

# A Note on the Occurrence of Fibonacci Terms in the Differences Between $n$ and $P_n$ of Pancake Graphs

Ryan O'Rourke

**Abstract.** In this paper I discuss a pattern observable in the  $n$  and  $P_n$  values of the first nineteen pancake graphs. This pattern could potentially hint at the as yet unknown diameters of the next graphs, and is at least viable for first seventy-four values of  $n$ , since it predicts values, in the aforementioned range, of  $P_n$  which fall between published lower and upper bounds. The pattern arises in sets of adjacent  $n$ s with equal differences between  $n$  and  $P_n$ , and is equivalent to a subsequence of the Fibonacci sequence. If one takes the known values of  $P_n$  and deletes  $n$  from each, one gets a difference value, which we call  $d$ , which allows one to arrange the numbers into corresponding blocks, so that the first block has 2 columns, the second 3, then 5 and then 8, and there a Fibonacci subsequence appears to be emerging (...2,3,5,8...). In this paper, I provide a formula for  $P_n$  for those  $n$ s that follow this pattern:  $h(n) = n + \lceil (\log_{\varphi}(-(n-1)(\sqrt{5} - \sqrt{5}\varphi) + \varphi^3)) - 4 \rceil - 1$ , and test it against published upper and lower bounds for  $P_n$  for  $n \leq 10000$ .

**Keywords.** Pancake sorting problem, Pancake graphs, Lower bound, Upper bound, Fibonacci sequence

## 1. Introduction

Pancake sorting is a well known problem in discrete mathematics which can be imagined as that of sorting a disordered stack of pancakes with the minimum of moves. The goal is by a certain operation (flipping substacks of pancakes) to arrive at an ordering in terms of size, with the smallest pancake at the top and the largest at the bottom. Any subset starting at the current top of the stack may be chosen and flipped. The pancakes may be formalized as graphs (the aptly named pancake graphs), or as numerical sequences in which flipping is performed as prefix reversals. The problem was first devised in this form by Jacob E. Goodman under the pseudonym “Harry Dweighter” [1] It has a range of applications.

Finding the pancake number  $P_n$  for any  $n$  is equivalent to ascertaining the diameter of a pancake graph with  $n$  nodes. The pancake number  $P_n$  has been shown to have a lower bound of  $\lfloor \frac{15n-9}{14} \rfloor, n \geq 6$ , and an upper bound of  $\frac{18n}{11}$  [2] [4], but the exact value remains an open problem, though it is known for graphs where  $n \leq 19$ . Above these we have been in indefinite territory. A stack of one pancake, or graph with one node, is trivial, since it already has the maximum order. The pancake number of a graph with nineteen nodes,  $P_{19}$ , is 22 and – as yet – the minimum number of flips required for all permutations of twenty pancakes has not been proven.

In this paper I propose a tentative connection between the pancake numbers and Fibonacci sequence, derived from an observation I made when having a feeling of familiarity on seeing a table showing the pancake numbers corresponding to the first nineteen pancake graphs and noticing what seemed to be a sequence of Fibonacci terms naturally appearing in sequences of *differences* between successive  $n$ s and  $P_n$ s

**Definition 1.** Let  $n$  be the number of nodes of a pancake graph,  $P_n$  the pancake number, which is the maximum number of flips required to sort a given number  $n$  of pancakes or the diameter of a pancake graph with  $n$  nodes, and let  $F_n$  be a term of the Fibonacci sequence, with  $F_0$  being 0,  $F_1$  being 1, and so on.

## 2. Preliminary discussion

Table 1 on page 3 relates some values of  $n$  to their pancake number  $P_n$ , where this is known. In addition, it shows hypothetical values for  $P_{20}$ ,  $P_{21}$  and  $P_{22}$ , which would be valid if this paper’s proposal holds true at least for these graphs. These values are  $P_{20} = 23$ ,  $P_{21} = 24$  and  $P_{22} = 25$ . It also gives the difference  $d$  between  $n$  and  $P_n$ . I have shaded cells sharing the same value of  $d$  with same value of gray, in order to make what appears to be

an emergent Fibonacci subsequence clearer. In addition, the table contains other data relating to the values of  $d$  which will be explained in this section.

