Title: Formulas for su(3), sl(3,C) Matrix Generators, the Mathematica Notebook
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Today's Date:
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Out[2]= 2025-12-11
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Abstract

This notebook presents formulas for matrices that form the bases of the generators of the irreducible representations of the Lie algebras sl(3,C) and su(3). The matrix generators are shown to satisfy the commutation relations of the algebras. For an irrep of the reader’s choosing, the notebook calculates the bases of both algebras, sl(3,C) and su(3). The numerical matrices are saved in files in the folder with this notebook.

Table of Contents

i. Glossary:

(p,q) and (pp,qq): two non-negative integers that identify the sl(3,C) and su(3)-irreps. Single letters p, q for numerical values, doubled letters pp, qq for variables.

Table of matrix generators in the TYUV basis of sl(3,C).

x	1	2	3	4	5	6	7	8
X	T3	*Y*	Tp	Tm	Up	Um	Vp	Vm

where *x* is the index and *X* is the matrix. We have renamed the generators to avoid superscripts: T^3 = T3, T^+ = Tp, T^- = Tm, etc.

T,Y matrices: T3,Y,Tp,Tm
U,V matrices: Up,Um,Vp,Vm

(a,b): a and b are integers, 0<=a<=q and 0<=b<=p. See Fig. 2 in Ref. [1].
kab: The kth place for (ak,bk) in the sequence of (a,b) parameters
The pair (a,b) is a ‘double index’. The letter ‘k’ is a ‘single index’.

Block Array, see Figs. 1&2 above.
(iROW,jCOL): The row/column indices in the array of blocks
n: Block type. The (i,j) block has block type n=0 for a diagonal block i=j, n=1 for an upper block i<j, and n=2 for a lower block i>j.

n	Type of Block	*i*, *j*
0	Diagonal	$i = j$
1	Upper	$i < j$
2	Lower	$i > j$

(ai,bi)(aj,bj): Double block indices. The indices (iROW,jCOL) for nonzero blocks are functions of two (a,b) pairs, one pair (ai,bi) for rows, one pair (aj,bj) for columns.

The components in a block
(α,β): row/column ‘spin’ indices of a component in the matrix of an (i,j) block
(α,β): -ti<=α<=+ti and -tj<=β<=+tj, for ‘spins’ ti = (ai+bj)/2 & tj = (aj+bj)/2
comp: The (α,β) *comp*onent has a value of comp[x,n,q,p,ai,bi,aj,bj,α,β]
comp[x,n,q,p,ai,bi,aj,bj,α,β]: the function that produces the value of the α,β component of the (ai,bi)(aj,bj) block in the matrix x, which is an n-type block

su(3) basis

F1 = (Tp+Tm)/2 , F2 = -i(Tp-Tm)/2 , F3 = T3 , F4 = (Vp+Vm)/2 ,

F5 = -i(Vp-Vm)/2 , F6 = (Up+Um)/2 , F7 = -i(Up-Um)/2 , F8 = \sqrt{3}Y/2

In[3]:=
```
(*Glossary continued*)
(*Componant Notation:
*)
(*
X_(α,β)^(ai,bi),(aj,bj) : The (α,β) component of the (iROW,jCOL) block of the matrix X,
where iROW is a function of (ai,bi) and jCOL is a function of (aj,bj).
*)
(*The terminology follows Ref. [1] *)
(*The "T,Y Matrices" are the generators T3,Y,Tp,
and Tm. The T,Y matrices are nonzero in diagonal blocks.  *)
(*The "U,V Matrices" are the generators Up,Um,Vp,Vm.  *)
(*The "U,V Matrices" are nonzero in off-diagonal upper and lower blocks.*)
(*See Figs. 1 and 2 below.*)
```

In[4]:=
```
Unprotect[Up](*In this Notebook, 'Up' is a matrix, so we disable the MMA function Up.*)
```

