Spinning Electrons on Pendulum-Paths in Hydrogen Atoms

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Abstract

While electronic orbitals with zero orbital angular momentum are a standard feature of modern quantum mechanics, the corresponding linear electron paths with zero orbital angular momentum ("pendulum-paths") were explicitly excluded in the "old quantum theory" because of concerns that an electron on such paths would collide with the atom's nucleus. More recently, some researchers hypothesized that models of spinning electrons allow for electrons on pendulum-paths without collisions with the nucleus. In the present work, the scenario of a spinning electron in a hydrogen atom on a pendulum-path was numerically simulated using the bi-level electron model. The resulting trajectories were evaluated by comparing time-averaged powers of the distance between electron and proton with corresponding time-averaged values in an improved variant of the Bohr-Sommerfeld model as well as with quantum mechanical expectation values. The numerical results for a spinning electron were in better agreement with quantum mechanical expectation values than the results for the improved Bohr-Sommerfeld model.

1 Introduction

While the Bohr-Sommerfeld model [Som23] predicted the fine-structure of energy levels of hydrogen-like atoms as accurately as Dirac's theory of the electron, it failed in many other cases. Some of these shortcomings were unavoidable considering that the original Bohr-Sommerfeld model treated electrons as spinless particles. Other shortcomings could have been avoided by relatively small amendments. This is especially true for the orbital angular momentum of electrons and the corresponding azimuthal quantum number l as discussed, for example, by Born and Jordan [BJ30], Pauling and Wilson [PW35], Pauling [Pau60], and Bucher [Buc08]. Specifically, it is preferable to choose values $l = 0, \ldots, n-1$ instead of $l = 1, \ldots, n$ (with the primary quantum number n) and to set the corresponding orbital angular momentum to $\hbar\sqrt{l(l-1)}$ instead of $\hbar l$ or $\hbar(l-1)$.

Bohr and Sommerfeld's actual reasons for their choice were probably rather complex as discussed in some detail in Section 2.2. However, the main reason that Bohr and Sommerfeld provided in their writings was the concern that an electron on an orbit with zero orbital angular momentum (i.e. a pendulum-path) would collide with the nucleus of the atom. Since then, researchers have been considering physical processes that would allow for pendulum-paths, e.g., elastic collisions [DL74], penetrations of finite-size nuclei [Buc06], or pendulum-paths as very narrow ellipses [Eps62]. However, none of these models has been particularly successful. More recently, Mac Gregor [Gre13] and Rivas [Riv24] hypothesized that a spinning electron could avoid collisions with another particle by spiraling around it on a helical trajectory; see Section 2.1 for more details.

The main objective of the present work was to numerically simulate this scenario in an idealized hydrogen atom with an infinitely heavy proton with negligible magnetic moment. To this end, the equations of motion of the bi-level electron model [Kra25a] were implemented as described in Section 3. The maximum distance of the simulated electron from the proton as well as the time period from perihelion to perihelion matched the expected values for pendulum-paths [DL74, Buc06] providing numerical evidence that Mac Gregor's and Rivas's hypothesis is correct.

The simulated trajectories were further analyzed by numerically computing time-averaged values of r, r^2, $1/r$, and $1/r^2$ with r denoting the distance between electron and proton. This approach was motivated by Pauling and Wilson's comparison of analytic expressions for these values in an improved Bohr-Sommerfeld model and corresponding quantum-mechanical expectation values [PW35]. Results are described and discussed in Section 4. Conclusions and future work are presented in Section 5.

2 Previous Works

2.1 Models of Spinning Electrons

The models of spinning electrons that are relevant for the present work describe an electron as a point-like charge (at the "center of charge") that spins around a "center of mass" (or "spin center"). Examples include models by Mac Gregor [Gre13], Rivas [Riv01, Riv24], and the bi-level electron model [Kra25a]. (More examples of such models are discussed by Rivas [Riv01, Chapter 5].) A common feature of these models is that the electromagnetic Lorentz force on the electron is computed for the position of the point-like charge, but the force accelerates the movement of the center of mass such that it moves approximately like an electron with spin.

