with

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{7}$$

and

$$[\Pi^\mu, \Pi^\nu] = ieF^{\mu\nu}, \tag{8}$$

which gives

$$(\gamma\Pi)^2 = -\Pi^2 + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}. \tag{9}$$

For the constant magnetic field chosen in the z-direction we have

$$F_{12} = -F_{21} = B = const \tag{10}$$

and

$$(\gamma\Pi)^2 = -\Pi^2 - eB\sigma^{12}, \tag{11}$$

where we designated by σ^{12} the following matrix:

$$\sigma^{12} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{12}$$

Using the last formulas, we write

$$G_+ = \frac{m - \gamma\Pi}{\Pi^2 - \kappa^2 - i\varepsilon}, \tag{13}$$

where

$$\kappa^2 \stackrel{d}{=} m^2 - eB\sigma^3 \tag{14}$$

and

$$G_+(x', x''|A) = \langle x'|\frac{m - \gamma\Pi}{\Pi^2 - \kappa^2 - i\varepsilon}|x''\rangle. \tag{15}$$

Now, we can evaluate the formula (15), or, solve eq. (2). Let us solve eq. (2). We use the ansatz

$$G_+(x', x''|A) = \Phi(x', x'')\left[m - \gamma^\nu\left(\frac{1}{i}\partial'_\nu - eA_\nu(x')\right)\right]\Delta_+(x', x''|A) \tag{16}$$

with

$$\Phi(x', x'') = \exp\left\{ie\int_{x''}^{x'} dx_\mu[A^\mu(x) + \frac{1}{2}F^{\mu\nu}(x'_\nu - x''_\nu)]\right\}, \tag{17}$$

where the integral is not dependent of the choice of the integration path because of vanishing rot of the integrand. If we choose the integration path in the form of the straight line

$$x(t) = x'' + t(x' - x''); \quad t \in < 0, 1 >, \tag{18}$$

we find that the second term of the integrand gives no contribution and we have instead of eq. (17):

$$\Phi(x', x'') = \exp\left\{ie\int_{x''}^{x'} dx_\mu A^\mu(x)\right\} \tag{19}$$

and the derivatives are as follows:

$$\frac{\partial\Phi(x',x'')}{\partial x'^{\mu}} = ie[A_\mu(x') - A'_\mu(x')]\Phi(x',x''), \tag{20}$$

where

$$A'_\mu(x')] = -\frac{1}{2}F_{\mu\nu}(x'-x'')^\nu. \tag{21}$$

After substitution of the ansatz (16) into eq. (2) we get

$$\left((\frac{1}{i}\partial' - eA')^2 + \kappa^2\right)\Delta_+(x',x''|A) = \delta(x'-x''), \tag{22}$$

where A' is defined by eq. (21). After modification of eq. (22) we have

$$(-\partial^2 + \kappa^2 - i\varepsilon - \frac{e^2}{4}x_\mu F^{2\mu\nu}x_\nu)\Delta_+(x|A') = \delta(x) \tag{23}$$

with

$$F^{2\mu\nu} = F^\mu_{\ \alpha}F^{\alpha\nu}. \tag{24}$$

We can solve the equation (23) by using the Fourier transform

$$\Delta_+(x|A') = \int\frac{(dk)}{(2\pi)^4}\Delta_+(k|A'), \tag{25}$$

which gives

$$\left(k^2 + \frac{e^2}{4}\frac{\partial}{\partial k^\mu}F^{2\mu\nu}\frac{\partial}{\partial k^\nu} + \kappa^2 - i\varepsilon\right)\Delta_+(k|A') = 1. \tag{26}$$

Using the ansatz

$$\Delta_+(x|A') = i\int_0^\infty ds e^{-M(is)}e^{-is(\kappa^2 - is)} \tag{27}$$

with

$$M(is) = k^\alpha X_{\alpha\beta}(is)k^\beta + Y(is); \quad X_{\alpha\beta} = X_{\beta\alpha}, \tag{28}$$

we get

$$i\int_0^\infty ds\left\{k\left[1 + e^2X(is)F^2X(is)\right]k - \frac{e^2}{2}\mathrm{tr}\,[F^2X] + \kappa^2 - i\varepsilon\right\} \quad \times$$

$$e^{-M(is)}e^{-is(\kappa^2 - is)} = 1, \tag{29}$$

or, in the equivalent form

$$i\int_0^\infty ds g(is)e^{-f(is)} = 1, \tag{30}$$

which is equation for X and Y.