Out[4]= {Up}

ii. Samples of Generators

$$T^{+} = \left(\begin{array}{c|cc|cc|ccc|ccc|cccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&\sqrt{2}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&\sqrt{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\sqrt{2}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&\sqrt{3}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\sqrt{3}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)$$

Figure 1. For the (p,q) = (2,1) -irrep, the generator T^+ splits into blocks.
Nonzero components are confined to diagonal blocks, i.e. (i,j) -blocks with i = j.

$$U^{+} = \left(\begin{array}{c|cc|cc|ccc|ccc|cccc}
0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 2\sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Figure 2. The generator U^{+} has nonzero components in upper blocks above the diagonal blocks and lower blocks below the diagonal blocks. Note that 'upper' (i,j)-blocks have $i < j$ and 'lower' blocks have $i > j$.

1. Introduction

The Special Unitary Lie group SU(3) is exemplified by the group of 3x3 unitary matrices with unit determinant combined by matrix multiplication. Here, we consider SU(3) irreducible representations (irreps) whose elements behave the same way as those 3x3 unitary matrices. There is one irrep for each pair of nonnegative integers (p,q). See [1-4] for better discussions of the topics mentioned in the abbreviated recap in this section.

The group has an algebra su(3). Each element g of the SU(3) Lie group is generated by an element F of the su(3) Lie algebra, $g = e^{iF}$, where the ‘generator’ F can be expressed in terms of a basis of eight generators Fj, j = 1,...,8. We have $F = \Sigma\theta_j F_j$ with real-valued θ_j. The generators satisfy a set of 28 commutation relations (CRs), which make the su(3) Lie algebra. The F_j matrix generators are hermitian and traceless, with complex components. Within similarity transformations, the algebra su(3) has one irrep for each pair of nonnegative integers (p,q).

An invertible linear transformation with complex coefficients applied to the basis F_j produces the TYUV basis of the sl(3,C) Lie algebra. sl(3,C) is the algebra for the SL(3,C) Lie group, the group whose elements multiply the way that 3x3 complex matrices with determinant one multiply. There is one irrep for each pair of nonnegative integers (p,q). Applying a similarity transformation to an irrep produces an equivalent irrep.

The TYUV basis for the (p,q) irrep of sl(3,C) can be traceless matrices with real-valued components.

The formulas in this article produce a set of TYUV matrices with real-valued components. The TYUV basis can then be transformed to the Fj basis of su(3), which is both traceless and hermitian self-conjugate.

Casimir operators of a Lie algebra commute with all elements of the algebra. For matrix irreps, they are proportional to the unit matrix, with a constant of proportionality dependent on the irrep identifiers (p,q). We highlight two Casimir operators for sl(3,C), C1 and C2, which are built from nonlinear combinations of basis generators. Other forms of the Casimir operators exist, but they are not independent of the two dealt with here. Being nonlinear, C1 is quadratic in F_j and C2 is cubic, the Casimir operators are not elements of the algebra.

For more details, see Ref. [5]. This notebook in its ready-to-run proprietary language can be accessed online, Ref. [6,7,8].

2. The TYUV basis of the (p,q) sl(3,C) Lie algebra

The (p,q) irrep of the sl(3,C) Lie algebra has a TYUV basis of eight generators T^3, Y, T^+; T^-, U^+, U^-, V^+, V^-. The matrices T^3 and Y are diagonal, while the other six are raising and lowering matrices. The four matrices T^3, Y, T^+; T^- form a basis for a u(2) Lie subalgebra and, so, they can be reduced to block diagonal form with a u(2) irrep in each block. The matrices representing the eight TYUV generators have dimension d , given by

$$d = \text{dimREP} = \frac{1}{2}(p+1)(q+1)(p+q+2)\ .$$

The dimension d is both the number of rows and the number of columns in a matrix representing the (p,q) irrep of the sl(3,C) algebra. It is also the number of simultaneous eigenvectors of T^3 and Y.