Based on his model of spinning electrons, Rivas hypothesized: "In the ground state of the Hydrogen atom the electron is in [an] S-state of orbital angular momentum $l = 0$. This implies, from the classical point of view, that the center of mass of the electron is going through the center of mass of the proton. This is impossible for the spinless point particle. Nevertheless this can be justified classically, because the center of mass and the center of charge of [...] spinning electrons are different points and their separation is greater than the estimated size of the proton. Then in the ground state of the atom the center of mass of the electron describes a straight trajectory passing through the center of mass of the proton" [Riv24, page 18, prediction no. 39]. In other words, Rivas argues that in the ground state of a hydrogen atom, the center of mass of a spinning electron is on a straight pendulum-path but the center of charge is not colliding with the proton because it is on a helical trajectory.

Mac Gregor described the effect of these helical trajectories in scattering experiments with electrons or positrons: "The large electron or large positron is *channeled* into forward scattering angles. This is the process that we refer to as *Mott channeling* or *helical channeling*" [Gre13, page 180]. Helical trajectories are also predicted by many other models of spinning electrons, including the bi-level electron model [Kra25a]. One of the distinguishing features of the latter model is that it spins half as fast at twice the radius in comparison to the model by Rivas.

2.2 The Rocky History of Pendulum-Paths

Since electrons in the Bohr-Sommerfeld model of atoms [Som23] move on classical trajectories around an atom's nucleus, one might expect that an electron's state of zero orbital angular momentum is represented by a linear path straight through the nucleus, i.e., a pendulum-path ("Pendelbahn" in German). Historically, however, this is not how the Bohr-Sommerfeld model represented such states. Bucher argued that this feature (or "flaw") of the Bohr-Sommerfeld model accelerated its downfall and the rise of modern quantum mechanics [Buc08]. Therefore, this section tries to trace the history of this feature of the Bohr-Sommerfeld model in quite some detail. Readers who are only interested in more recent research on pendulum-paths (after 1920) are encouraged to skip forward to the last paragraph of this section.

What motivated Bohr and Sommerfeld to exclude pendulum-paths? Sommerfeld stated that "the electron, in describing this orbit would fall into the nucleus. Owing to the permanence of atoms we regard this as impossible" [Som23, page 238]. Similarly, Bohr stated that "for certain external fields such motions cannot be regarded as physical realisable stationary states of the atom, since in the course of the perturbations the electron would collide with the nucleus" [Boh18, page 56]. Were Bohr and Sommerfeld just too narrow minded to consider the possibility of pendulum-paths?

Interestingly, some of Bohr's earlier publications [Boh14, Boh15, Boh16] and at least one of Sommerfeld's publications [Som17] show that they had accepted the possibility of pendulum-paths just a few years earlier. And they were not alone: Epstein (who coined the term "Pendelbahn") was sure that pendulum-paths had been observed indirectly in experiments by Stark [Eps16, Eps19]; and even the grandfather of quantum physics, Max Planck, did not dismiss the idea [Pla20]. Thus, physicists at the time had been more than open to the idea of pendulum-paths. What else could have changed Bohr's and Sommerfeld's minds in the years around 1917?

In 1930, Born and Jordan described Bohr's and Sommerfeld's decision in this way: "In Bohr's theory—due to the quantum conditions for the orbital angular momentum quantum number, which was usually referred to as the azimuthal quantum number k—all values $k = 0, 1, 2, \ldots, n$ were initially permissible for a given n. [...] However, the number $n + 1$ of these values is 1 more than observed in experiments. Therefore, the case $k = 0$ (which would have corresponded to a pendulum-path with

the electron passing through the nucleus) was excluded by a special 'extra rule;' however, this gave rise to other difficulties"[1] [BJ30, pages 189–190]. (Some of these "other difficulties" were described in 1926 by Pauli [Pau26].) Thus, according to Born and Jordan, experimental results (specifically the number of spectral lines in observations of the Stark effect and the Zeeman effect) confronted Bohr and Sommerfeld with the choice of either excluding pendulum-paths or excluding circular orbits. Since it was unclear why pendulum-paths would not result in collisions with the nucleus, Bohr and Sommerfeld decided to introduce an extra rule that excluded pendulum-paths.

