

Einstein Equations from a Timeless Euclidean Model: Operational Reconstruction and the Compensation Principle

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Abstract

We derive the equations of General Relativity as emergent from a timeless Euclidean model on \mathbb{E}^4 with a single fundamental field Φ satisfying $\Delta_{\mathbb{E}^4}\Phi = 0$. Building on the operational reconstruction of Lorentz transformations for observable transformations M on a fixed slice, we allow slow variations of the foliation direction $n_A(x)$ and introduce the effective metric $g_{AB} = -h_{AB} + v_{\max}^{-2}n_A n_B$ with $h_{AB} = \delta_{AB} - n_A n_B$. The compensation principle $\delta S_{\text{eff}} + \delta S_g = 0$, under locality, diffeomorphism invariance, and second order in derivatives, together with the invariance of the null cone under M on the slice, uniquely fixes the second-order local density; via the Gauss–Codazzi identities it is equivalent to the Einstein–Hilbert action with the Gibbons–Hawking–York boundary term. Variation of the total action yields $G_{AB} + \Lambda g_{AB} = 8\pi G T_{AB}^{\text{eff}}$ with $\nabla^A T_{AB}^{\text{eff}} = 0$; the normalization of G is set by the Newtonian limit. The requirement of causal reconstruction implies the scale hierarchy $L_{\text{field}} \ll L_{\text{fol}}$ and the recovery of the Newtonian limit, as well as the universality of gravity: all effective fields couple minimally through the same g , which renders field-dependent cones observationally inadmissible and ensures falsifiability. In addition, we obtain a local upper bound on the hierarchy parameter $\varepsilon = L_{\text{field}}/L_{\text{fol}} \leq \varepsilon_{\max} \ll 1$ that yields testable bounds on deviations from the SR/GR regime.

Contents

1 Introduction

3

1.1	Background and Motivation	4
1.2	From Lorentz Transformations to Curvature	4
1.3	Goal and Scope	5
2	Fundamental Euclidean Model and Foliation Structure	6
2.1	Laplace Equation and Admissible Configurations	6
2.2	Local Foliations and Induced Geometry	6
2.3	Curvature from Variations of the Normal Field	8
3	Operational Consistency and the Equivalence Principle	10
3.1	Effective Fields as Operational Degrees of Freedom	10
3.2	Local Consistency of Observable Transformations on a Slice	11
3.3	Acceleration as a Tilt of the Foliation and the Equivalence Principle	12
4	Variational Derivation of Einstein’s Equations	13
4.1	Constraints and Symmetries of the Variational Setup	14
4.2	Compensation Principle and Variational Setup	15
4.3	General Form of the Lagrangian on the Foliation	17
4.4	Gauss–Codazzi Decomposition and the Passage to the Einstein–Hilbert Action	19
4.5	Variations and Equations	20
4.6	Noether Energy–Momentum Currents in the SR Regime	20
4.7	Energy–Momentum from T_{AB}^{eff} : Densities and Killing Charges	21
4.8	Checks: SR Regime and Newtonian Limit	22
4.9	Section Summary	23
5	Scale Hierarchy and the Weak-Field Regime	23
5.1	Two-Scale Separation	23
5.2	Weak Field and the Newtonian Limit	24
5.3	Structural Origin of the Hierarchy	25
5.4	Ban on Strong Fields in Observable Regions	25
6	Discussion and Outlook	26
7	Conclusion	28
A	Hypersurface-Deformation Algebra and Fixing the Coefficients	30
A.1	Geometry of deformations	30
A.2	Canonical variables and constraints	30
A.3	Closure of the algebra and fixing the coefficients	30

1 Introduction

In the previous work [1] we studied a timeless Euclidean model on \mathbb{E}^4 with a real scalar field $\Phi(x)$ satisfying the Laplace equation

$$\Delta_{\mathbb{E}^4}\Phi = 0, \quad \Delta_{\mathbb{E}^4} := \delta^{AB}\partial_A\partial_B,$$

on a class of admissible configurations defined by the observer’s operational constraints. No explicit solutions were constructed; rather, properties of the class were analyzed. We obtained:

- a foliation together with a mode decomposition gives rise to an *operational* inertial frame (IFR) with its own event structure, causality, and inertia;
- two types of transformations between IFRs were distinguished: *direct* (geometric, acting on global configurations) and *observable* (preserving the event structure in the observer’s description);
- a proof that events lying outside the observer’s own event structure are unreachable;
- an operational reconstruction of the postulates of SR: the equivalence of all IFRs and the existence of a finite IFR-invariant limiting speed v_{\max} ;
- a demonstration that observable transformations take the Lorentz form with invariant v_{\max} ; the Galilean limit is excluded;
- it was established that, upon changing the IFR, the reconstructed event set may gain or lose elements; nevertheless, the causal structure remains internally consistent within each IFR.

In this paper we further develop the model and derive the equations of General Relativity. We pass from foliations by hyperplanes to the general case, where the foliation direction $\mathbf{n}(x)$ (components $n_A(x)$) may vary slowly and the slices need not be hyperplanes. The curvature of the foliation arises as a requirement of operational causal reconstruction: in order for the local consistency condition (invariance of the null cone under observable transformations M , see §3.2) to hold between neighboring slices, the geometry must adapt. In this formulation, the very requirement of consistent reconstruction serves as the source of curvature, whereas “energy–momentum” is an emergent quantity of the effective description on the slices.

Effective metric. We introduce the *effective metric form*

$$g_{AB}(x) = -h_{AB}(x) + v_{\max}^{-2} n_A(x)n_B(x), \quad h_{AB}(x) = \delta_{AB} - n_A(x)n_B(x), \quad (1)$$

where $n_A n^A = 1$, and h_{AB} is the projector onto the slice Σ_s . When $K_{ab} = 0$ (equivalently, $\mathcal{L}_n h_{ab} = 0$; for a precise definition see §4.3), the observable transformations M are strictly Lorentzian; locally the kinematics of SR is recovered with signature $\text{sig}(g) = (+, -, -, -)$ and light cone $|\mathbf{r}| = v_{\max} t$ (henceforth we identify $v_{\max} \equiv c$).

Beyond gravity, the model yields *effective fields* as modal degrees of freedom on Σ_s . Their dynamics is constructed by the principle of minimal coupling via g ; the universality of this coupling will be derived from the variational *compensation principle* $\delta S_{\text{eff}} + \delta S_g = 0$ together with the invariance of the null cone on the slices. The scale hierarchy $L_{\text{field}} \ll L_{\text{fol}}$ arises naturally and ensures the weak-field limit (see §5).

1.1 Background and Motivation

Basic setup:

- Euclidean space \mathbb{E}^4 with no fundamental time;
- a real scalar field $\Phi(x)$ satisfying the Laplace equation $\Delta_{\mathbb{E}^4} \Phi = 0$ (see (2));
- events as discrete detections by the observer on the slices of the foliation $\Sigma_s^{(\mathbf{n})}$.

In the approximation we call the classical regime, the event set $\mathcal{C}_{\mathbf{n}}$ is the same for all observers at rest in a given inertial frame of reference (IFR), and the observable transformations M between IFRs take the Lorentz form at finite v_{\max} [1]. Local variations of the direction $n_A(x)$ are operationally indistinguishable from observer acceleration; the requirement of consistent reconstruction between neighboring slices (invariance of the null cone on the slice and under transport; see (24), (25)) imposes geometric constraints on admissible fields $n_A(x)$.

1.2 From Lorentz Transformations to Curvature

For $K_{ab} = 0$ (equivalently, $\nabla_A n_B = 0$) the observable transformations M are strictly Lorentzian (the SR regime) [1]. When $\nabla_A n_B \neq 0$, the normal field $n_A(x)$ and the induced geometry on the slices define a local effective metric

$$g_{AB} = -h_{AB} + v_{\max}^{-2} n_A n_B, \quad \text{sig}(g) = (+, -, -, -),$$

see (10). Smooth variations of $n_A(x)$ induce curvature of g .

We then show that the local consistency of observable transformations M on a slice (24) and under transport (25), together with the compensation principle $\delta S_{\text{eff}} + \delta S_g = 0$, uniquely fixes the geometric action to the Einstein–Hilbert form and leads to the equations (42). The two-scale structure with parameter $\varepsilon := L_{\text{field}}/L_{\text{fol}} \ll 1$ ensures the Newtonian limit and the operational exclusion of observable strong-field regimes; see §5 and §5.4.

1.3 Goal and Scope

The goal is to rigorously derive the equations for the effective metric $g[n]$ (that is, a metric g locally parameterized by the normal field n_A) from the requirement of operational consistency of causal reconstruction, and to show their equivalence to Einstein’s equations with an effective source.

Tasks:

- to fix a variational setup that combines the geometric functional for $g[n]$ with the operational constraints of transport, under explicit assumptions: locality, at most second order in derivatives, 3-diffeomorphism invariance on Σ_s , invariance under reparametrizations $s \mapsto f(s)$, the SR regime for $\nabla n = 0$, and the absence of extra modes (see §4.2, §4.3);
- to establish the local Lorentz limit and the equivalence principle (see §3);
- to obtain the Newtonian limit in the weak-field, two-scale regime (see §5).

Absence of extra modes as a consequence. We do *not* postulate the number of degrees of freedom. From the operational admissibility criteria — (i) locality and at most second order in derivatives, (ii) 3-diffeomorphism invariance on Σ_s and invariance under reparametrizations $s \mapsto f(s)$, (iii) the SR limit for $\nabla n = 0$ and invariance of the null cone for observable M , (iv) the absence of kinetic terms for N and N^a , (v) closure of the hypersurface deformation algebra (Dirac–Teitelboim), — it *follows* that the scalar mode is excluded and $\zeta = 1$ is fixed. As a result, exactly two tensor polarizations remain in the linearized spectrum. Details: §4.3; see also [7, 5, 6, 14]. Any $\zeta \neq 1$ violates the closure of the algebra or compatibility with the cone, whereas adding terms beyond second order leads to Ostrogradsky instability and contradicts (i).