Let us try to put

$$g(is) = f'(is). \tag{31}$$

Then,

$$i\int_0^\infty dsg(is)e^{-f(is)} = e^{-f(0)} - e^{-f(i\infty)}. \tag{32}$$

From the comparison of eq. (32) with eq. (30) the requirement follows

$$f(0) = 0; \quad \mathrm{Re}\ f(i\infty) = \infty. \tag{33}$$

The relation $g = f'$ reads in our case

$$k\left[1 + e^2X(is)F^2X(is)\right]k - \frac{e^2}{2}\mathrm{tr}\ [F^2X(is)] = M'(is), \tag{34}$$

which enables to write after rotation of the integration path according to $is \to s$, the following equations:

$$1 + e^2X(s)F^2(s) = \dot{X}(s) \tag{35}$$

$$-\frac{e^2}{2}\mathrm{tr}\ [F^2X(s)] = \dot{Y}(s). \tag{36}$$

The solutions of eqs. (35) and (36) are

$$X(s) = (eF)^{-1}\tan(eFs) \tag{37}$$

$$Y(s) = \frac{1}{2}\mathrm{tr}\ \ln\cos(eFs), \tag{38}$$

which can be verified by differentiation. At the same time it can be verified the first condition in (33) is fulfilled because of $X(0) = Y(0) = 0$.

To write $X(s)$ and $Y(s)$ in the explicit form, we use the advantage of the special form of the strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{39a}$$

$$(iF)^2 = B^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{39b}$$

Then, we have

$$X(is) = (eF)^{-1}\tan(ieFs) \quad =$$

$$is\left[\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\frac{\tan(eBs)}{(eBs)}\right]. \quad (40)$$

Now, let us introduce the notation for the specific vectors as follows

$$a_{\|}\stackrel{d}{=}(a^0,0,0,a^3);\quad a_{\perp}\stackrel{d}{=}(0,a^1,a^2,0) \quad (41)$$

$$(ab)_{\|}\stackrel{d}{=}-a^0b^0+a^3b^3\quad (ab)_{\perp}\stackrel{d}{=}a^1b^1+a^2b^2. \quad (42)$$

Then, ($X=X_{\alpha}^{\beta}$)

$$kX(is)k\ =is\left[k_{\|}^2+\frac{\tan(eBs)}{eBs}k_{\perp}^2\right]. \quad (43)$$

For $Y(is)$ we get by the similar way

$$Y(is)=\frac{1}{2}\text{tr }\ln\cos(ieFs)\quad =$$

$$\frac{1}{2}\ln\det\left[\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\cos(eBs)\right]\quad =$$

$$\ln\cos(eBs). \quad (44)$$

At this stage we are prepared to compute $\exp(-f(i\infty))$. Let us recall that

$$f(is)=M(is)+is(\kappa^2-is)=kX(is)k+Y(is)+is(\kappa^2-is). \quad (45)$$

Then,

$$e^{-f(i\infty)}=\lim_{s\to\infty}e^{-f(is)}\quad =$$

$$\lim_{s\to\infty}\exp\left\{-is\left(k_{\|}^2+k_{\perp}^2\ \frac{\tan(eBs)}{eBs}\right)\right\}\frac{e^{-is(\kappa^2-is)}}{\cos(eBs)}=0. \quad (46)$$

If we introduce

$$z=eBs, \quad (47)$$

we get from eqs. (25), (28), and (38)

$$\Delta(x',x''|A)=i\int_0^{\infty}ds\Phi(x',x'')\int\frac{(dk)}{(2\pi)^4}e^{ik(x'-x'')}\quad\times$$

$$\frac{1}{\cos z}\exp\left\{-is\left(k_{\|}^2+k_{\perp}^2\ \frac{\tan(z)}{z}\right)\right\}e^{-is(\kappa^2-is)}. \quad (48)$$