Number of pancakes $n$	1	2	3	4	5	6	7	8	9	10	11
Number of flips $P_n$	0	1	3	4	5	7	8	9	10	11	13
$d = P_n - n$	-1	-1	0	0	0	1	1	1	1	1	2
$ C _{Setd}$	2	2	3	3	3	5	5	5	5	5	8
$F_N$	$F_3$	$F_3$	$F_4$	$F_4$	$F_4$	$F_5$	$F_5$	$F_5$	$F_5$	$F_5$	$F_6$

  

n	12	13	14	15	16	17	18	19	20	21	22	...
$P_n$	14	15	16	17	18	19	20	22	<b>23</b>	<b>24</b>	<b>25</b>	...
d	2	2	2	2	2	2	2	3	3	3	3	...
$ C _{Setd}$	8	8	8	8	8	8	8	13	13	13	13	...
$F_N$	$F_6$	$F_7$	$F_7$	$F_7$	$F_7$	...						

Table 1: Number of nodes of pancake graphs  $n$  with their pancake numbers  $P_n$  and the differences  $d$  between them, cardinality of  $Set_d$  and its corresponding  $F_n$

**Definition 2.** Let us define the difference  $d$  between  $P_n$  and  $n$  as  $d = (P_n - n)$ , thus  $P_n = n + d$

Making a list of consecutive graphs of  $n$  nodes along with corresponding values of  $d$  will allow one to count the number of times any particular  $d$  repeats. Observing the values of  $d$  for graphs with known diameter, and if our conjecture is to hold, it seems that any equivalent values for  $d$  should appear adjacently to each other and that these groupings be in sequential order. It is in these blocks or sets of equivalent differences between a number of adjacent  $P_n$  and  $n$  numbers that the Fibonacci sequence is seen to emerge. With reference to Table 1, one can see that  $n = 1$  and  $n = 2$  have a pancake number  $P_n$  that corresponds to  $n - 1$ , the difference  $d$  being thus  $-1$ . Continuing, we see that  $n = 3, n = 4, n = 5$  all share a  $P_n$  value corresponding to  $n - 0$ , in these three cases  $P_n$  being equal to 0. The sequence continues, with a difference of 1 between the next five values of  $n$  and  $P_n$ . So at the beginning of the sequence, we see sets of two, three and five values of  $d$ , these being the Fibonacci numbers  $F_3$  to  $F_5$ . The pattern continues and holds for all the known values of  $P_n$ . Whether it breaks off after the last known pancake number at  $n = 19$  or continues for a while is not clear. As we will show later, the pattern reaches a point at  $n = 74$ , after which it becomes nonviable for many, and most likely the majority of the later values

of  $n$ .

**Definition 3.** We can define sets  $S_d$  as containing all  $n$  values having equal values of  $d$ , where  $d = P_n - n$ . These groupings can be understood as sets of adjacent values of  $n$  which have equal values of  $d$ :

$$S_d = \{d, P_n \in \mathbb{Z}, n \in \mathbb{Z}^+ : P_n - n = d\}$$

...so that the first three such sets we generate are:  $\{1,2\}$ ,  $\{3,4,5\}$ ,  $\{6,7,8,9,10\}$ . The sizes or cardinalities  $|C|$  of these sets is the key factor here: the cardinality of these sets is 2,3 and 5 respectively. Table 2 on page 4 gives the first five of these sets, the last of which would contain values for  $P_n$  for graphs 20 to 31 which would be true if the pattern here described holds.

$Set_d$	$\{1,2\}$	$\{3,4,5\}$	$\{6,7,8,9,10\}$	$\{11...18\}$	$\{19...31\}$
Cardinality	2	3	5	8	13

Table 2: Sets of  $n$ -values with equivalent  $d$ -values and their cardinalities

At this point we can define our proposition:

**Proposition 1:** that differences  $d$  between  $P_n$  and  $n$  continue progressing from  $n = 1$  in consequent sets of  $n$  values with equal values of  $d$ , with the sizes of those sets growing according to the terms of the Fibonacci sequence, starting with the the fourth term,  $F_3 = 2$ .