In §2 we formalize the foliation field and kinematics; in §3 we establish the local equivalence principle; in §4 we carry out the variational derivation and

show equivalence to Einstein's equations; in §5 we analyze the scale hierarchy and the Newtonian limit; in §6 we discuss limitations and prospects.

In addition, we obtain a *strong-field ban* in observable regions in the form of the bound $\varepsilon \leq \varepsilon_{\max} \ll 1$ (see §5.4), which follows from invariance of the null cone and local operational consistency; this simultaneously addresses the hierarchy problem and ensures falsifiability of the model.

Notation remark: the symbols $\Sigma_s, h_{ab}, K_{ab}, R^{(3)}[h], N, N^a, \alpha, \beta, \zeta$ are introduced in §2–§4.3; they are used here only to preview the structure.

2 Fundamental Euclidean Model and Foliation Structure

In this section we fix the basic objects of the model, introduce local foliations and the induced geometry on the slices, and make precise how smooth variations of the foliation direction give rise to curvature.

2.1 Laplace Equation and Admissible Configurations

Let \mathbb{E}^4 be endowed with the Euclidean metric δ_{AB} and global coordinates x^A , $A = 0, 1, 2, 3$. The Laplacian is defined by $\Delta_{\mathbb{E}^4} := \delta^{AB} \partial_A \partial_B$. The fundamental field is a real-valued function $\Phi : \mathbb{E}^4 \rightarrow \mathbb{R}$, $\Phi \in C^2$, satisfying

$$\Delta_{\mathbb{E}^4} \Phi(x) = 0 \quad \text{for all } x \in \mathbb{E}^4. \quad (2)$$

This is the *sole fundamental equation of the model*. No fundamental variational or energy functional for Φ is introduced.

When necessary, we restrict to a domain $U \subset \mathbb{E}^4$ with C^1 boundary ∂U and impose well-posed boundary conditions (Dirichlet/Neumann/Robin) compatible with the local transport law from [1]. No fundamental variational functional for Φ is introduced.

Conventions. Repeated indices are summed; indices are raised/lowered with δ^{AB}, δ_{AB} .

2.2 Local Foliations and Induced Geometry

A smooth field of unit normals $n_A(x)$ is specified on \mathbb{E}^4 ,

$$n_A n^A = 1, \quad n_A \in C^2(\mathbb{E}^4). \quad (3)$$

We assume *local integrability* of the normal field: there exists a scalar function $s(x)$ such that

$$n_A \propto \partial_A s, \quad n^A \partial_A s = 1, \quad (4)$$

equivalently, the Frobenius condition $P_{[A}^C P_{B]}^D \nabla_C n_D = 0$ holds. The level sets

$$\Sigma_s := \{ x \in \mathbb{E}^4 \mid s(x) = \text{const} \} \quad (5)$$

form a local foliation. The projector onto the tangent subspace of a slice,

$$P_{AB} := \delta_{AB} - n_A n_B \quad (6)$$

defines the induced three-dimensional metric

$$h_{AB} := P_{AB}, \quad h_{AB} n^B = 0, \quad (7)$$

and a covariant derivative D_a compatible with h_{ab} : $D_c h_{ab} = 0$. Here and below, ∇ denotes the Levi-Civita connection of the flat metric δ_{AB} ; h_{AB} is the four-dimensional projector, and h_{ab} is its pullback to Σ_s . In a neighborhood of a point we use coordinates (s, y^a) , where s satisfies (4) and y^a parametrize Σ_s ; any vector decomposes as

$$V^A = (V \cdot n) n^A + P^A_B V^B \equiv V_\perp n^A + V_\parallel^A, \quad V_\parallel^A n_A = 0. \quad (8)$$

The extrinsic curvature of a slice is

$$K_{ab} := \frac{1}{2} \mathcal{L}_n h_{ab}, \quad K := h^{ab} K_{ab}. \quad (9)$$

Effective 4-metric and the SR regime. We introduce the effective metric form

$$g_{AB}(x) := -h_{AB}(x) + v_{\max}^{-2} n_A(x) n_B(x), \quad \text{sig}(g) = (+, -, -, -), \quad (10)$$

compatible with the local light cone $|\mathbf{r}| = v_{\max} t$ and with the isotropy of the metric h_{ab} on Σ_s . The operational time is normalized so that a displacement by length ℓ along n^A corresponds to $t = \ell/v_{\max}$; henceforth we identify $v_{\max} \equiv c$. If $n_A = \text{const}$ (equivalently, $\nabla \mathbf{n} = 0$), then g_{AB} is flat, and the *observable* transformations M between IFRs are strictly Lorentzian with invariant v_{\max} [1].

Observable vs direct transformations. Local consistency conditions are formulated for the observable transformations M , which by definition preserve the event structure and the null cone (10) (for $\nabla \mathbf{n} = 0$ they locally form the Lorentz group). Direct transformations D rearrange the family of events $\{\mathcal{C}_s\}$ and are not used for local kinematics.

Operational motivation for joint reconstruction. Both the events and the foliation are *jointly reconstructed* from the same data. On each slice Σ_s , events are defined as local critical points of the operational sensitivity functional $S[\Phi; \Sigma_s]$ (see [1]). We start with hyperplanes ($\nabla \mathbf{n} = 0$) and construct the events $\mathcal{C}_s = \text{Crit } S[\Phi; \Sigma_s]$. We then check the local transport law on Σ_s and introduce the transport misfit

$$\mathfrak{E}[\Phi; \Sigma_s; n, h] := (\text{transfer predicted by the observable law}) - (\text{actual linkage of events}).$$

In the general case with $\nabla \mathbf{n} = 0$, one obtains $\mathfrak{E} \neq 0$ on a finite observation domain $\Omega \subset \Sigma_s$. We then allow *curving of the foliation* (slow variations of $n_A(x)$ and h_{ab}) and require exact cancellation of the misfit:

$$\mathfrak{E}[\Phi; \Sigma_s; n, h] = 0 \quad \text{on } \Omega, \quad (11)$$

subject to the constraints

$$n_A n^A = 1, \quad D_c h_{ab} = 0, \quad g_{AB} = -h_{AB} + v_{\max}^{-2} n_A n_B,$$

and preservation of the null cone for the observable transformations M on Σ_s . A solution with $K_{ab} \neq 0$ removes the violations of the transport equation. Thus there is *no priority*: the events \mathcal{C}_s and the geometry (n, h) are determined jointly as a solution of $\mathfrak{E} = 0$; the “slowness” of foliation variations follows from the stability of consistent reconstruction on finite domains Ω .

Intuitive explanation. Foliation curvature is introduced as a compensation for the transport misfit: gravity is the geometric correction that enforces the transport law for *all* effective fields on the slices. Hence: (i) gravity and other effective fields are of different nature; (ii) gravity acts on all fields defined on the foliations through a single metric g .

2.3 Curvature from Variations of the Normal Field

Throughout this subsection, $\nabla \equiv \nabla[g]$ denotes the Levi–Civita connection of the metric g , and indices intrinsic to a slice are raised/lowered with h_{ab} .

Let the normal field $n_A(x)$ satisfy (3) and vary smoothly. In the gauge $N = 1$, $N^a = 0$ the extrinsic curvature of the slices

$$K_{ab} := P_a^A P_b^B \nabla_A n_B = \frac{1}{2} \mathcal{L}_n h_{ab}, \quad K := h^{ab} K_{ab}, \quad (12)$$

induces curvature of the effective metric g_{AB} from (10). For $n_A = \text{const}$ (i.e., $\nabla \mathbf{n} = 0$) we have $K_{ab} = 0$, $R[g] = 0$, and the observable transformations M between IFRs are strictly Lorentzian with invariant v_{\max} [1].

Gauss–Codazzi decomposition. The relation between the curvature of g and (h_{ab}, K_{ab}) is given by the identities

$$R_{abcd}^{(4)}[g] = R_{abcd}^{(3)}[h] + K_{ac}K_{bd} - K_{ad}K_{bc}, \quad (13)$$

$$D_b K^b{}_a - D_a K = P_a{}^A n^B R_{AB}^{(4)}[g], \quad (14)$$

$$R[g] = R^{(3)}[h] + K_{ab}K^{ab} - K^2 + 2\nabla_A(n^A K - n_B K^{AB}). \quad (15)$$

Hence any smooth variation of n_A ($\nabla n \neq 0$) produces $R_{ABCD}[g] \neq 0$ through K_{ab} and its derivatives.

Cone consistency for observable transformations. Local consistency of causal reconstruction on a fixed slice Σ_s is formulated as preservation of the null cone of the metric g under a change of the observational chart:

$$(\Delta x)^A g_{AB}(\Sigma_s)(\Delta x)^B = 0 \implies (\Delta x')^A g_{AB}(\Sigma_s)(\Delta x')^B = 0, \quad (16)$$

where $(\Delta x') = M(\Delta x)$ for an observable transformation M *within* Σ_s . Transport between neighboring slices is described by a separate evolution operator $U(s+ds, s)$ and is not part of M .