While this description by Born and Jordan is plausible, it cannot be the whole story: If Sommerfeld's primary objective had been to match the experimental results, then he would have had no reason to change his opinion about pendulum-paths after 1917 when he had accepted the possibility of pendulum-paths in observations of the Stark effect as described by Epstein, who wrote in 1916: "the electron oscillates in a straight line along the x-axis between two fixed points, coming infinitely close to the nucleus. Without an external electric field, such an orbit (a Keplerian ellipse of eccentricity 1) is impossible for relativistic-energetic reasons, which is why Sommerfeld does not consider it when calculating the doublets of spectral lines. However, observations of the Stark effect show that they undoubtedly occur here, even though the components associated with them are extremely weak"[2] [Eps16]. Sommerfeld appeared to agree with Epstein's statement in 1917 when he wrote: "when an electric field is applied, however, pendulum-paths appear to be able to occur with weak intensity"[3] [Som17]. A few years later, Epstein employed concepts of modern quantum mechanics to show that his explanation of the Stark effect from 1916 (confirmed in 1919 [Eps19]) was as good as possible within the framework of the Bohr-Sommerfeld model [Eps26]. Thus, whatever motivated Sommerfeld to change his opinion about pendulum-paths, it was neither supported by experimental observations of the Stark effect nor by modern quantum mechanics.

Is it possible that another expert on the Stark effect convinced Sommerfeld after 1917 to mistrust Epstein's explanation? At the time, the only serious alternative to Epstein's explanation of the Stark effect had been published by the famous Karl Schwarzschild [Sch16]. The story of how Sommerfeld caused Epstein and Schwarzschild to work on the Stark effect at the same time was recounted by Epstein in an interview in 1962 [Eps62]. Sommerfeld was well aware that Epstein reached the correct equation describing the Stark effect first, and that Schwarzschild had to revise his own article before publication. Nonetheless, if Schwarzschild had tried to convince Sommerfeld that Epstein was wrong about pendulum-paths, Sommerfeld might have trusted Schwarzschild's well-known expertise in celestial mechanics. However, it is very unlikely that Schwarzschild changed Sommerfeld's opinion in 1917 or later, because Schwarzschild had tragically died in May 1916.

Who else could have changed Sommerfeld's opinion? The quote by Born and Jordan includes another hint: the rule that excluded pendulum-paths [BJ30]. Epstein mentioned this extra rule in his 1926 publication and attributed it to Bohr without providing a source [Eps26]. Fortunately, Sommerfeld provided more details in the 2nd edition of his book „Atombau und Spektrallinien"[Som21]: "From a historical perspective, it should be noted that our discussion regarding the reality or unreality of the components deviates somewhat from Epstein's original one. We relied on our theoretically based selection principle together with Bohr's rule prohibiting vanishing azimuthal quantum numbers. Epstein, on the other hand, used a more empirical selection principle (previously advocated by the author but now abandoned)"[4] [Som21, pages 451–452]. Sommerfeld provided the source for Bohr's rule in a

[1] Original text in German: „In der Bohrschen Theorie waren nämlich auf Grund der Quantenbedingungen für die Impulsquantenzahl, die gewöhnlich als ‚Nebenquantenzahl' k bezeichnet wurde, bei vorgegebenem n zunächst alle Werte $k = 0, 1, 2, \ldots, n$ zulässig. [...] Die Anzahl $n + 1$ dieser Werte ist aber gegenüber der experimentellen Erfahrung um 1 zu groß. Man hat deshalb den Fall $k = 0$ (der einer „Pendelbahn" mit Durchgang des Elektrons durch den Kern entsprochen hätte) durch ein besonderes "Zusatzverbot" ausgeschlossen; dabei ergaben sich jedoch anderweitige Schwierigkeiten." [BJ30, pages 189–190]

[2] Original text in German: „[D]as Elektron pendelt in der x-Achse geradlinig zwischen zwei festen Punkten hin und her, wobei es dem Kern unendlich nahe kommt. Ohne ein äußeres elektrisches Feld ist eine solche Bahn (eine Keplerellipse von der Exzentrizität 1) aus relativistisch-energetischen Gründen unmöglich, weshalb sie auch Sommerfeld bei Berechnung der Dubletten von Spektrallinien nicht in Betracht zieht. Die Beobachtungen über den Starkeffekt zeigen aber, daß sie hier ohne jeden Zweifel auftreten, wenn auch die zu ihnen gehörenden Komponenten äußerst schwach sind" [Eps16].