The job of applying order to the rows, columns and eigenvectors is taken by a special sequence of integers n, with n = 1,...,d. The sequence n is a function of three parameters a,b,α. We have

$$n(a,b,\alpha) = \frac{1}{2}\left[1 + (a+b+1)^2 + q\, b(q+b+2) - 2\,\alpha\right]\ ,$$

where $0 <= a <= q$, $0 <= b <= p$, $+(a+b)/2 >= \alpha >= -(a+b)/2$. In Ref. 5, the sequence is shown to be a consequence of the properties of the raising and lowering generators T^+, T^-, U^+, U^-, V^+, V^- .

Let r and c be the row and column indices for a square matrix M^{rc} with dimension d. Since r and c are integers in the range $1 <= r,c <= d$, there exists parameters a_r, b_r, α_r and a_c, b_c, α_c with

$$r = n(a_r, b_r, \alpha_r) \quad \text{and} \quad c = n(a_c, b_c, \alpha_c)\ .$$

We can use the six parameters a_r, b_r, α_r, a_c, b_c, α_c in the formulas for the values of the components M^{rc}. The formulas for the TYUV basis can be written in this way:

The formulas for the components of the TYUV matrices for the (p,q) sl(3,C) irrep. Each formula depends on three of the six parameters. Sometimes these are the parameters for the row index r, i.e. a,b,$\alpha = a_r$, b_r, α_r and sometimes the column index c, i.e. a,b,$\beta = a_c$, b_c, α_c.

T^3: $T^{3\,rc} = \alpha$; $r = c = n(a,b,\alpha)$
where $a = 0, ..., q$; $b = 0, ..., p$; $\alpha = (a+b)/2,\ (a+b)/2-1, \ldots, -(a+b)/2$

Y: $Y^{rc} = b - a - 2(p-q)/3$; $r = c = n(a,b,\alpha)$;
where $a = 0, ..., q$; $b = 0, ..., p$; $\alpha = (a+b)/2,\ (a+b)/2-1, \ldots, -(a+b)/2$

T^{+}: $T^{+\,rc}$ = $[(t+\alpha)(1+t-\alpha]^{1/2}$; r = n(a,b,α) ; c = n(a,b,α-1) ;
where a = 0, ..., q ; b = 0, ..., p ; α = (a+b)/2, (a+b)/2-1, ... , -(a+b)/2 + 1

T^{-}: $T^{-\,rc}$ = $[(t-\alpha)(1+t+\alpha]^{1/2}$; r = n(a,b,α) ; c = n(a,b,α+1) ;
where a = 0, ..., q ; b = 0, ..., p ; α = (a+b)/2 - 1, (a+b)/2-1, ... , -(a+b)/2

U^{+}: $U^{+\,rc}$ = $[g(a,b)(t+\beta)]^{1/2}$; r = n(a-1,b,β-1/2) ; c = n(a,b,β) ;
where a = 1, ..., q ; b = 0, ..., p ; β = (a+b)/2, (a+b)/2-1, ... , -(a+b)/2+1
$U^{+\,rc}$ = $[h(a,b)(t-\alpha)]^{1/2}$; r = n(a,b,α) ; c = n(a,b-1,α+1/2) ;
where a = 0, ..., q ; b = 1, ..., p ; α = (a+b)/2-1, ... , -(a+b)/2

U^{-}: $U^{-\,rc}$ = $[h(a,b)(t-\beta)]^{1/2}$; r = n(a,b-1,β+1/2) ; c = n(a,b,β) ;
where a = 0, ..., q ; b = 1, ..., p ; β = (a+b)/2-1, ... , -(a+b)/2
$U^{-\,rc}$ = $[g(a,b)(t+\beta)]^{1/2}$; r = n(a,b,α) ; c = n(a-1,b,α-1/2) ;
where a = 1, ..., q ; b = 0, ..., p ; α = (a+b)/2, ... , -(a+b)/2+1