Adiabatic regime and curvature scale. We define the foliation curvature radius L_{fol} by the estimates

$$\|K_{ab}\| \sim \mathcal{O}(L_{\text{fol}}^{-1}), \quad \|D_c K_{ab}\| \sim \mathcal{O}(L_{\text{fol}}^{-2}), \quad (17)$$

with norms induced by h_{ab} on a local observation domain. For $\varepsilon := L_{\text{field}}/L_{\text{fol}} \ll 1$ gravitational corrections to local SR kinematics are suppressed, and in terms of orders

$$R^{(3)}[h] = \mathcal{O}(L_{\text{fol}}^{-2}), \quad R[g] = \mathcal{O}(L_{\text{fol}}^{-2}). \quad (18)$$

A detailed analysis of the two-scale asymptotics is given in §5; there it is shown that the causal reconstruction requirement implies the hierarchy $L_{\text{field}} \ll L_{\text{fol}}$, ensuring the Newtonian limit and excluding observable strong-field regimes ($\varepsilon \sim 1$).

Summary.

$$\nabla n = 0 \implies K_{ab} = 0, R[g] = 0, \text{ and } M \text{ is strictly Lorentzian;} \quad \nabla n \neq 0 \implies K_{ab} \neq 0, R[g] \neq 0,$$

that is, curvature of g is a direct consequence of variations of the normal field $n_A(x)$.

3 Operational Consistency and the Equivalence Principle

In this section:

- we introduce the minimally required definition of effective fields on the slices and their local evolution;
- we formulate the local consistency of observable transformations M as preservation of the null cone under a *change of the observational chart* on a fixed slice;
- we derive an operational form of the equivalence principle.

3.1 Effective Fields as Operational Degrees of Freedom

Let $\{\varphi_\alpha^{(n)}(y; s)\}$ be an admissible local modal basis on Σ_s (see [1, §3.2]). We define effective fields as local functionals of the fundamental field:

$$\psi_I(s, y) = \Psi_I[\Phi; \Sigma_s](y), \quad \psi_I = W_I^\alpha(s, y) a_\alpha^{(n)}(s) + \mathcal{O}(Da, DW), \quad (19)$$

where $a_\alpha^{(n)}(s)$ are the modal coefficients, D is the covariant derivative on Σ_s , and $W_I^\alpha(s, y)$ are locally invertible on the working subspace.

Their *evolution along the foliation* is specified by a local operator

$$a_\alpha^{(n)}(s+ds) = A_\alpha^\beta[\Phi; s] a_\beta^{(n)}(s) + \mathcal{O}(ds^2), \quad \psi_I(s+ds, y) = U_I^J(s, y) \psi_J(s, y) + \mathcal{O}(ds^2, D\psi, DW), \quad (20)$$

where $U := WAW^{-1}$ is defined pointwise in (s, y) . The equivalence of the descriptions via a_α and ψ_I is discussed in Appendix B.

We require the representation $R(M)$ of observable transformations M on ψ to be finite-dimensional; to the working order,

$$R(M)U = UR(M) + \mathcal{O}(\varepsilon), \quad \varepsilon := \frac{L_{\text{field}}}{L_{\text{fol}}} \quad (\text{see §5}). \quad (21)$$

Minimal conditions (as consequences of the model). From locality on Σ_s , transport invariance, and the adiabaticity of the foliation it follows that:

1. *Locality and second order in y^a* : no terms beyond two D -derivatives appear at the working order.
2. *M -covariance and 3-diffeomorphism invariance* on Σ_s ; compatibility is expressed by (21).

3. *Adiabaticity*: dependences on K_{ab} and DK_{ab} enter as $\mathcal{O}(L_{\text{fol}}^{-1})$ and higher.
4. *Reality of observables*: for complex ψ , the action and observables are Hermitian. For the present discussion it is immaterial whether the effective fields are real or complex.

Optional complexification. When needed we introduce complex combinations of real fields:

$$\psi_c = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \quad (\psi_1, \psi_2) \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (22)$$

i.e., an effective $U(1)$ phase is equivalent to an internal $SO(2)$ rotation on a pair of real fields. The on-slice evolution is specified pointwise in y by an orthogonal operator $U \in O(N)$; a unitary form appears only after complexification and is not required for the derivations in Secs. 4–5.

Effective action and source. We adopt the minimal local form

$$S_{\text{eff}}[g, \psi] = \int d^4x \sqrt{|g|} \mathcal{L}_{\text{eff}}(\psi, \nabla\psi; g), \quad T_{AB}^{\text{eff}} := -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{eff}}}{\delta g^{AB}}. \quad (23)$$

In the SR regime ($\nabla n = 0$, $g = \eta$) (20) is equivalent to the Euler–Lagrange equations, and T_{AB}^{eff} coincides with the symmetrized Noether tensor. In the curved case, the Bianchi identity implies $\nabla^A T_{AB}^{\text{eff}} = 0$.

Sufficiency for deriving GR. The definitions (19)–(23) suffice to: (i) justify minimal coupling of the effective degrees of freedom to g ; (ii) obtain T_{AB}^{eff} as the unique compatible source in (42); (iii) recover energy and momentum in the SR limit via Noether charges.

3.2 Local Consistency of Observable Transformations on a Slice

Definition. Let $p \in \Sigma_s$. An *observable transformation* M is a linear change of the observational chart at p , i.e., a linear operator $M : T_p\mathbb{E}^4 \rightarrow T_p\mathbb{E}^4$ acting on coordinates (t, \mathbf{r}) adapted to $g(p)$, and on observables $\psi_I \mapsto R_I^J(M)\psi_J$. The consistency requirement is

$$(\Delta x)^A g_{AB}(p) (\Delta x)^B = 0 \implies (\Delta x')^A g_{AB}(p) (\Delta x')^B = 0, \quad \Delta x'^A := M^A_B \Delta x^B, \quad \Delta x \neq 0, \quad (24)$$

i.e., M preserves the null cone in $T_p\mathbb{E}^4$. We additionally assume preservation of spatial and temporal orientation. Evolution in s is governed by the transport operator $U(s+ds, s)$ from (20) and is not part of M . When $\nabla\mathbf{n} = 0$, the class of admissible M locally coincides with $SO^+(1, 3)$.

Cone consistency under transport. Let $\Pi^A_B(s+ds, s) : T_p\mathcal{U} \rightarrow T_{p_+}\mathcal{U}$ be the linear transport operator induced by the joint reconstruction procedure between neighboring slices $\Sigma_s \rightarrow \Sigma_{s+ds}$. We require preservation of the null cone under transport:

$$(\Delta x)^A g_{AB}(\Sigma_s)(\Delta x)^B = 0 \Rightarrow (\Delta x_+)^A g_{AB}(\Sigma_{s+ds})(\Delta x_+)^B = 0, \quad \Delta x_+^A := \Pi^A_B(s+ds, s) \Delta x^B + \mathcal{O}(d) \quad (25)$$

Here Π acts on tangent vectors and *does not coincide* with U from (20), which acts on effective fields; M acts on $T_p\mathcal{U}$ on a fixed slice, see (24).

By admissible transformations we henceforth mean those observable M which (i) preserve the null cone of g ; (ii) preserve spatial orientation ($\det M > 0$); (iii) preserve the chosen time orientation determined by n^A ; (iv) preserve the time-scale gauge $t = \ell/v_{\max}$ (equivalently, $(Mn)^A(Mn)_A = n^A n_A = 1$).

In the SR regime ($\nabla\mathbf{n} = 0$) such M are locally of Lorentz form; thus the set of admissible M is isomorphic to the proper orthochronous component of the Lorentz group ($SO^+(1, 3)$) *in the sense of local structure*. Here the notation $SO^+(1, 3)$ is used as a shorthand, rather than as a pre-established global group classification.

Theorem 1 (SR regime for M). *If $\nabla n = 0$, then $g = \eta$, and the set of all admissible M in a neighborhood of any point p coincides with the proper orthochronous Lorentz group $SO^+(1, 3)$ with invariant v_{\max} [11].*

Proof. Work at the point p with $g(p) = \eta$. An admissible M is linear on $T_p\mathbb{E}^4$ and preserves the null cone: $\eta_{AB}x^A x^B = 0 \Rightarrow \eta_{AB}(Mx)^A(Mx)^B = 0$. Hence $M^T \eta M = \lambda \eta$ for some $\lambda \neq 0$. Condition (iv) (preservation of the gauge/norm of n) gives $\lambda = 1$, therefore $M \in O(1, 3)$. Preservation of orientations yields $M \in SO^+(1, 3)$. \square

3.3 Acceleration as a Tilt of the Foliation and the Equivalence Principle

The projector is $P^A_B = \delta^A_B - n^A n_B$, with $h_{AB} = P_{AB}$. The *operational acceleration* along an integral curve of n^A is

$$a_A := P_A^B (n^C \bar{\nabla}_C n_B) = P_A^B \mathcal{L}_n n_B, \quad (26)$$

where $\bar{\nabla}$ is the Levi–Civita connection of the flat metric δ_{AB} . The equality follows from $\mathcal{L}_n n_B = n^C \bar{\nabla}_C n_B + n_C \bar{\nabla}_B n^C$ and $n_C \bar{\nabla}_B n^C = \frac{1}{2} \bar{\nabla}_B (n_C n^C) = 0$.

The extrinsic curvature is $K_{ab} = \frac{1}{2} \mathcal{L}_n h_{ab}$; the Gauss–Codazzi decompositions relate K_{ab} to the curvature of g [11, 13].

Theorem 2 (Operational Equivalence Principle). *For any point p there exists a sufficiently small neighborhood and a local gauge in which*

$$g_{AB}(p) = \eta_{AB}, \quad \partial_C g_{AB}(p) = 0, \quad n_A = \text{const}, \quad K_{ab}(p) = 0.$$

In this neighborhood, all results of local experiments defined by a fixed on-slice evolution operator are indistinguishable from those in a uniformly accelerated system in flat space. The trajectories of free test bodies satisfy

$$u^B \nabla_B u^A = 0, \tag{27}$$

where ∇ is the Levi–Civita connection of g . In this gauge the observable transformations M act as local Lorentz transformations and preserve the null cone.