[3] Original text in German: „bei Anlegung eines elektrischen Feldes dagegen scheinen Pendelbahnen mit schwacher Intensität auftreten zu können" [Som17].

[4] Original text in German: „In historischer Hinsicht sei noch bemerkt, daß unsere Diskussion bezüglich der Realität oder Irrealität der Komponenten von der ursprünglichen Epsteinschen etwas abweicht. Wir stützten uns auf unser theoretisch begründetes Auswahlprinzip zusammen mit dem Bohrschen Verbot der verschwindenden äquatorialen Quantenzahlen. Epstein dagegen benutzte ein (vom Verfasser früher vertretenes, jetzt fallen gelassenes) mehr empirisches

footnote on page 431, namely the 2nd part of Bohr's work "On the Quantum Theory of Line-spectra" [Boh18]. In the same footnote, Sommerfeld summarized Bohr's reason for this extra rule: the eventual collision of the electron with the nucleus.

It is probably worth repeating that even Planck was not too worried about the electron colliding with the nucleus [Pla20], and Sommerfeld appeared to have overcome his concerns by 1917 [Som17]. Epstein was asked in an interview [Eps62] whether "the problem of eventual collision in some of these orbits [was] considered to be a grave one." Epstein replied: "Not to my knowledge, except by mathematicians. You see, I remember that I talked about this point to Weyl, that is whether it makes sense to use such an orbit, and he thought no, it should be excluded. But we just kept it in" [Eps62]. The main reason for the extra rule excluding pendulum-paths that was provided by Bohr was the concern about an eventual collision with the nucleus, which had not stopped him in earlier publications [Boh14, Boh15, Boh16]. Thus, the issue of an eventual collision with the nucleus was probably not considered a compelling scientific reason (not even by Bohr and Sommerfeld) but a mechanical analogy to make plausible the exclusion of pendulum-paths. As mentioned by Born and Jordan, it also allowed Bohr and Sommerfeld to keep circular orbits in their model of hydrogen-like atoms without having to explain their reasons. For Sommerfeld, in particular, it was convenient to be able to refer to Bohr's extra rule [Boh18] when writing his book on atomic structure and spectral lines [Som21].

For Bohr, on the other hand, there was an event in early 1916 that might have contributed to his extra rule: Sommerfeld had sent him some early articles on the fine structure of spectral lines of the hydrogen atom, which caused Bohr to withdraw at least one planned publication [Eck13, page 213]. In Sommerfeld's articles [Som15b, Som15a], he excluded pendulum-paths based on concerns about collisions with the nucleus and infinite velocities [Som15b]. He also stated his strong intention to include circular orbits: "the circular orbit, which we will in any case declare to be possible"[5] [Som15b, page 445]. As mentioned, these reasons did not stop Sommerfeld from changing his mind about pendulum-paths in 1916 or 1917 [Som17]. However, they might have convinced Bohr to propose an extra rule in 1918 [Boh18], which then encouraged Sommerfeld to exclude pendulum-paths again in his book [Som21]. (Even some figures from earlier publications [Som16, Fig. 4 on page 23] were edited to remove pendulum-paths in corresponding figures of the book [Som21, Fig. 75 on page 272][Som23, Fig. 72 on page 240].)

Would Bohr's extra rule from 1918 have carried the same weight with Sommerfeld in 1919, if Sommerfeld had assumed that Bohr's rule was an echo of his own earlier opinion from 1915 that he had revised in 1916 or 1917 due to Epstein's work explaining observations of the Stark effect? Probably not. What else would have changed? Sommerfeld might have quoted Bohr's extra rule less often, which was what actually happened in later editions of his book [Som23], but he still argued that pendulum-paths should be excluded due to collisions with the nucleus. In any case, the anecdote might serve as a reminder that even great physicists (like Sommerfeld and Bohr) might fall victim to confirmation bias and echo chambers.