### 3. Derivation of a possible formula $h(n) = P_n$

We will be using the definitions for  $n$ ,  $P_n$  and  $F_n$  from Definition 1. and of  $d$  as the difference from Definition 2.

**Definition 4.** We also need a way to define the Fibonacci numbers. It is known that  $F_n$  can be found by rounding:  $F_n = \lfloor \frac{\varphi^n}{\sqrt{5}} \rfloor, n \geq 0$  where  $\varphi$  is the golden ratio:  $\varphi = \frac{1}{2}(1 + \sqrt{5})$

The Fibonacci sequence we are working does not start from the first term,  $F_0$ , but from the fourth one,  $F_3$ , or 2. It continues with the fifth Fibonacci number  $F_4$ , which is 3, and so on... If one admitted graphs with a negative or zero number of nodes, one could make use of the Fibonacci numbers below  $F_3$ .

Since the connection between the Fibonacci sequence and the pancake

numbers  $P_n$  is not as direct as would be the case if the sequence in question assigned one Fibonacci number for every one  $n$ , we need to perform a couple of intermediate steps.

In summary, there exist an infinite number of pancake graphs with  $n$  nodes, the diameters  $P_n$  of which are currently known only for first 19 graphs. Apart from those cases, our problem becomes finding an unknown pancake number for any given number of nodes. We have noticed a pattern in the differences between the currently known values of  $n$  and  $P_n$  which consists of discrete blocks of differences, the sizes of which correspond to the terms of a Fibonacci subsequence starting at  $F_3$ . These can be expressed as cardinalities of sets  $S_d$ , as defined before.

**Definition 5.** It can be observed that the value of the difference  $d$  is equal to the index  $n(F)$ , lowered by 3, of the Fibonacci number corresponding to the cardinality of the set  $S_d$  in question, minus 1:  $P_n = n + (n(F) - 3) - 1$ . It will become clear later on why the expression is written this way.

$n$	1	<b>2</b>	3	4	<b>5</b>	6	7	8	9	<b>10</b>	11	12	13	14	15	16	17	<b>18</b>	19
$P_n$	0	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19	20	22
$d$	-1	-1	0	0	0	1	1	1	1	1	2	2	2	2	2	2	2	2	3
$x$		0			1					2								3	
$\lceil x \rceil$	0	0	1	1	1	2	2	2	2	2	3	3	3	3	3	3	3	3	4
$F_n$	$F_3$	$F_3$	$F_4$	$F_4$	$F_4$	$F_5$	$F_5$	$F_5$	$F_5$	$F_5$	$F_6$	$F_7$							

Table 3: Boundary conditions, with  $B_n$  in bold font

Considering that we begin by only knowing values of  $n$ , we need a way to find the corresponding values of  $d$ , as we have called them in this model, and preferably to do so without having to list them manually, since we are trying to model a function for this behaviour. This is easy enough if we notice that the value of  $n$  at the right boundary of each block of  $d$ -values is equal to a sum of preceding Fibonacci numbers starting from  $F_3$ , so for instance,  $n = 10$  is equivalent to the sum of the third, fourth and fifth such numbers (see Table 3 on page 5):

$$n = F_3 + F_4 + F_5 = 2 + 3 + 5 = 10$$

We can express these sums as partial sums of terms in the Fibonacci sequence, using Definition 4. and remembering that we begin at the third term:

$$\lfloor \sum_{N=3}^{x+3} \frac{\varphi^N}{\sqrt{5}} \rfloor$$

...where  $x$  refers to the boundary in such a way that the first  $x$ , corresponding to the highest  $n$  of a block  $ds$ , is equal to  $x = 0$ , the second  $x = 1$ , etc. In other words, it indicates the position of the boundaries between the various groups of  $ds$ . We will call the highest values of  $n$  for each block of  $d$  values  $B_n$ . The  $x$ -value is equal to the index of the associated Fibonacci number minus 3, so that at  $n = 2$ , i.e. just to the left of the boundary between the first two blocks of differences, a value of  $x = 0$  can be observed, corresponding to Fibonacci number  $F_3 - 4$ , that is  $x = 3 - 3 = 0$ . In other words,  $x = n(F) - 3$