Sketch of the proof of Theorem 2. In Riemann normal coordinates for g at p one has $g_{AB}(p) = \eta_{AB}$ and $\partial_C g_{AB}(p) = 0$ [11]. By parallel transport with $\bar{\nabla}$ choose n_A so that $\bar{\nabla}_C n_B(p) = 0$. Then $\mathcal{L}_n h_{ab}(p) = 0$ and hence $K_{ab}(p) = \frac{1}{2} \mathcal{L}_n h_{ab}(p) = 0$. The null cone of g at p coincides with that of η , therefore admissible M are locally Lorentz and preserve the cone. The geodesic equation gives $u^B \nabla_B u^A = 0$; it follows that in a sufficiently small neighborhood the results of local experiments are indistinguishable from those in a uniformly accelerated system in flat space. \square

Remark. Locally, curvature of the foliation is a smooth tilt/rotation of the normal n^A ; uniform acceleration is the same tilt considered in a locally inertial gauge with $g_{AB}(p) = \eta_{AB}$ and $\partial_C g_{AB}(p) = 0$.

4 Variational Derivation of Einstein’s Equations

The goal is to derive the geometric action and the equations for the effective metric g_{AB} (10) from: (i) the local consistency of observable transformations M (preservation of the null cone on the slices), and (ii) the foliation structure $(\Sigma_s, h_{ab}, K_{ab})$, understood as a *joint reconstruction* of events and geometry: variations of g are chosen so as to cancel the event transport misfit $\mathfrak{E}[\Phi; \Sigma_s; n, h]$ (see §2, Eq. (11)). In the model there is a single fundamental equation $\Delta_{\mathbb{E}^4} \Phi = 0$; all other fields are *effective* degrees of freedom on Σ_s , minimally coupled via g .

4.1 Constraints and Symmetries of the Variational Setup

We work within the class of configurations satisfying (3), (7), (12), and (10); here $h_{ab} := \iota^*(h_{AB})$ is the pullback of h_{AB} to Σ_s , and $\iota : \Sigma_s \hookrightarrow \mathbb{E}^4$ is the natural embedding. The covariant derivative on a slice is denoted D_a and is compatible with h_{ab} ($D_c h_{ab} = 0$).

We require:

1. **Locality and second order.** The geometric Lagrangian is local on Σ_s , contains at most two D -derivatives with respect to y^a , and depends on h_{ab} and K_{ab} at most quadratically.
2. **Invariances.** Invariance under three-dimensional diffeomorphisms on Σ_s and under reparametrizations $s \mapsto f(s)$; observable transformations M preserve the light cone $|\mathbf{r}| = v_{\max} t$ (which fixes the normalization of v_{\max}).
3. **SR regime.** When $\nabla \mathbf{n} = 0$ ($K_{ab} = 0$), the metric g is flat ($R[g] = 0$), the action yields no dynamics, and observable M are strictly Lorentzian with invariant v_{\max} .
4. **Absence of extra degrees of freedom.** In the linear regime only two tensor polarizations are present.

Adiabatic regime. By the adiabatic regime we mean a slow variation of the foliation:

$$\varepsilon := \frac{L_{\text{field}}}{L_{\text{fol}}} \ll 1, \quad \|K_{ab}\| \sim \mathcal{O}(L_{\text{fol}}^{-1}), \quad \|D_c K_{ab}\| \sim \mathcal{O}(L_{\text{fol}}^{-2}),$$

and we expand in powers of ε . Here D_a is the covariant derivative compatible with h_{ab} .

Regularity and observational scale. We assume $n, h \in C^2$ and smallness estimates on the observation scale L_{obs} :

$$\|K\| L_{\text{obs}} \ll 1, \quad \|DK\| L_{\text{obs}}^2 \ll 1,$$

and we neglect terms of order $\mathcal{O}(L_{\text{obs}}^2/L_{\text{fol}}^2)$.

4.2 Compensation Principle and Variational Setup

On the equations-of-motion shell for the effective fields ψ (i.e., with $\delta\psi = 0$) the variation of their action with respect to the metric is

$$\delta S_{\text{eff}} \Big|_{\delta\psi=0} = -\frac{1}{2} \int_{\mathcal{U}} d^4x \sqrt{|g|} T_{AB}^{\text{eff}} \delta g^{AB}. \quad (28)$$

The joint reconstruction of events and foliation is formulated as the vanishing of the transport misfit $\mathfrak{E}[\Phi; \Sigma_s; n, h] = 0$ (see §2, Eq. (11)). Then, for all variations δg compatible with the parametrization $g[h, n]$ and the gauge $N = 1$, $N^a = 0$, we require the compensation

$$\delta S_{\text{eff}} + \delta S_g = 0, \quad (29)$$

which is equivalent to

$$\frac{\delta S_g}{\delta g^{AB}} = \frac{1}{2} \sqrt{|g|} T_{AB}^{\text{eff}}. \quad (30)$$

In other words, S_g is chosen so that $\delta S_{\text{tot}}[g, \psi] = \delta(S_g + S_{\text{eff}}) = 0$ for all admissible δg is equivalent to the condition $\mathfrak{E} = 0$.

Set

$$\mathcal{E}_{AB}[g] := \frac{2}{\sqrt{|g|}} \frac{\delta S_g}{\delta g^{AB}}.$$

Diffeomorphism invariance of S_g implies the identity $\nabla^A \mathcal{E}_{AB} \equiv 0$. Together with (30) this yields $\nabla^A T_{AB}^{\text{eff}} = 0$. (After determining the form of S_g in Lemma 1, this also follows immediately from the Bianchi identity.)

theorem 1 (Universality of the Gravitational Action). *Let a set of effective fields $\{\psi_i\}$ on the slices have local Lagrangians compatible with 3-diffeomorphisms on Σ_s , with reparametrizations of the foliation parameter $s \mapsto f(s)$, with adiabaticity, and with preservation of the null cone by observable transformations M (24) as well as under transport (25). Then the compensation (29) on the class of metrics $g[h, n]$ is achievable only with a universal minimal coupling of all $\{\psi_i\}$ through one and the same metric g :*

$$S_{\text{eff}}[g, \{\psi_i\}] = \int \sqrt{|g|} \mathcal{L}_{\text{eff}}(\{\psi_i\}, \nabla\{\psi_i\}; g) d^4x,$$

and therefore

$$\frac{\delta S_g}{\delta g^{AB}} = \frac{1}{2} \sqrt{|g|} \sum_i T_{AB}^{\text{eff}(i)} \iff \frac{\delta S_g}{\delta g^{AB}} = \frac{1}{2} \sqrt{|g|} T_{AB}^{\text{eff}}, \quad T_{AB}^{\text{eff}} := \sum_i T_{AB}^{\text{eff}(i)}.$$

Proof. If different fields were assigned different metrics $g^{(i)}$, then

$$\delta S_{\text{eff}} \Big|_{\delta\psi=0} = -\frac{1}{2} \sum_i \int \sqrt{|g^{(i)}|} T_{AB}^{\text{eff}(i)} \delta g^{(i)AB}.$$

The compensation (29) requires a single gravitational functional S_g depending on one metric variation. Since g is parametrized by (h, n) , and preservation of the null cone (24), (25) fixes a unique cone both on a slice and under transport, independent variations of several $g^{(i)}$ are incompatible with the class $g[h, n]$. Hence only universal minimal coupling via a single g is admissible, which leads to summing the contributions $T_{AB}^{\text{eff}(i)}$ in (30). \square

Corollary 1 (Falsifiability of Universality). *If in a causally accessible region one observes robust effects that require field-dependent light cones or metrics (for example, different front velocities for different fields under joint detection, or different free-fall trajectories all else being equal), then the conditions (24), (25) are violated and the model is ruled out.*

Remark 1 (Comparison with GR). *In General Relativity the universality of gravitational action is taken as part of the equivalence principle and the postulate of minimal coupling. In the present model the same universality is derived from the compensation principle (29) and from the uniqueness of the class of metrics $g[h, n]$ compatible with cone preservation. The empirical consequences coincide with GR, but the justification is constructive rather than axiomatic.*

Corollary 2 (Local Equivalence Principle). *Under universal minimal coupling (Prop. 1) and in the SR regime at a point there exists a local inertial coordinate system (Fermi/Riemann normal) along any timelike worldline in which*

$$g_{AB}(p) = \eta_{AB}, \quad \partial_C g_{AB}(p) = 0, \quad g_{AB} = \eta_{AB} + \mathcal{O}(R \cdot x^2).$$

Localized wave packets of effective fields and test detectors with the minimal-length action $S_{\text{det}} = -m \int ds \sqrt{g_{AB} \dot{x}^A \dot{x}^B}$ obey the equation of motion

$$\frac{D^2 x^A}{d\tau^2} + \Gamma^A_{BC} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau} = 0,$$

and, in the eikonal limit of wave equations, phase fronts propagate along (null) geodesics of g . It follows that free fall is independent of internal structure and mass at order $\mathcal{O}(\varepsilon^0)$; differences appear only through tidal terms $\mathcal{O}(R \cdot L_{\text{obs}}^2)$.