However, because of

$$\Delta_+(x', x''|A) = \langle x'|\frac{1}{\Pi^2 + \kappa^2 - i\varepsilon}|x''\rangle \quad =$$

$$i\int_0^\infty ds \langle x'|e^{-is\Pi^2}|x''\rangle e^{-is(\kappa^2 - is)}, \tag{49}$$

we get after comparison of eq. (49) with eq. (48)

$$\langle x'|e^{-is\Pi^2}|x''\rangle \quad =$$

$$\Phi(x', x'') \int \frac{(dk)}{(2\pi)^4} e^{ik(x'-x'')} \frac{1}{\cos z} e^{-is(k_\parallel^2 + k_\perp^2 \frac{\tan(z)}{z})}. \tag{50}$$

The formula (50) may be used as a starting point in derivation of the further formulas. For instance, putting $s \to a_\perp s$ in eq. (50) we get

$$\langle x'|e^{-isa_\perp\Pi^2}|x''\rangle \quad =$$

$$\Phi(x', x'') \int \frac{(dk)}{(2\pi)^4} e^{ik(x'-x'')} \frac{1}{\cos(a_\perp z} \exp\left\{-is\left(k_\parallel^2 + k_\perp^2 \frac{\tan(a_\perp z)}{a_\perp z}\right) a_\perp\right\}. \tag{51}$$

Further, because of

$$\langle x'|e^{-isa_\perp\Pi^2}\Pi_\mu|x''\rangle = (i\partial'' - eA(x''))_\mu \langle x'|e^{-isa_\perp\Pi^2}|x''\rangle, \tag{52}$$

we get

$$\langle x'|\exp\left\{-is\left(a^0\Pi_0\Pi^0 + a^3\Pi_3\Pi^3 + a_\perp\Pi_\perp^2\right)\right\}|x''\rangle \quad =$$

$$\langle x'|e^{-isa_\perp\Pi^2} \exp\left\{-is\left((a^0 - a_\perp)\Pi_0\Pi^0 + (a^3 - a_\perp)\Pi_3\Pi^3\right)\right\}|x''\rangle \quad =$$

$$\exp\left\{-is\left(-(a^0 - a_\perp)\left(i\partial''^0 - eA^0(x'')\right)^2 + (a^3 - a_\perp)\left(i\partial''^3 - eA^3(x'')\right)^2\right)\right\}$$

$$\times \quad \langle x'|e^{-isa_\perp\Pi^2}|x''\rangle. \tag{53}$$

Using

$$\partial''^\mu \Phi(x', x'') = -ie\left(A^\mu(x'') - \frac{1}{2}F^{\mu\nu}(x' - x'')_\nu\right)\Phi(x', x'') \tag{54}$$

and

$$\exp\left\{-is\left(-(a^0 - a_\perp)\left(i\partial''^0\right)^2 + (a^3 - a_\perp)\left(i\partial''^3\right)^2\right)\right\} e^{ik(x'-x'')} \quad =$$

$$\exp\left\{-is\left(-(a^0 - a_\perp)\left(k^0\right)^2 + (a^3 - a_\perp)\left(k^3\right)^2\right)\right\} e^{ik(x'-x'')}, \tag{55}$$

we get from eqs. (52) and (53)

$$\langle x'|\exp\left\{-is\left(a^0\Pi_0\Pi^0+a^3\Pi_3\Pi^3+a_\perp\Pi_\perp^2\right)\right\}\Pi^\mu|x''\rangle \quad =$$

$$\Phi(x',x'')\int\frac{(dk)}{(2\pi)^4}e^{ik(x'-x'')} \quad \times$$

$$\frac{1}{\cos(a_\perp z)}\exp\left\{-is\left(a^0k_0k^0+a^3k_3k^3+a_\perp k_\perp^2\frac{\tan(a_\perp)z}{a_\perp z}\right)\right\}. \tag{56}$$