If we were attempting to determine an unknown pancake number, we would have to assume that we know the value of  $n$  but not the value of  $x$ . Since at this phase of our investigation, we are interested in the highest  $ns$  for each block of  $d$  values, we can use a partial sum to solve for  $x$  for the nearest upper boundary  $B_n$  to  $n$ :

$$B_n = \lfloor \frac{1}{\sqrt{5}} \sum_{N=3}^{x+3} \varphi^N \rfloor$$

...and solve this by using a closed form for a partial geometric sum:

$$\sum_{N=0}^x ar^N = a \left( \frac{1-r^{x+1}}{1-r} \right)$$

...which needs to be modified since the sum begins at  $N = 3$ , not  $N = 0$ :

$$\sum_{N=3}^{x+3} ar^N = \sum_{N=0}^{x+3} ar^N - \sum_{N=0}^{2+1} ar^N = a \left( \frac{1-r^{x+3+1}}{1-r} \right) - a \left( \frac{1-r^{2+1}}{1-r} \right)$$

$$\Rightarrow B_n = \frac{1}{\sqrt{5}} \left( \frac{1-\varphi^{x+3+1}}{1-\varphi} \right) - \frac{1}{\sqrt{5}} \left( \frac{1-\varphi^{2+1}}{1-\varphi} \right)$$

$$\Rightarrow B_n = \left( \frac{1-\varphi^{x+4}}{\sqrt{5}(1-\varphi)} \right) - \left( \frac{1-\varphi^3}{\sqrt{5}(1-\varphi)} \right)$$

$$\Rightarrow B_n = \left( \frac{1-\varphi^{x+4} - (1-\varphi^3)}{\sqrt{5}(1-\varphi)} \right)$$

$$\Rightarrow B_n = \left\lfloor \left( \frac{-\varphi^{x+4} + \varphi^3}{\sqrt{5}(1-\varphi)} \right) \right\rfloor$$

$$\Rightarrow B_n - 0.5 \leq \left( \frac{-\varphi^{x+4} + \varphi^3}{\sqrt{5}(1-\varphi)} \right) < B_n + 0.5$$

$$\Rightarrow (B_n - 0.5)\sqrt{5}(1-\varphi) \leq -1(\varphi^{x+4} - \varphi^3) < (B_n + 0.5)\sqrt{5}(1-\varphi)$$

$$\Rightarrow -(B_n - 0.5)\sqrt{5}(1-\varphi) \leq (\varphi^{x+4} - \varphi^3) < -(B_n + 0.5)\sqrt{5}(1-\varphi)$$

$$\begin{aligned}
&\Rightarrow -(B_n - 0.5)\sqrt{5}(1 - \varphi) + \varphi^3 \leq \varphi^{x+4} < -(B_n + 0.5)\sqrt{5}(1 - \varphi) + \varphi^3 \\
&\Rightarrow -(B_n - 0.5)\sqrt{5}(1 - \varphi) + \varphi^3 \leq \varphi^{x+4} < -(B_n + 0.5)\sqrt{5}(1 - \varphi) + \varphi^3 \\
&\Rightarrow (\log_\varphi - (B_n - 0.5)\sqrt{5}(1 - \varphi) + \varphi^3) - 4 \leq x < (\log_\varphi - (B_n + 0.5)\sqrt{5}(1 - \varphi) + \varphi^3) - 4
\end{aligned}$$

Since  $x$  is related to the highest value of  $n$  for each block of values of  $d$ , all  $n$ s between  $x$  and, but not including,  $x - 1$  will share the same value of  $x$ , and for this reason we need use only the left side of the inequality and can replace  $B_n$  by  $n$  since in the next step we will be applying a ceiling operation:

$$x \geq \log_\varphi - (n - 0.5)\sqrt{5}(1 - \varphi) + \varphi^3 - 4$$

Adding a ceiling operation to  $x$  allows us to create an equivalent structure to the sequence of differences discussed prior, and which can be seen in Table 3 on page 5:

$$\lceil x \rceil = \lceil \log_\varphi - (n - 0.5)\sqrt{5}(1 - \varphi) + \varphi^3 - 4 \rceil$$