Remark 2 (Equivalence of Gravity and Acceleration). *In the neighborhood of a worldline with 4-acceleration a^i , the Rindler metric in flat space and a weak gravitational field with potential ϕ from (58) yield equivalent local physics: $g_{00} \simeq 1 + 2\phi/v_{\max}^2 \simeq 1 + 2a_i x^i/v_{\max}^2$ up to $\mathcal{O}(R \cdot x^2)$. Gravitational redshift and free-fall acceleration follow from $\nabla_i g_{00}$.*

theorem 1 (Uniqueness of S_g under Locality and Invariances). *Suppose S_g is local, diffeomorphism invariant, yields equations of at most second order, has an SR limit when $\nabla n = 0$, and introduces no extra modes in the linearized spectrum. Then from (30) it follows that $G_{AB} + \Lambda g_{AB} = 8\pi G T_{AB}^{\text{eff}}$, and $S_g = \frac{1}{16\pi G} \int \sqrt{|g|} (R - 2\Lambda) d^4x$ up to boundary terms.*

Sketch of the proof. From (29) we have $\delta S_g = \frac{1}{2} \int \sqrt{|g|} T_{AB}^{\text{eff}} \delta g^{AB}$. Hence $\mathcal{E}_{AB} := \frac{2}{\sqrt{|g|}} \frac{\delta S_g}{\delta g^{AB}}$ is symmetric and satisfies $\nabla^A \mathcal{E}_{AB} \equiv 0$ (diffeomorphism invariance). In 4D, by Lovelock's theorem [10], under locality and second order in derivatives, $\mathcal{E}_{AB} = c_1 G_{AB} + c_2 g_{AB}$. The existence of the SR limit excludes other combinations and fixes the action to $S_g = \frac{c_1}{2} \int \sqrt{|g|} (R - 2\Lambda) d^4x$ up to boundary terms. The normalization $c_1 = \frac{1}{8\pi G}$ is determined in the weak-field (Newtonian) limit [11, 12], whence $G_{AB} + \Lambda g_{AB} = 8\pi G T_{AB}^{\text{eff}}$.

Alternatively, in the ADM representation [7] the general local second-order Lagrangian $\sqrt{h} N [\alpha (K_{ab} K^{ab} - \zeta K^2) + \beta R^{(3)} - 2\Lambda]$ together with closure of the hypersurface deformation algebra (Dirac–Teitelboim) [5, 6, 14] and the absence of extra modes yields $\zeta = 1$ and $\alpha = \beta = \frac{1}{16\pi G}$. With the Gauss–Codazzi identities this is equivalent to the Einstein–Hilbert action plus the Gibbons–Hawking–York boundary term [8, 9]. \square

4.3 General Form of the Lagrangian on the Foliation

Foliation decomposition. In coordinates (s, y^a) with $n^A \partial_{As} = 1$ the metric takes the form

$$ds^2 = v_{\max}^2 N^2 ds^2 - h_{ab} (dy^a + N^a ds) (dy^b + N^b ds), \quad (31)$$

where N is the lapse and N^a the shift; they are gauge functions (with no kinetic terms). In the gauge aligned with n^A and with the normalization of the cone $|\mathbf{r}| = v_{\max} t$,

$$N \equiv 1, \quad N^a \equiv 0, \quad (32)$$

and

$$g_{AB} = -h_{AB} + v_{\max}^{-2} n^A n_B, \quad \text{sig}(g) = (+, -, -, -). \quad (33)$$

Henceforth we identify $v_{\max} \equiv c$.

Lemma (joint reconstruction \Rightarrow gauge fixing). Preservation of the null cone *for* observable transformations M on the slices and the normalization $t = \ell/v_{\max}$ in the misfit-cancellation problem $\mathfrak{E} = 0$ imply $N \equiv 1$, $N^a \equiv 0$ in the chart with $n^A \partial_A s = 1$. *Proof:* if $N \neq 1$ the local cone is deformed and event transport consistency is violated; if $N^a \neq 0$ the cone tilts in the (s, y^a) coordinates, producing an inconsistency $\mathfrak{E} \neq 0$.

Kinematic invariants on a slice. Define the extrinsic curvature

$$K_{ab} = \frac{1}{2} \mathcal{L}_n h_{ab}, \quad K := h^{ab} K_{ab}, \quad (34)$$

and the scalar curvature of the slice $R^{(3)}[h]$. We write $\sqrt{h} := \sqrt{\det h_{ab}}$. In the adiabatic regime only local invariants with at most two D -derivatives are admissible.

Most general local Lagrangian. Assuming 3-diffeomorphism invariance on Σ_s and reparametrizations $s \mapsto f(s)$, the general form of the geometric density is

$$\mathcal{L}_g = \sqrt{h} N \left[\alpha (K_{ab} K^{ab} - \zeta K^2) + \beta R^{(3)}[h] - 2\Lambda \right], \quad (35)$$

and in the gauge (32)–(33)

$$\mathcal{L}_g = \sqrt{h} \left[\alpha (K_{ab} K^{ab} - \zeta K^2) + \beta R^{(3)}[h] - 2\Lambda \right]. \quad (36)$$

Linearization about the SR limit shows the appearance of an extra scalar mode when $\zeta \neq 1$; requiring its absence fixes

$$\zeta = 1 \quad (\text{details: Appendix A}).$$

Why the coefficients are fixed. The generators of normal and tangential deformations $(\mathcal{H}_\perp, \mathcal{H}_a)$ must close to the Dirac–Teitelboim algebra with structure function h^{ab} , as determined by the geometry of embeddings [5, 6, 7, 14]. Closure $\{\mathcal{H}_\perp, \mathcal{H}_\perp\} \sim \mathcal{H}_a[h^{ab}(\dots)]$ eliminates the scalar mode and selects the unique quadratic combination $K_{ab} K^{ab} - K^2$. Then (i) an exact SR regime for $\nabla \mathbf{n} = 0$ (flat g , no dynamics), (ii) isotropy on the slice, and (iii) matching of normalizations in the deformation algebra fix the overall scale:

$$\alpha = \beta = \frac{1}{16\pi G},$$

where G is determined from the Newtonian limit in the linear regime (see §4.8). Thus

$$\mathcal{L}_g = \frac{\sqrt{h}}{16\pi G} \left[(K_{ab}K^{ab} - K^2) + R^{(3)}[h] - 2\Lambda \right], \quad (37)$$

which is equivalent to the 4D density $\sqrt{|g|} R[g]$ up to a boundary term (see §4.4).

Comments. (i) $R^{(3)}$ is the only local second-order scalar on Σ_s . (ii) Any other local second-order terms are either reducible to a boundary term or violate the deformation algebra. (iii) Higher-than-second-order terms are suppressed by adiabaticity. All requirements in this subsection pertain to *observable* transformations M , which preserve event structure and the null cone; direct transformations D rearrange $\{\mathcal{C}_s\}$ and are not used in constructing the local Lagrangian.

4.4 Gauss–Codazzi Decomposition and the Passage to the Einstein–Hilbert Action

We work in the gauge (32)–(33). Let $\mathcal{U} \simeq \bigcup_s \Sigma_s$ be a 4D region, and let $\partial\mathcal{U}$ be its smooth boundary with induced metric γ_{ij} . The Gauss–Codazzi identities yield

$$\sqrt{|g|} R[g] = \sqrt{h} (R^{(3)} + K_{ab}K^{ab} - K^2) + \partial_A \mathcal{V}^A, \quad (38)$$

where \mathcal{V}^A is a total divergence depending on n^A, K_{ab}, h_{ab} [8, 9]. Integrating over \mathcal{U} and applying Stokes’ theorem, we obtain

$$\int ds d^3y \frac{\sqrt{h}}{16\pi G} (R^{(3)} + K_{ab}K^{ab} - K^2) = \frac{1}{16\pi G} \int_{\mathcal{U}} \sqrt{|g|} R[g] d^4x - \frac{1}{8\pi G} \int_{\partial\mathcal{U}} \sqrt{|\gamma|} \varepsilon \mathcal{K} d^3x, \quad (39)$$

where \mathcal{K} is the trace of the extrinsic curvature of $\partial\mathcal{U}$ (with outward-pointing unit normal), and $\varepsilon = +1$ for a timelike boundary and $\varepsilon = -1$ for a spacelike boundary.

To make the variation with respect to g^{AB} well-posed for fixed γ_{ij} on $\partial\mathcal{U}$, we add the Gibbons–Hawking–York boundary functional [8, 9]

$$S_{\text{GHY}} = \frac{1}{8\pi G} \int_{\partial\mathcal{U}} \sqrt{|\gamma|} \varepsilon \mathcal{K} d^3x, \quad (40)$$

which cancels the boundary contribution in (39). As a result,

$$S_g[g] = \frac{1}{16\pi G} \int_{\mathcal{U}} \sqrt{|g|} (R[g] - 2\Lambda) d^4x + S_{\text{GHY}}. \quad (41)$$

Remarks. (i) For non-smooth boundaries, additional “corner” terms (Hayward) must be included; this does not affect the variational calculation in the text. (ii) We assume variations with compact support inside \mathcal{U} while keeping γ_{ij} fixed on $\partial\mathcal{U}$. (iii) The decomposition (38) is geometric and independent of the chosen gauge; the choice (32)–(33) serves to align notation.

4.5 Variations and Equations

Take S_g from (41). The joint reconstruction condition $\mathfrak{E} = 0$ (see (11)) fixes the gauge $N = 1$, $N^a = 0$ and the class of admissible variations, but does not add dynamical equations. The variational derivation is performed with respect to the metric g_{AB} with γ_{ij} fixed on $\partial\mathcal{U}$.

Consider the total action

$$S_{\text{tot}}[g, \psi] = S_g[g] + S_{\text{eff}}[g, \psi],$$

where S_{eff} and T_{AB}^{eff} are given in (23). The variations δg^{AB} have compact support in \mathcal{U} ; boundary contributions are canceled by S_{GHY} . We vary with respect to g as an independent variable; the parametrization $g[h, n] = h - v_{\text{max}}^{-2} n n$ is a subsequent reconstruction and imposes no restrictions on the variations.