V^{+}: $V^{+\,rc}$ = $-[g(a,b)(t-\beta)]^{1/2}$; r = n(a-1,b,β+1/2) ; c = n(a,b,β) ;
where a = 1, ..., q ; b = 0, ..., p ; β = (a+b)/2-1, ... , -(a+b)/2
$V^{+\,rc}$ = $[h(a,b)(t+\alpha)]^{1/2}$; r = n(a,b,α) ; c = n(a,b-1,α-1/2) ;
where a = 0, ..., q ; b = 1, ..., p ; β = (a+b)/2, ... , -(a+b)/2+1

V^{-}: $V^{-\,rc}$ = $[h(a,b)(t+\beta)]^{1/2}$; r = n(a,b-1,β-1/2) ; c = n(a,b,β) ;
where a = 0, ..., q ; b = 1, ..., p ; β = (a+b)/2, ... , -(a+b)/2+1
$V^{-\,rc}$ = $-[g(a,b)(t-\alpha)]^{1/2}$; r = n(a,b,α) ; c = n(a-1,b,α+1/2) ;
where a = 1, ..., q ; b = 0, ..., p ; β = (a+b)/2-1, ... , -(a+b)/2 .

The functions g(a,b) and h(a,b) are
g(a,b) = a(p+a+1)(q-a+1)/[(a+b)a+b+1)]
h(a,b) = b(p-b+1)(q+b+1)/[(a+b)a+b+1)] .

The TYUV basis satisfies the following commutation relations of the sl(3,C) Lie algebra.

$$[T^{+}, T^{-}] = 2\,T^{3}\,;\ [T^{3}, T^{\pm}] = \pm T^{\pm}\,;\ [Y, T^{\pm}] = 0\,;\ [Y, T^{3}] = 0$$
$$[T^{\pm}, U^{\mp}] = \mp\tfrac{1}{2}\,U^{\pm}\,;\ [T^{3}, V^{\pm}] = \pm\tfrac{1}{2}\,V^{\pm}\,;\ [Y, U^{\pm}] = \pm U^{\pm}\,;\ [Y, V^{\pm}] = \pm V^{\pm}$$
$$[U^{+}, U^{-}] = \tfrac{3}{2}\,Y - T^{3}\,;\ [V^{+}, V^{-}] = \tfrac{3}{2}\,Y + T^{3}\,;\ [U^{\pm}, V^{\mp}] = \pm T^{\mp}\,;\ [U^{\pm}, V^{\pm}] = 0\,.$$

The formulas for the components of the Fj matrices for the (p,q) su(3) irrep are given implicitly. The transformation from the TYUV basis of sl(3,C) to the Fj basis of su(3) is

F1 = (Tp+Tm)/2 , F2 = -i(Tp-Tm)/2 , F3 = T3 , F4 = (Vp+Vm)/2 ,
F5 = -i(Vp-Vm)/2 , F6 = (Up+Um)/2 , F7 = -i(Up-Um)/2 , F8 = $\sqrt{3}$ Y/2 .

These matrices are traceless and they are self-Hermitian conjugates: $F_j^{*T} = F_j$.

3. Formulas for the TYUV basis generators of the (p,q) sl(3C) irrep

The formulas are written in block-matrix notation. The component comp = M^{rc} with r = n(a_r, b_r, α_r) and c = n(a_c, b_c, α_c) can be written instead as

$$\text{comp} = M^{rc} = M_{(\alpha_r,\alpha_c)}{}^{(a_r,b_r),(a_c,b_c)} = M_{(\alpha_r,\alpha_c)}{}^{i,j} \;;$$

$$i = 1 + a_r + b_r(q+1)\,,\; j = 1 + a_j + b_j(q+1)\,;$$

We say that the i,j block of the matrix M has the component $M_{(\alpha_r,\alpha_c)}{}^{i,j}$ in row α_r and column α_c.