Soon after the Bohr-Sommerfeld model of atoms was published, modern quantum mechanics established the existence of states of hydrogen-like atoms with zero orbital angular momentum [PW35]. Pendulum-paths are not only the natural representation of such states in the Bohr-Sommerfeld model but also in many interpretations of quantum mechanics that include electron trajectories, e.g., ensemble interpretations [Bal70]. Therefore, it is not surprising that several researchers revisited the idea of pendulum-paths in hydrogen-like atoms since the 1920s, for example, Nicholson [Nic23], Lindsay [Lin27], Pauling [PW35, Pau60], Dankel and Levy [DL74], Bucher [Buc06, Buc08], and Rivas [Riv24], while Mac Gregor [Gre13] discussed similar paths in scattering experiments. Interested readers may find many more details about the history of pendulum-path (including pendulum-paths in two-electron systems) in Bucher's work [Buc08].

3 Simulation of Spinning Electrons on Pendulum-Paths

According to Rivas [Riv24], a spinning electron on a pendulum-path in a hydrogen atom spirals around the nucleus without colliding with it (see Section 2.1). The numerical simulation of this scenario presented here was based on the bi-level electron model [Kra25a]. The simulated equation of motion

Auswahlprinzip." [Som21, pages 451–452]

[5] Original text in German: „die Kreisbahn, die wir jedenfalls als möglich erklären werden" [Som15b, page 445]

for the center of mass $\mathbf{r}_{\mathrm{CM}}$ of an electron of charge q_{CC} in an electric field $\mathbf{E}(t, \mathbf{r}_{\mathrm{CC}})$ and magnetic field $\mathbf{B}(t, \mathbf{r}_{\mathrm{CC}})$ (with $\mathbf{r}_{\mathrm{CC}}$ and $\mathbf{v}_{\mathrm{CC}}$ denoting the position and velocity of the center of charge) is:

$$\frac{\mathrm{d}^2\mathbf{r}_{\mathrm{CM}}}{\mathrm{d}t^2} = \frac{q_{\mathrm{CC}}}{\gamma_{\mathrm{CM}} m_{\mathrm{CM}}} \frac{\omega_{\mathrm{C}}}{2\pi} \int_{t-2\pi/\omega_C}^{t} \left(\mathbf{E}(t', \mathbf{r}_{\mathrm{CC}}(t')) + \mathbf{v}_{\mathrm{CC}}(t') \times \mathbf{B}(t', \mathbf{r}_{\mathrm{CC}}(t'))\right) \mathrm{d}t', \tag{1}$$

with the Compton angular frequency $\omega_{\mathrm{C}} \stackrel{\mathrm{def}}{=} m_{\mathrm{CM}} c^2/\hbar$ and the factor $\gamma_{\mathrm{CM}} \stackrel{\mathrm{def}}{=} 1/\sqrt{1 - |\mathbf{v}_{\mathrm{CM}}|^2/c^2}$, which is missing in the non-relativistic version of the bi-level electron model [Kra25a].

The equation of motion for the center of charge $\mathbf{r}_{\mathrm{CC}}$ (which is spinning around $\mathbf{r}_{\mathrm{CM}}$) is:

$$\frac{\mathrm{d}^2\mathbf{r}_{\mathrm{CC}}}{\mathrm{d}t^2} = \omega_{\mathrm{C}}^2 \left(\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_{\mathrm{CC}}\right) + \frac{q_{\mathrm{CC}}}{m_{\mathrm{CC}}} \mathrm{proj}_{\mathbf{v}_{\mathrm{CC}} \times (\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_{\mathrm{CC}})} \left(\mathbf{E}(t, \mathbf{r}_{\mathrm{CC}}) + \mathbf{v}_{\mathrm{CC}} \times \mathbf{B}(t, \mathbf{r}_{\mathrm{CC}})\right) \tag{2}$$

with the mass $m_{\mathrm{CC}} \stackrel{\mathrm{def}}{=} m_{\mathrm{CM}}/2$ resisting the acceleration by the external fields and

$$\mathrm{proj}_{\mathbf{b}} \mathbf{a} \stackrel{\mathrm{def}}{=} \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}. \tag{3}$$

Furthermore, $\mathbf{v}_{\mathrm{CC}}$ is constrained by $|\mathbf{v}_{\mathrm{CC}}| = c$, i.e., $\mathbf{r}_{\mathrm{CC}}$ is always moving at the speed of light.