Using the definition of the energy–momentum tensor and the Bianchi identity,

$$\delta S_g = \frac{1}{16\pi G} \int_{\mathcal{U}} \sqrt{|g|} (G_{AB} + \Lambda g_{AB}) \delta g^{AB} d^4x, \quad \delta S_{\text{eff}} = -\frac{1}{2} \int_{\mathcal{U}} \sqrt{|g|} T_{AB}^{\text{eff}} \delta g^{AB} d^4x,$$

and from $\delta S_{\text{tot}} = 0$ for all δg^{AB} we obtain

$$G_{AB}[g] + \Lambda g_{AB} = 8\pi G T_{AB}^{\text{eff}}. \quad (42)$$

From $\nabla^A G_{AB} \equiv 0$ it follows that $\nabla^A T_{AB}^{\text{eff}} = 0$.

4.6 Noether Energy–Momentum Currents in the SR Regime

Let $\nabla n = 0$, so that $g_{AB} = \eta_{AB}$ and the observable transformations M are strictly Lorentzian. For an effective field (or a set of fields) ψ the action

$$S_{\text{eff}}[\eta, \psi] = \int d^4x \mathcal{L}_{\text{eff}}(\psi, \partial\psi; \eta)$$

is invariant under translations $x^A \mapsto x^A + \epsilon^A$. By Noether’s theorem there exists the canonical tensor [2]

$$\Theta^A_B = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial(\partial_A \psi)} \partial_B \psi - \delta^A_B \mathcal{L}_{\text{eff}}, \quad \partial_A \Theta^A_B = 0 \quad (\text{on shell}). \quad (43)$$

Symmetrizing à la Belinfante–Rosenfeld [3, 4] yields Θ_{AB}^{sym} , which coincides with the metric definition under minimal coupling:

$$T_{AB}^{\text{eff}}|_{g=\eta} = -\frac{2}{\sqrt{|\eta|}} \frac{\delta S_{\text{eff}}}{\delta \eta^{AB}} = \Theta_{AB}^{\text{sym}}. \quad (44)$$

(For spinors one uses the tetrad and spin connection; the equality holds with the usual improvement terms.)

Choose n^A as the four-velocity of a rest observer and $P^A_B := \delta^A_B - n^A n_B$ as the projector onto Σ_s . The local energy and momentum densities are

$$\rho_{\text{eff}} = T_{AB}^{\text{eff}} n^A n^B, \quad \pi_a = -T_{AB}^{\text{eff}} n^A P^B_a. \quad (45)$$

Let τ^A be a timelike Killing vector and $e^A_{(i)}$ form a basis of spatial Killing vectors in $(\mathbb{R}^{1,3}, \eta)$. Then the integral charges

$$E = \int_{\Sigma_s} d^3y \sqrt{h} T_{AB}^{\text{eff}} \tau^A n^B, \quad P_i = \int_{\Sigma_s} d^3y \sqrt{h} T_{AB}^{\text{eff}} e^A_{(i)} n^B, \quad (46)$$

are independent of the choice of slice in the absence of flux through the boundary (or with sufficient falloff at infinity), since $\partial_A(T^{\text{eff}A}_B \xi^B) = 0$ for any Killing vector ξ^A . In the gauge $\tau^A = v_{\text{max}}^{-1} n^A$ and in an orthonormal basis $e^A_{(i)}$ on Σ_s ,

$$E = \int_{\Sigma_s} d^3y \sqrt{h} \rho_{\text{eff}}, \quad P_i = \int_{\Sigma_s} d^3y \sqrt{h} \pi_i.$$

4.7 Energy–Momentum from T_{AB}^{eff} : Densities and Killing Charges

Let $\nabla^A T_{AB}^{\text{eff}} = 0$. For any vector field ξ^A define the current

$$J^A[\xi] := T^{\text{eff}A}_B \xi^B. \quad (47)$$

If ξ^A is a Killing vector ($\nabla_{(A} \xi_{B)} = 0$), then $\nabla_A J^A[\xi] = 0$, and the integral charge

$$Q_{\Sigma}[\xi] := \int_{\Sigma_s} d^3y \sqrt{h} J^A n_A = \int_{\Sigma_s} d^3y \sqrt{h} T_{AB}^{\text{eff}} \xi^A n^B \quad (48)$$

is independent of the choice of slice Σ_s in the absence of flux through $\partial\Sigma_s$ (or with sufficient falloff at infinity).

Energy and momentum (SR regime). When $\nabla n = 0$ and $g = \eta$, there exist global Killing vectors: a timelike τ^A and three spatial $e_{(i)}^A$. Then

$$E = Q_\Sigma[\tau] = \int_{\Sigma_s} d^3y \sqrt{h} T_{AB}^{\text{eff}} \tau^A n^B, \quad (49)$$

$$P_i = Q_\Sigma[e_{(i)}] = \int_{\Sigma_s} d^3y \sqrt{h} T_{AB}^{\text{eff}} e_{(i)}^A n^B. \quad (50)$$

In the gauge $\tau^A = v_{\text{max}}^{-1} n^A$ and in an orthonormal basis $e_{(i)}^A$ on Σ_s , the formulas (49)–(50) reduce to (45).

Curved case. If there are no global Killing vectors, then: (i) local measurements are specified by (45) for a chosen observer four-velocity u^A (typically $u^A = n^A$); (ii) quasi-charges are defined via (48) for conformal or approximate Killing vectors, or via fluxes of J^A through the boundary; (iii) the on-slice stress tensor $S_{ab} := T_{CD}^{\text{eff}} P^C_a P^D_b$ describes momentum flux within Σ_s .

Consistency with Noether. In the flat case, the canonical Noether tensor symmetrized à la Belinfante–Rosenfeld coincides with the metric tensor: $T_{AB}^{\text{eff}}|_{g=\eta} = \Theta_{AB}^{\text{sym}}$, which ensures that (49)–(50) agree with the standard expressions.

4.8 Checks: SR Regime and Newtonian Limit

SR. For $\nabla \mathbf{n} = 0$ ($K_{ab} = 0$) the metric is flat, $g = \eta$, the observable M are strictly Lorentzian, and (42) with $T_{AB}^{\text{eff}} = 0$ is identically satisfied.

Weak field (Newtonian limit). Assume: (i) a weak field $|\phi|/v_{\text{max}}^2 \ll 1$; (ii) slow motion of sources $|\mathbf{v}| \ll v_{\text{max}}$; (iii) quasistaticity $\partial_t \phi \approx 0$; (iv) small pressure $p \ll \rho_{\text{eff}} v_{\text{max}}^2$. In the gauge

$$g_{00} = 1 + \frac{2\phi}{v_{\text{max}}^2}, \quad g_{0i} = 0, \quad g_{ij} = -\left(1 - \frac{2\phi}{v_{\text{max}}^2}\right) \delta_{ij}, \quad (51)$$

where $i, j = 1, 2, 3$, and δ_{ij} is the Euclidean metric on the slice Σ_s , the equations (42) reduce to

$$\nabla^2 \phi = 4\pi G \rho_{\text{eff}}, \quad \rho_{\text{eff}} := T_{00}^{\text{eff}}, \quad (52)$$

with $\nabla^2 := \delta^{ij} \partial_i \partial_j$ the three-dimensional Laplacian on Σ_s . This *fixes* G : we require the coefficient in (52) to be 4π .

EEP check. In view of Corollary 2, at any point there exists a local IFR with $g = \eta + \mathcal{O}(Rx^2)$; any gravitational effects can be eliminated locally by choosing an inertial frame, up to tidal corrections. This is consistent with (58) and with the Newtonian limit (59).

4.9 Section Summary

Taken together, the premises of this section—locality and at most second order in derivatives, 3-diffeomorphism invariance and reparametrizations of s , the SR regime for $\nabla n = 0$, and the absence of extra modes—combined with the compensation principle $\delta S_{\text{eff}} + \delta S_g = 0$ and the gauge $N = 1$, $N^a = 0$ (preservation of the cone for M) uniquely fix the geometric density (37). Via the Gauss–Codazzi identities this is equivalent to the 4D Einstein–Hilbert action with the Gibbons–Hawking–York boundary term, see (41). Variation of the total action $S_g + S_{\text{eff}}$ with respect to g^{AB} at fixed γ_{ij} on $\partial\mathcal{U}$ yields Einstein’s equations (42) with emergent T_{AB}^{eff} and the condition $\nabla^A T_{AB}^{\text{eff}} = 0$. The normalization of G is fixed in the weak-field limit as in §4.8.

5 Scale Hierarchy and the Weak-Field Regime

Introduce the small parameter of the two-scale asymptotics

$$\varepsilon := \frac{L_{\text{field}}}{L_{\text{fol}}} \ll 1,$$

where L_{fol} is the characteristic curvature radius of the foliation, and L_{field} is the coherence length of the effective-field modes on the slice Σ_s . For $\varepsilon \ll 1$ the geometry varies slowly while the effective fields on the slices vary rapidly; to leading order one obtains the Newtonian limit, with corrections suppressed by powers of ε .

5.1 Two-Scale Separation

Definitions of scales. Let $\Omega \subset \Sigma_s$ be a compact set with smooth boundary. Define

$$L_{\text{fol}}^{-2} \sim \max \left\{ \|K_{ab}K^{ab}\|, |R^{(3)}[h]| \right\}, \quad L_{\text{field}}^{-2} \sim \sup_{\psi} \frac{\int_{\Omega} \sqrt{h} |D\psi|^2}{\int_{\Omega} \sqrt{h} \psi^2}, \quad (53)$$

where norms are taken with respect to h_{ab} , and the supremum is over those modes ψ admitted by \mathcal{L}_{eff} with boundary conditions on $\partial\Omega$ compatible with the setup. Then $D\psi = \mathcal{O}(L_{\text{field}}^{-1})$ and $D^2\psi = \mathcal{O}(L_{\text{field}}^{-2})$.