In fact it looks like we can create a function for finding known and hypothetical values of  $P_n$  which we are calling  $h(n)$  using definition 5. and  $\lceil x \rceil$  :

$$h(n) = n + \lceil x \rceil - 1$$

**Formula 1.**

$$h(n) = n + \lceil \log_\varphi - (n - 0.5)\sqrt{5}(1 - \varphi) + \varphi^3 - 4 \rceil - 1$$

△

#### 4. Data and Testing

In an older paper [6] I gave a formula for any  $P_n$  following this pattern in the form  $h(n) = n + \lceil \log_\varphi [\varphi^{\lceil \log_\varphi - (n+3)\sqrt{5}(1-\varphi)-1 \rceil} + 1] \rceil - 4$ . I soon realised that I have made a number of errors in that paper and after testing the formula found it did not work to generate the known  $P_n$  numbers, partially because it was based on a function that does not precisely generate the Fibonacci subsequence in question. In the present paper, I have calculated another function that generates the correct result and is based on a function that precisely generates the Fibonacci sequence:  $h(n) = n + \lceil \log_\varphi - (n - 0.5)\sqrt{5}(1 - \varphi) + \varphi^3 - 4 \rceil - 1$  I have tested it for accuracy in generating the Fibonacci subsequence from  $F_3$ . This testing has not been included here, since it does not add to the argument.

What is important for the premise of this paper, though, is to test whether the values generated by Formula 1. lie within the bounds proven to contain  $P_n$  for all  $n$ , and if so, for which values of  $n$ . To this end, I wrote a computer program in JavaScript that generates pancake numbers using the conjectured function for the first ten thousand values of  $n$  and tested if they stayed within the upper and lower bounds given in the literature. Results for only the first eighty-nine values are shown, since this is sufficient to disprove the conjecture.

I tested the results my formula for  $P_n$  gave against the lower bounds proven by Heydari and Sudborough [3], as well as the newer result by Peczarski[4]. These are respectively  $f(n) \geq 15n/14$  for all multiples of 14, and  $\geq \lfloor (15n + 9)/14 \rfloor, n \geq 6$ . All the numbers tested fall within the upper bound  $18n/11$  given by Chitturi et al. [2], though I have not included this data as I feel it would be redundant but can easily be calculated.

The data is presented below in Table 4 in six columns, the first two showing respectively, for each  $n$  from 1 to 89 the value for  $P_n$  that the formula presented in this paper generates;  $n$ s for which there is a known value of  $P_n$  are given in a green font and those with an unproven value are black. The third column is populated by values for the lower bound put forward by Heydari et al. at each relevant  $n$  (since the lower bound of Heydari et al. is valid only for any  $n$  divisible by 14). The fourth column is either blank where an entry would be irrelevant for the last-mentioned reason, or contains the word “yes” if  $P_n$  falls above the above mentioned bound, as is thus at least possible, and “no” in a red font if it does not. The fifth column shows the value, for each  $n$ , of the lower bound as defined by Peczarski, and the sixth is similar to column 4 in that it either contains a blank space where irrelevant (since the bound is valid for  $n \geq 6$ ), a “yes” when the calculated value falls above the bound, or a “no”, in red, if it doesn’t.

Table 4

$n$	(Hypothesized Pancake number $P(n)$ )	Lower bound of Heydari et al.	$P(n) \geq$ bound of Heydari et al.?	Peczarski's lower bound	$P(n) \geq$ Peczarski's bound?
1	0			1	
2	1			2	
3	3			3	yes
4	4			4	yes
5	5			6	
6	7			7	yes
7	8			8	yes
8	9			9	yes
9	10			10	yes
10	11			11	yes
11	13			12	yes
12	14			13	yes
13	15			14	yes
14	16	15	yes	15	yes
15	17			16	yes
16	18			17	yes
17	19			18	yes
18	20			19	yes
19	22			21	yes
20	23			22	yes
21	24			23	yes
22	25			24	yes
23	26			25	yes
24	27			26	yes
25	28			27	yes
26	29			28	yes
27	30			29	yes
28	31	30	yes	30	yes
29	32			31	yes
30	33			32	yes
31	34			33	yes
32	36			34	yes
33	37			36	yes
34	38			37	yes
35	39			38	yes
36	40			39	yes
37	41			40	yes