In[14]:=

```
(*Nonzero Components:
  The nonzero components are functions of three parameters (a,b) and α.
    diagonal blocks:
  T3_(α,α)^(a,b),(a,b) ,Y_(α,α)^(a,b),(a,b) ,Tp_(α,α-1)^(a,b),(a,b) Tm_(α,α+1)^(a,b),(a,b)
   upper blocks:
  Up_(β-1/2,β)^(a-1,b),(a,b);  Vp_(β+1/2,β)^(a-1,b),(a,b); Um_(β+1/2,β)^(a,b-1),(a,b);  Vm_(β-1/2,β)^(a,b-1),(a,b)
   lower blocks:
  Up_(α,α+1/2)^(a,b),(a,b-1);  Vp_(α,α-1/2)^(a,b),(a,b-1); Um_(α,α-1/2)^(a,b),(a-1,b);  Vm_(α,α+1/2)^(a,b),(a-1,b)
  The ranges allowed to (a,b) and α are given
 with each function comp[x,n,q,p,ai,bi,aj,bj,α,β].
  Components that are not listed above vanish.
*)
```

In[15]:=
```
Clear[g, h]
```

In[16]:=
```
(*g, h - two functions defined here. *)
(*g,h  are used to write the components of the Up,Um,Vp,Vm matrices *)
g[qq_, pp_, a_, b_] := ((+a) (1 + pp + a) (1 + qq - a) / ((a + b) (a + b + 1)))
h[qq_, pp_, a_, b_] := ((+b) (1 + pp - b) (1 + qq + b) / ((a + b) (a + b + 1)))
```

In[18]:=
```
(*kab:The place of the pair of parameters
  (a,b) in the sequence of (ak,bk) parameter pairs*)
kab[qq_, pp_, a_, b_] := 1 + a + (qq + 1) b
```