Power counting. In the regime (53) we have

$$K_{ab} = \mathcal{O}(L_{\text{fol}}^{-1}), \quad D_c K_{ab} = \mathcal{O}(L_{\text{fol}}^{-2}), \quad D_a \psi = \mathcal{O}(L_{\text{field}}^{-1}), \quad D^2 \psi = \mathcal{O}(L_{\text{field}}^{-2}), \quad D^2 := h^{ab} D_a D_b. \quad (54)$$

Accordingly, the Lagrangian densities scale as

$$\mathcal{L}_{R^{(3)}} \sim \mathcal{O}(L_{\text{fol}}^{-2}), \quad \mathcal{L}_{K^2} \sim \mathcal{O}(L_{\text{fol}}^{-2}), \quad \mathcal{L}_{\text{eff}} \sim \mathcal{O}(L_{\text{field}}^{-2}). \quad (55)$$

Mixed terms of the type $K |D\psi|^2$ give

$$\mathcal{L}_{\text{mix}} \sim K |D\psi|^2 \sim \frac{1}{L_{\text{fol}} L_{\text{field}}^2}. \quad (56)$$

To compare dimensionless orders we use the coordinate rescaling $\tilde{y}^a := y^a / L_{\text{field}}$ on Ω (or, equivalently, coarse-graining on the scale L_{field}). Then

$$\frac{\mathcal{L}_{\text{mix}}}{\mathcal{L}_{\text{eff}}} = \mathcal{O}(\varepsilon), \quad \varepsilon = \frac{L_{\text{field}}}{L_{\text{fol}}}. \quad (57)$$

Adiabatic expansion. For a fixed observational scale $L_{\text{obs}} \ll L_{\text{fol}}$, solutions and functionals admit expansions

$$g_{AB} = g_{AB}^{(0)} + \varepsilon g_{AB}^{(1)} + \varepsilon^2 g_{AB}^{(2)} + \dots, \quad \psi = \psi^{(0)} + \varepsilon \psi^{(1)} + \dots,$$

where $g^{(0)}$ is a locally flat metric on scales L_{field} , consistent with Sec. 3.2; the corrections $g^{(k)}$ vary slowly on L_{fol} .

5.2 Weak Field and the Newtonian Limit

Weak-field gauge. Let

$$g_{AB} = \eta_{AB} + \gamma_{AB}, \quad |\gamma_{AB}| \ll 1,$$

and choose the harmonic gauge

$$\partial^A \bar{\gamma}_{AB} = 0, \quad \bar{\gamma}_{AB} := \gamma_{AB} - \frac{1}{2} \eta_{AB} \gamma, \quad \gamma := \eta^{CD} \gamma_{CD}.$$

In the Newtonian regime we assume: slow sources $|\mathbf{v}| \ll v_{\text{max}}$, quasistaticity $\partial_0 \gamma_{AB} \approx 0$, weak field $|\phi| / v_{\text{max}}^2 \ll 1$, and small pressure $p \ll \rho_{\text{eff}} v_{\text{max}}^2$. We use the gauge

$$g_{00} = 1 + \frac{2\phi}{v_{\text{max}}^2}, \quad g_{0i} = 0, \quad g_{ij} = -\left(1 - \frac{2\phi}{v_{\text{max}}^2}\right) \delta_{ij}, \quad (58)$$

where $i, j = 1, 2, 3$, and δ_{ij} is the Euclidean metric on Σ_s .

Then from (42) in harmonic gauge and under quasistaticity it follows that

$$\nabla^2 \phi = 4\pi G \rho_{\text{eff}}, \quad \rho_{\text{eff}} := T_{00}^{\text{eff}}, \quad \nabla^2 := \delta^{ij} \partial_i \partial_j, \quad (59)$$

which reproduces the Newtonian limit.

Estimate of corrections. Corrections to (59) due to finite foliation curvature and slow nonstationarity are estimated as

$$\delta(\nabla^2\phi) = \mathcal{O}\left(\frac{\phi}{L_{\text{fol}}^2}\right) + \mathcal{O}\left(\frac{\partial_0^2\phi}{v_{\text{max}}^2}\right), \quad (60)$$

and for an experiment with spatial scale L_{obs} and duration T_{obs} , given a characteristic metric-variation time L_T , one has

$$\frac{|\delta(\nabla^2\phi)|}{|\nabla^2\phi|} = \mathcal{O}\left(\frac{L_{\text{obs}}^2}{L_{\text{fol}}^2}\right) + \mathcal{O}\left(\frac{T_{\text{obs}}^2}{L_T^2}\right). \quad (61)$$

5.3 Structural Origin of the Hierarchy

Operational consistency. Let L_{obs} be the characteristic spatial size of a measurement procedure on Σ_s (detector aperture, interferometer baseline, etc.). The joint conditions of preserving the null cone *on the slice* and *under transport* (see (24) and (25), §3.2) yield the estimates

$$\|K_{ab}\|_{L^\infty(\Omega)} L_{\text{obs}} \ll 1, \quad \|D_c K_{ab}\|_{L^\infty(\Omega)} L_{\text{obs}}^2 \ll 1, \quad (62)$$

where norms are taken with respect to h_{ab} on a chosen domain $\Omega \subset \Sigma_s$, $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}$, and D_c is the covariant derivative compatible with h_{ab} ; we assume regularity $n_A, h_{ab} \in C^2$. Setting $L_{\text{obs}} \sim L_{\text{field}}$ gives

$$\varepsilon := \frac{L_{\text{field}}}{L_{\text{fol}}} \ll 1, \quad (63)$$

i.e., small ε is a consequence of operational consistency rather than parameter fine-tuning (see also the upper bound (64)).

Physical content. Under (63): (i) the geometric sector is described by S_g and provides a quasi-static background metric on the scale L_{field} ; (ii) the effective fields evolve on the background g with minimal coupling and source T_{AB}^{eff} ; (iii) the observable transformations M realize SR kinematics with corrections of order $\mathcal{O}(L_{\text{obs}}^2/L_{\text{fol}}^2)$.

5.4 Ban on Strong Fields in Observable Regions

The requirements of preserving the null cone *on the slice* and *under transport* (24), (25) for a fixed class of transport and observation procedures on a domain $\Omega \subset \Sigma_s$ entail the existence of an upper bound

$$\varepsilon \leq \varepsilon_{\text{max}}(\delta_{\text{cone}}, C_{\text{reg}}) \ll 1, \quad (64)$$

where $\varepsilon := L_{\text{field}}/L_{\text{fol}}$, δ_{cone} is the admissible threshold for cone misfit, and C_{reg} encodes the regularity constants $n_A, h_{ab} \in C^2$ and the norms $\|K_{ab}\|_{L^\infty(\Omega)}$, $\|D_c K_{ab}\|_{L^\infty(\Omega)}$ (with norms taken with respect to h_{ab}). This is a local estimate within a causally accessible region, not a universal constant of nature.

Consequently, strong-field regimes with $\varepsilon = \mathcal{O}(1)$ are *excluded* in observable regions: all corrections to SR/GR kinematics are suppressed by at least $\mathcal{O}(\varepsilon^2)$ for $L_{\text{obs}} \sim L_{\text{field}}$ (see §5.1).

Falsifiability. The observation of reproducible effects that require $\varepsilon \sim 1$ in a causally accessible region under controlled transport contradicts the model.

6 Discussion and Outlook

Summary. We have shown that

- the Einstein–Hilbert action and the equations (42) follow from the operational consistency of observable transformations M and the foliation structure $(\Sigma_s, h_{ab}, K_{ab})$;
- Lorentzian kinematics for M is exactly recovered when $\nabla n = 0$ (the SR regime);
- a natural scale hierarchy $L_{\text{field}} \ll L_{\text{fol}}$ arises, ensuring the Newtonian limit;
- in observable regions there exists an operational upper bound on the hierarchy parameter ε of the form $\varepsilon \leq \varepsilon_{\text{max}} \ll 1$ (see (62), (63), (64)), which renders corrections to the SR/GR regime parametrically small and ensures falsifiability;
- the source T_{AB}^{eff} is emergent and satisfies $\nabla^A T_{AB}^{\text{eff}} = 0$.

Interpretation. The compensation principle of §4.2 yields $\delta S_{\text{eff}} + \delta S_g = 0$ for admissible variations; thus G_{AB} is interpreted as the variational response that compensates δS_{eff} under transport $U(s+ds, s)$ between neighboring slices. Curvature is associated not with fundamental matter, but with the condition of operational reconstruction. The energy and momentum of effective fields arise as Noether charges in the SR regime and extend covariantly via T_{AB}^{eff} when $\nabla n \neq 0$.

Domain of applicability and operational hierarchy. The emergent geometry g is applicable as long as

$$\varepsilon := \frac{L_{\text{field}}}{L_{\text{fol}}} \ll 1, \quad \|K_{ab}\| L_{\text{field}} \ll 1, \quad \|D_c K_{ab}\| L_{\text{field}}^2 \ll 1,$$

which is equivalent to preservation of the null cone under transport $U(s+ds, s)$ between neighboring slices (see (24), (25)). Regimes where ε approaches the local bound $\varepsilon_{\text{max}} < 1$ (64) mark the end of applicability of the effective theory g ; beyond this one requires a description in terms of the fundamental variables (Φ, n) .