Similarly, we can derive

$$\langle x'|\exp\left\{-is\left(a^0\Pi_0\Pi^0+a^3\Pi_3\Pi^3+a_\perp\Pi_\perp^2\right)\right\}|x''\rangle \quad =$$

$$\Phi(x',x'')\int\frac{(dk)}{(2\pi)^4}e^{ik(x'-x'')}\frac{1}{\cos(a_\perp z)}\left[k^\mu-ea_\perp sF^{\mu\nu}k_\nu\frac{\tan(a_\perp z)}{a_\perp z}\right] \quad \times$$

$$\exp\left\{-is\left(a^0k_0k^0+a^3k_3k^3+a_\perp k_\perp^2\frac{\tan(a_\perp)z}{a_\perp z}\right)\right\}. \tag{57}$$

From this equation and using

$$\gamma_\mu F^{\mu\nu}k_\nu\tan(a_\perp z)=(\gamma_1k_2-\gamma_2k_1)B\tan(a_\perp z) \quad =$$

$$\frac{i}{\cos(a_\perp z)}B\sin(a_\perp z\sigma^3)(\gamma k)_\perp \tag{58}$$

and

$$(k\gamma)_\perp-esa_\perp(\gamma_\mu F^{\mu\nu}k_\nu)\frac{\tan(a_\perp z)}{a_\perp z}=\frac{1}{\cos(a_\perp z)}e^{-ia_\perp z\sigma^3}(\gamma k)_\perp, \tag{59}$$

we get the following relations

$$\langle x'|\exp\left\{-is\left(a^0\Pi_0\Pi^0+a^3\Pi_3\Pi^3+a_\perp\Pi_\perp^2\right)\right\}\left(1,\gamma^0\Pi^0,\gamma^3\Pi^3,(\gamma\Pi)_\perp\right)|x''\rangle \quad =$$

$$\Phi(x',x'')\int\frac{(dk)}{(2\pi)^4}e^{ik(x'-x'')}\frac{1}{\cos(eBsa_\perp z)} \quad \times$$

$$\exp\left\{-is\left(a^0k_0k^0+a^3k_3k^3+a_\perp k_\perp^2\frac{\tan(eBsa_\perp)}{eBsa_\perp}k_\perp^2\right)\right\} \quad \times$$

$$\left(1,\gamma^0k^0,\gamma_3k^3,\frac{1}{\cos(eBsa_\perp)}e^{-ieBsa_\perp\sigma^3}(\gamma k)_\perp\right). \tag{60}$$

For the propagator $G_+(x',x'')$ which we write in the form

$$G_+(x',x')=\langle x'|\frac{m-\gamma\Pi}{\Pi^2-\kappa^2-i\varepsilon}|x''\rangle \quad =$$

$$i\int_0^\infty dse^{-is(\kappa^2-is)}\langle x'|e^{-is\Pi^2}(m-\gamma\Pi)|x''\rangle e^{-is(\kappa^2-is)}, \tag{61}$$

we get

$$G_+(x',x') = \Phi(x',x'')\int \frac{(dk)}{(2\pi)^4}e^{ik(x'-x'')}G_+(k), \tag{62}$$

where

$$G(k) = i\int_0^\infty ds\exp\left\{-is\left(m^2 - i\varepsilon + k_\parallel^2 + \frac{\tan z}{z}k_\perp^2\right)\right\} \quad \times$$

$$\frac{e^{i\sigma^3 z}}{\cos z}\left(m - \gamma k_\parallel - \frac{e^{-i\sigma^3 z}}{\cos z}\gamma k_\perp\right). \tag{63}$$

This representation of the Green function of electron in the constant magnetic field will be used in calculation of the mass operator and the polarization tensor in the constant magnetic field.

3 Discussion

The standard explanation of the Green function which differs from the new Schwinger approach can be found in the famous monograph and textbook on quantum electrodynamics (Akhiezer et al., 1965; Berestetskii et al., 1982). The introduction to the Schwinger source theory is presented in the well-known monographs (Dittrich, 1978; Dittrich at al., 1985; Schwinger, 1969; 1970; 1973; 1989).

The purpose of this paper was to present a complete and explicit result by using a different approach that is an extension of the simple and transparent method proposed by Tsai and Erber (1974) to calculate the photon mass operator in an external homogeneous magnetic field (Tsai et al., 1974).

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