In[19]:=
```
(*For matrix generators*)
(*double index (ai,bi) for block in ith row,
where i is the single index also called 'r'.*)
(*double index (aj,bj) for block in jth column,
where j is the single index also called 'c'.*)
(*Below, n = 0,1,2 indicates the type of ij block,
n = 0 for a diagonal, n = 1 for an upper, n = 2 for a lower block.*)
(*Above, we used 'n' for the sequence of integers 1,...,
d as a function of the parameters a, b, α.*)
```

```
In[20]:= (*x = 1 indicates matrix X = T3. *)
(*n = 0 indicates  On-Diagonal blocks: (aj,bj) = (ai,bi).*)
(*Min,Max a:*)
axnMin[1, 0, qq_, pp_] := 0; axnMax[1, 0, qq_, pp_] := qq;
(*Min,Max b:*)
bxnMin[1, 0, qq_, pp_] := 0; bxnMax[1, 0, qq_, pp_] := pp;
(*Row/column block address: *)
iROW[1, 0, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
jCOL[1, 0, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
(*Nonzero component T3_(α,α)^(a,b),(a,b) = *)
comp[1, 0, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] := (α KroneckerDelta[β - α] KroneckerDelta[aj - ai] KroneckerDelta[bj - bi]);

In[25]:= (*x = 2 indicates matrix X = Y. *)
(*n = 0 indicates  On-Diagonal blocks: (aj,bj) = (ai,bi).*)
axnMin[2, 0, qq_, pp_] := 0;
axnMax[2, 0, qq_, pp_] := qq;
bxnMin[2, 0, qq_, pp_] := 0;
bxnMax[2, 0, qq_, pp_] := pp;
iROW[2, 0, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
jCOL[2, 0, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
(*Nonzero component Y_(α,α)^(a,b),(a,b) = *)
comp[2, 0, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] :=
  ((bi - ai - 2 (pp - qq) / 3) KroneckerDelta[β - α] KroneckerDelta[aj - ai] KroneckerDelta[bj - bi]);

In[32]:= (*x = 3 indicates matrix X = Tp. *)
(*n = 0 indicates  On-Diagonal blocks: (aj,bj) = (ai,bi).*)
axnMin[3, 0, qq_, pp_] := 0;
axnMax[3, 0, qq_, pp_] := qq;
bxnMin[3, 0, qq_, pp_] := 0;
bxnMax[3, 0, qq_, pp_] := pp;
iROW[3, 0, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
jCOL[3, 0, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
(*Nonzero component Tp_(α,α-1)^(a,b),(a,b) = *)
comp[3, 0, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] :=
  (((ai + bi) / 2 + α) (1 + (ai + bi) / 2 - α))^1/2 KroneckerDelta[β - (α - 1)] KroneckerDelta[aj - ai] KroneckerDelta[bj - bi];

In[39]:= (*x = 4 indicates matrix X = Tm. *)
(*n = 0 indicates  On-Diagonal blocks: (aj,bj) = (ai,bi).*)
axnMin[4, 0, qq_, pp_] := 0;
axnMax[4, 0, qq_, pp_] := qq;
bxnMin[4, 0, qq_, pp_] := 0;
bxnMax[4, 0, qq_, pp_] := pp;
iROW[4, 0, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
jCOL[4, 0, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
(*Nonzero component Tm_(α,α+1)^(a,b),(a,b) = *)
comp[4, 0, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] :=
  (((ai + bi) / 2 - α) (1 + (ai + bi) / 2 + α))^1/2 KroneckerDelta[β - (α + 1)] KroneckerDelta[aj - ai] KroneckerDelta[bj - bi];

In[46]:= (*x = 5 indicates matrix X = Up. *)
(*n = 1 indicates Upper Off-diagonal blocks:*)
(*axn is aj, bxn is bj*)
axnMin[5, 1, qq_, pp_] := +1;
axnMax[5, 1, qq_, pp_] := qq;
bxnMin[5, 1, qq_, pp_] := 0;
bxnMax[5, 1, qq_, pp_] := pp;
(*double index for block row (ai,bi)*)
aixn[5, 1, qq_, pp_, ai_, bi_, aj_, bj_] := aj - 1
bixn[5, 1, qq_, pp_, ai_, bi_, aj_, bj_] := bj
(*double index for block column (aj,bj)*)
ajxn[5, 1, qq_, pp_, ai_, bi_, aj_, bj_] := aj
bjxn[5, 1, qq_, pp_, ai_, bi_, aj_, bj_] := bj
iROW[5, 1, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, aj - 1, bj];
jCOL[5, 1, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, aj, bj];
(*Nonzero component Up_(α,α+1/2)^(a-1,b),(a,b) = *)
comp[5, 1, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] := ((g[qq, pp, aj, bj] ( ((aj + bj + 1) / 2 + α)))^1/2
    KroneckerDelta[β - (α + 1 / 2)] KroneckerDelta[(aj - 1) - ai] KroneckerDelta[bj - bi]);
```

In[57]:=
```
(*x = 5 indicates matrix X = Up. *)
(*n = 2 indicates Lower Off-diagonal blocks:*)
(*(aj,bj) = (ai,bi-1); (ai,bi) = (aj,bj+1) *)
(**)
axnMin[5, 2, qq_, pp_] := 0;
axnMax[5, 2, qq_, pp_] := qq;
bxnMin[5, 2, qq_, pp_] := 1;
bxnMax[5, 2, qq_, pp_] := pp;
aixn[5, 2, qq_, pp_, ai_, bi_, aj_, bj_] := ai
bixn[5, 2, qq_, pp_, ai_, bi_, aj_, bj_] := bi
ajxn[5, 2, qq_, pp_, ai_, bi_, aj_, bj_] := ai
bjxn[5, 2, qq_, pp_, ai_, bi_, aj_, bj_] := bi - 1
iROW[5, 2, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
jCOL[5, 2, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi - 1];
(*Nonzero component Up_(α,α+1/2)^(a,b),(a,b-1) = *)
comp[5, 2, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] :=
  ((h[qq, pp, ai, bi] ((ai + bi) / 2 - α))^(1/2) KroneckerDelta[β - (α + 1 / 2)] KroneckerDelta[aj - ai] KroneckerDelta[bj - (bi - 1)]);
```