The external electric field $\mathbf{E}$ was set to a static Coulomb field of a proton's electric charge. Setting the external magnetic field $\mathbf{B}$ to a proton's magnetic dipole field had only a very small effect on the electron's trajectory. Therefore, the magnetic field was set to 0 for the results reported below. (However, the small effect of the proton's magnetic moment might be important to explain the spherical symmetry of the quantum mechanical ground state of the hydrogen atom.)

For the actual numerical integration method, an explicit midpoint method was implemented. The step size was set to 10^{-22}s, which ensured that there were several integration steps per period of the spin motion of $\mathbf{r}_{\mathrm{CC}}$. The total energy of the center-of-mass motion and the radius of the spin motion were constrained as described in earlier work [Kra25b]. (While these constraints are of critical importance for the robustness of the simulation, there was only a minor difference between the numerical results with the non-relativistic correction method and the relativistic correction method.)

4 Results

The first result obtained by the simulation was that a spinning electron on a pendulum-path in a hydrogen atom is in fact spiraling around the nucleus without "falling" into it. The maximum distance of the simulated electron on a pendulum-path from the nucleus as well as the time period from perihelion to perihelion numerically matched the expected values for pendulum-paths [DL74, Buc06]. This provided numerical evidence that spinning electrons can serve as a plausible explanation of how electrons move on pendulum-paths—as already hypothesized by Rivas [Riv24] and Mac Gregor [Gre13].

Inspired by Pauling and Wilson [PW35], the trajectories of a simulated electron on pendulum-paths (azimuthal quantum number $l = 0$) in a hydrogen atom (atomic number $Z = 1$) for values 1, 2, and 3 of the principal quantum number n were further analyzed by evaluating time-averaged values $\overline{r}$, $\overline{r^2}$, $\overline{r^{-1}}$, and $\overline{r^{-2}}$ of powers of the distance r between the fixed position of the nucleus and the center of charge of the spinning electron. (Using the center of mass instead of the center of charge resulted in worse results, specifically for $\overline{r^{-1}}$ and in particular for $\overline{r^{-2}}$, which is not surprising considering that the center of mass moves straight through the nucleus.)

Pauling and Wilson [PW35, page 144] provided the time-averaged value $\overline{r_{nk}}$ for a Bohr-Sommerfeld orbit with principal quantum number n and azimuthal quantum number k as

$$\overline{r_{nk}} = \frac{n^2 a_0}{Z} \left(1 + \frac{1}{2}\left(1 - \frac{k^2}{n^2}\right)\right) \tag{4}$$

with the Bohr radius a_0. For an improved version of the Bohr-Sommerfeld model, Pauling and Wilson suggested replacing k^2 by $l(l+1)$ with values $l = 0, \ldots, n-1$. The resulting expression is identical to the expectation value of r_{nlm} in a basic hydrogen model in non-relativistic quantum mechanics (i.e., "wave mechanics") [PW35, page 144].

In all simulated scenarios ($Z = 1$; $n = 1, 2, 3$; $l = k = 0$), numerical results for $\overline{r}$ matched the expected values with deviations of less than 1%.

For the time-averaged value $\overline{r_{nk}^2}$ in the Bohr-Sommerfeld model, Pauling and Wilson [PW35, page 145] provided the expression

$$\overline{r_{nk}^2} = \frac{a_0^2 n^4}{Z^2}\left(1 + \frac{3}{2}\left(1 - \frac{k^2}{n^2}\right)\right) \tag{5}$$

while the non-relativistic quantum-mechanical model predicts [PW35, page 144]

$$\overline{r_{nlm}^2} = \frac{a_0^2 n^4}{Z^2}\left(1 + \frac{3}{2}\left(1 - \frac{l(l+1) - \frac{1}{3}}{n^2}\right)\right). \tag{6}$$

Thus, even for $k = l = 0$, $\overline{r_{nk}^2}$ is less than $\overline{r_{nlm}^2}$ by $a_0^2 n^4 Z^{-2}/(2n^2)$.