$n$	Hypothesized Pancake number $P(n)$	Lower bound of Heydari et al.	$P(n) \geq$ bound of Heydari et al.?	Peczarski's lower bound	$P(n) \geq$ Peczarski's bound?
38	42			41	yes
39	43			42	yes
40	44			43	yes
41	45			44	yes
42	46	45	yes	45	yes
43	47			46	yes
44	48			47	yes
45	49			48	yes
46	50			49	yes
47	51			51	yes
48	52			52	yes
49	53			53	yes
50	54			54	yes
51	55			55	yes
52	56			56	yes
53	58			57	yes
54	59			58	yes
55	60			59	yes
56	61	60	yes	60	yes
57	62			61	yes
58	63			62	yes
59	64			63	yes
60	65			64	yes
61	66			66	yes
62	67			67	yes
63	68			68	yes
64	69			69	yes
65	70			70	yes
66	71			71	yes
67	72			72	yes
68	73			73	yes
69	74			74	yes
70	75	75	yes	75	yes
71	76			76	yes
72	77			77	yes
73	78			78	yes
74	79			79	yes
75	80			81	no
76	81			82	no

$n$	Hypothesized Pancake number $P(n)$	Lower bound of Heydari et al.	$P(n) \geq$ bound of Heydari et al.?	Peczarski's lower bound	$P(n) \geq$ Peczarski's bound?
77	82			83	no
78	83			84	no
79	84			85	no
80	85			86	no
81	86			87	no
82	87			88	no
83	88			89	no
84	89	90	no	90	no
85	90			91	no
86	91			92	no
87	93			93	yes
88	94			94	yes
89	95			96	no

## 5. Conclusion and final comments

I tested ten thousand values of  $P_n$  (hypothetical except the first nineteen) against the lower bound given by Heydari and Sudborough, as well as against the improved lower bound from Peczarski's work. The fact that the function is not contained within the accepted lower bound for all  $ns$  disproves the conjecture and we restrict ourselves to commenting on its partial viability. It is, indeed, viable within the first seventy-four values of  $n$  and this might not be insignificant. The proposed function plunges below Peczarski's lower bound between  $n = 74$  and  $n = 75$ . It does at least twice, at  $n = 87$  and  $n = 88$ , rise momentarily above the bound again. Though I have not tested this thought, this could be due to the use of ceiling and rounding operations present in the formula put forward in this paper, as well as the floor operation used in the formula for the afore-mentioned lower bound, causing sufficient deviation from a continuous function and thus allowing a small "space" for the function to skip above the lower limit momentarily. However, it then trends away and it seems clear that the function I have proposed does not after this point ever fit within the lower bound again.

The problem discussed in this paper has been proven to be NP-hard [5] by Bulteau, Fertin and Rusu, so it is no surprise that a simple function would not provide a universal key to the diameter of pancake graphs.

The similarity to the Fibonacci sequence that arises in the discussed parameters of the first nineteen pancake graphs, the continuation of which

for, possibly, the first seventy-four values of  $n$  is not disproven, is somewhat striking. What is clear is that, even if the pattern here described continued after  $n = 19$ , at some point before  $n = 74$  it would have to shift to another pattern. Possibly the approach could be modified. It is pure conjecture, but I could imagine that there could be another cyclic algorithm playing out behind the one generating Fibonacci terms in this case, so that some or all of the first seventy-four values of  $P_n$  correspond to a first "loop" of this second algorithm; a sort of looping algorithm within whose turning some modifications of the Fibonacci sequence are generated, the first "bare" version of the Fibonacci numbers appearing only during the first iteration, with the following iterations obscuring its underlying presence.

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