Strong fields: black holes. Solutions of (42) with a horizon are admissible and describe strong curvature of g . Singularities of curvature invariants (e.g., $R_{ABCD}R^{ABCD}$) pertain to the effective metric g , whereas the fundamental field Φ satisfies (2). Inside the horizon, ε approaches $\varepsilon_{\text{max}} < 1$; the transport $U(s+ds, s)$ is then inapplicable and one needs a description at the level of (Φ, n) . Outside the horizon, where $\varepsilon \ll 1$, the transport $U(s+ds, s)$ is consistent and the emergent description g is applicable.

Strong fields: cosmology. For homogeneous–isotropic foliations $L_{\text{fol}} \sim H^{-1}$; looking back to early epochs, $\varepsilon(s)$ grows toward the operational boundary $\varepsilon_{\text{max}} < 1$. The FRW singularity is interpreted as the boundary of applicability of the emergent effective theory g , while Φ remains a smooth solution of (2). The effective description is valid in late-time regimes with $\varepsilon \ll 1$.

Testable consequences. Corrections to the Newtonian limit are suppressed as $\mathcal{O}(L_{\text{obs}}^2/L_{\text{fol}}^2)$ for experiments of scale L_{obs} . In the linear regime only tensor modes are excited; the absence of a scalar mode follows from §4.3 and Appendix A. Quasi-charges and boundary fluxes are specified via $J^A[\xi] = T^{\text{eff}A}{}_B \xi^B$ given appropriate boundary conditions on the integration domain.

Falsifiability: upper bounds on ε can be extracted from weak lensing, dynamics of binary systems, time delays, Einstein rings, pulsar timing arrays (PTA), and gravitational-wave detectors.

Universality of gravity. The absence of field-dependent cones/metrics is a strict result of the model (see Prop. 1); its violation under joint observations falsifies the model. Unlike GR, where universality (the equivalence principle) is postulated, here it follows from the condition of operational consistency of transport.

Equivalence principle. In GR, universality and local inertia are postulated; here they follow from the compensation principle and the uniqueness of the class $g[h, n]$ (Prop. 1, Cor. 2).

Relation to alternative approaches. In contrast to entropic and induced-gravity schemes, the proposed construction derives the geometric action from operational transport-consistency conditions rather than from statistical assumptions, while the SR regime for M is exactly maintained at $\nabla \mathbf{n} = 0$. The origin of the cosmological constant remains open; a possible link to the finite spectral sensitivity of the observer is a topic for future work.

7 Conclusion

We have shown that the Einstein–Hilbert action (41) and the equations (42) arise in a timeless Euclidean model with foliation $(\Sigma_s, h_{ab}, K_{ab})$ and a single fundamental field Φ as a consequence of the compensation principle $\delta S_{\text{eff}} + \delta S_g = 0$ under locality, diffeomorphism invariance, and second order in derivatives, together with preservation of the null cone on slices for observable transformations M . The effective degrees of freedom couple minimally to the metric g .

In the SR regime ($\nabla \mathbf{n} = 0$) the kinematics is strictly Lorentzian; under slow variation of the foliation the Newtonian limit is recovered with corrections $\mathcal{O}(L_{\text{obs}}^2/L_{\text{fol}}^2)$, and the normalization of G is fixed in this limit. The Einstein tensor is interpreted as the variational response compensating the contribution δS_{eff} ; from the Bianchi identity it follows that $\nabla^A T_{AB}^{\text{eff}} = 0$. In the linearized regime only tensor polarizations are present; the extra scalar mode is absent.

Universality of gravity is derived as a consequence of the uniqueness of the class of metrics $g[h, n]$ and of the compensation principle (see Prop. 1): all effective fields use the same metric g . An observed field-dependent cone structure falsifies the model.

The variational principle is formulated with the induced metric γ_{ij} fixed on the boundary $\partial\mathcal{U}$; the Gibbons–Hawking–York boundary functional ensures a well-posed variation. The working assumptions—locality and second order, smoothness $n, h \in C^2$, and the hierarchy $L_{\text{field}} \ll L_{\text{fol}}$ —are treated as consequences of operational transport consistency between neighboring slices rather than external postulates. This yields a local upper bound $\varepsilon = L_{\text{field}}/L_{\text{fol}} \leq \varepsilon_{\text{max}} \ll 1$; robust observations with $\varepsilon \sim 1$ in causally accessible regions, or violations of cone preservation, lead to rejection of the model.

These results indicate the consistency and testability of the timeless approach with operational reconstruction of geometry; open questions include, in particular, the origin of Λ and the description beyond the domain of applicability of the emergent metric g .

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A Hypersurface-Deformation Algebra and Fixing the Coefficients

Conventions. Signature $(+, -, -, -)$. On the foliation $\{\Sigma_s\}$: h_{ab} is the induced metric, n^A the unit normal, $K_{ab} := \frac{1}{2}\mathcal{L}_n h_{ab}$, and D_a is compatible with h_{ab} . The gauge $N=1, N^a=0$ is fixed in §4.3 and is not varied here.

A.1 Geometry of deformations

A local deformation of a slice specified by the pair (N, N^a) acts as

$$\delta_{(N, N^a)} h_{ab} = 2N K_{ab} + \mathcal{L}_N h_{ab}, \quad (65)$$

$$\delta_{(N, N^a)} K_{ab} = -D_a D_b N + N \left(R_{ab}^{(3)} + K K_{ab} - 2K_{ac} K^c_b \right) + \mathcal{L}_N K_{ab}. \quad (66)$$

The commutator of deformations is again a deformation:

$$[(N_1, N_1^a), (N_2, N_2^a)] = (\hat{N}, \hat{N}^a), \quad \hat{N} = N_1^b \partial_b N_2 - N_2^b \partial_b N_1, \quad \hat{N}^a = N_1 D^a N_2 - N_2 D^a N_1 + [N_1, N_2]^a, \quad (67)$$

which yields the kinematical deformation algebra (Dirac–Teitelboim).

A.2 Canonical variables and constraints

The most general 3D local second-order Lagrangian in the gauge $N=1, N^a=0$ is

$$\mathcal{L}_g = \sqrt{h} \left[\alpha (K_{ab} K^{ab} - \zeta K^2) + \beta R^{(3)}[h] - 2\Lambda \right]. \quad (68)$$

The canonical momentum is

$$\pi^{ab} := \frac{\partial \mathcal{L}_g}{\partial \dot{h}_{ab}} = \sqrt{h} \alpha (K^{ab} - \zeta K h^{ab}). \quad (69)$$

The smeared constraints are

$$\mathcal{H}_\perp = \frac{1}{\sqrt{h}} \left(\pi_{ab} \pi^{ab} - \frac{\zeta}{1-3\zeta} \pi^2 \right) - \sqrt{h} \beta R^{(3)} + 2\Lambda \sqrt{h}, \quad (70)$$

$$\mathcal{H}_a = -2D_b \pi^b_a. \quad (71)$$

A.3 Closure of the algebra and fixing the coefficients

We require the Poisson brackets of the constraints to realize (67):

$$\{\mathcal{H}_\perp[N], \mathcal{H}_\perp[M]\} = \mathcal{H}_a [h^{ab} (N \partial_b M - M \partial_b N)], \quad (72)$$

$$\{\mathcal{H}_a[V^a], \mathcal{H}_\perp[N]\} = \mathcal{H}_\perp[\mathcal{L}_V N], \quad \{\mathcal{H}_a[V^a], \mathcal{H}_b[W^b]\} = \mathcal{H}_a[\mathcal{L}_V W^a]. \quad (73)$$

A standard calculation (see [5, 7, 6, 14]) gives:

$$\zeta = 1 \quad \Rightarrow \quad \pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2, \quad \alpha = \beta = \frac{1}{16\pi G}.$$

The resulting density is

$$\mathcal{L}_g = \frac{\sqrt{h}}{16\pi G} \left[(K_{ab}K^{ab} - K^2) + R^{(3)}[h] - 2\Lambda \right]. \quad (74)$$

By the Gauss–Codazzi identities, the s -integration is equivalent to the 4D density $\sqrt{|g|} R[g]$ with the Gibbons–Hawking–York boundary term [8, 9].

Role of the observable transformations M . Preservation of the null cone for *observable* M and the time normalization fix the gauge $N=1$, $N^a=0$ (see §3.2, §4.3); direct transformations D do not preserve event structure and do not participate in the algebra (67).

B Equivalence of Descriptions via Modes and Effective Fields

theorem 2 (Local equivalence of evolutions). *Let a local mode basis $\{\varphi_\alpha\}$ be given on Σ_s , and suppose there exists an operator $W_I^\alpha(s, y)$ that is local in y and C^1 in s such that*

$$\psi_I = W_I^\alpha(s, y) a_\alpha(s) + \mathcal{O}(\partial_y a, \partial_y W).$$

If the transport of modes is local,

$$a_\alpha(s+ds) = A_\alpha^\beta(s) a_\beta(s) + \mathcal{O}(ds^2),$$

then the evolution of ψ is local and has the form

$$\psi_I(s+ds, y) = U_I^J(s, y) \psi_J(s, y) + \mathcal{O}(ds^2, \partial_y), \quad U = WAW^{-1} + \Gamma ds, \quad \Gamma := (\partial_s W) W^{-1}.$$

If W and A are covariant under the observable transformations M to the working order, then $[R(M), U] = \mathcal{O}(\varepsilon)$.

Sketch of proof. $\psi(s+ds) = W(s+ds) a(s+ds) = [W + \partial_s W ds][Aa] + \dots = (WAW^{-1} + \Gamma ds) \psi + \dots$. Locality of U follows from the locality of W, A ; covariance yields commutation with $R(M)$ up to $\mathcal{O}(\varepsilon)$.