Numerical results of the simulated scenarios matched $\overline{r_{nk}^2}$ of the improved Bohr-Sommerfeld model with deviations of less than 1%; therefore, they systematically underestimated the value of the wave-mechanical $\overline{r_{nlm}^2}$.

The time-averaged value of $\overline{r_n^{-1}}$ in the Bohr-Sommerfeld model as well as in the wave-mechanical model is $Z/(a_0 n^2)$ [PW35, page 145]. Numerical results matched these values with deviations of less than 3%.

For $\overline{r^{-2}}$, the wave-mechanical model predicts [PW35, page 145]

$$\overline{r_{nlm}^{-2}} = \frac{Z^2}{a_0^2 n^3 (l + \frac{1}{2})}, \tag{7}$$

while the Bohr-Sommerfeld model predicts

$$\overline{r_{nk}^{-2}} = \frac{Z^2}{a_0^2 n^3 k}. \tag{8}$$

Thus, $\overline{r_{nk}^{-2}}$ is undefined in the case $l = k = 0$ of the improved Bohr-Sommerfeld model considered here. Apparently, $\overline{r_{nk}^{-2}}$ is finite for $k = 1$ in the original Bohr-Sommerfeld model, which, however, does not include pendulum-paths.

In comparison to the wave-mechanical value of $\overline{r_{nlm}^{-2}}$, the obtained numerical results were too large by a factor of almost 4. While these deviations are substantial, they are an improvement compared to the undefined predictions of the improved Bohr-Sommerfeld model employed by Pauling and Wilson [PW35]. (Arguably, the original Bohr-Sommerfeld model provides better predictions for $\overline{r^{-2}}$ by excluding the case $k = 0$. However, it provides worse predictions for $\overline{r}$ and $\overline{r^2}$. Furthermore, it features additional shortcomings as discussed by Bucher [Buc08].)

The numerical simulations were also run for a spin motion of twice the Compton angular frequency at half the spin radius (as suggested by the model of Rivas [Riv01, Riv24]). The numerical results were very similar to the discussed results and, therefore, cannot be used to decide which spin frequency or radius is preferable.

5 Conclusions and Future Work

The main objective of this work was to provide numerical evidence for Rivas's hypothesis [Riv24] that spinning electrons can move on paths of zero orbital angular momentum (i.e., pendulum-paths) with the center of mass of the electron moving straight through the nucleus. The implementation of the simulation led to useful numerical integration techniques [Kra25b] and the development of the bi-level electron model [Kra25a]. The numerical simulations themselves showed not only that electrons on pendulum-paths spiral around the nucleus at a safe distance, but also that time-averaged values of powers of the distance between the nucleus and the center of charge of electrons on pendulum-paths are closer to the corresponding quantum mechanical expectation values than for an improved variant of the Bohr-Sommerfeld model [PW35, Buc06]. Furthermore, the discussion in Section 2.2 of the historical circumstances of the exclusion of pendulum-paths in the original Bohr-Sommerfeld model shows that there might have never been any compelling scientific reason to exclude them in the "old quantum theory." In modern quantum mechanics, on the other hand, the reality of particle trajectories in general is often denied or the concept is declared meaningless for quantum mechanics. However, some

interpretations of quantum mechanics (in particular ensemble interpretations) assume the existence of particle trajectories [Bal70]; thus, pendulum-paths might still have an important role to play for quantum mechanics.

Future work includes improvements to and further applications of the numerical simulation, for example, electronic orbits with non-zero angular momentum in hydrogen atoms [Buc14].

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A Revisions

- An earlier version based on [Kra24] (instead of [Kra25a]) was submitted to vixra.org on April 8, 2025, but it was withdrawn because it employed a questionable approximation ($t - t_y \approx 0$).
- Original version submitted to vixra.org on December 5, 2025.