In[68]:=
```
(*x = 6 indicates matrix X = Um. *)
(*n = 1 indicates Upper Off-diagonal blocks:*)
(*(aj,bj) = (ai,bi+1) *)
(*(axn,bxn) = (aj,bj) for (x,n) = (6,1)*)
axnMin[6, 1, qq_, pp_] := 0;
axnMax[6, 1, qq_, pp_] := qq;
bxnMin[6, 1, qq_, pp_] := 1;
bxnMax[6, 1, qq_, pp_] := pp;
aixn[6, 1, qq_, pp_, ai_, bi_, aj_, bj_] := aj
bixn[6, 1, qq_, pp_, ai_, bi_, aj_, bj_] := bj - 1
ajxn[6, 1, qq_, pp_, ai_, bi_, aj_, bj_] := aj
bjxn[6, 1, qq_, pp_, ai_, bi_, aj_, bj_] := bj
jCOL[6, 1, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, aj, bj];
iROW[6, 1, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, aj, bj - 1];
(*Nonzero component Um_(α,α-1/2)^(a,b-1),(a,b) = *)
comp[6, 1, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] := ((h[qq, pp, aj, bj] ( ((aj + bj + 1) / 2 - α)))^(1/2)
    KroneckerDelta[β - (α - 1 / 2)] KroneckerDelta[aj - ai] KroneckerDelta[(bj - 1) - bi]);
```

In[78]:=
```
(*x = 6 indicates matrix X = Um. *)
(*n = 2 indicates Lower Off-diagonal blocks:*)
(*(aj,bj) = (ai-1,bi); (ai,bi) = (aj-1,bj) *)
(*(axn,bxn) = (ai,bi) for (x,n) = (6,2)*)
axnMin[6, 2, qq_, pp_] := 1;
axnMax[6, 2, qq_, pp_] := qq;
bxnMin[6, 2, qq_, pp_] := 0;
bxnMax[6, 2, qq_, pp_] := pp;
aixn[6, 2, qq_, pp_, ai_, bi_, aj_, bj_] := ai
bixn[6, 2, qq_, pp_, ai_, bi_, aj_, bj_] := bi
ajxn[6, 2, qq_, pp_, ai_, bi_, aj_, bj_] := ai - 1
bjxn[6, 2, qq_, pp_, ai_, bi_, aj_, bj_] := bi
iROW[6, 2, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
jCOL[6, 2, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai - 1, bi];
(*Nonzero component Um_(α,α-1/2)^(a,b),(a-1,b) = *)
comp[6, 2, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] :=
  ((g[qq, pp, ai, bi] ((ai + bi) / 2 + α))^(1/2) KroneckerDelta[β - (α - 1 / 2)] KroneckerDelta[aj - (ai - 1)] KroneckerDelta[bj - bi]);
```

In[89]:=
```
(*x = 7 indicates matrix X = Vp. *)
(*n = 1 indicates Upper Off-diagonal blocks:*)
(*(aj,bj) = (ai+1,bi); (ai,bi) = (aj-1,bj) *)
(**)
axnMin[7, 1, qq_, pp_] := 1;
axnMax[7, 1, qq_, pp_] := qq;
bxnMin[7, 1, qq_, pp_] := 0;
bxnMax[7, 1, qq_, pp_] := pp;
aixn[7, 1, qq_, pp_, ai_, bi_, aj_, bj_] := aj - 1
bixn[7, 1, qq_, pp_, ai_, bi_, aj_, bj_] := bj
ajxn[7, 1, qq_, pp_, ai_, bi_, aj_, bj_] := aj
bjxn[7, 1, qq_, pp_, ai_, bi_, aj_, bj_] := bj
jCOL[7, 1, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, aj, bj];
iROW[7, 1, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, aj - 1, bj];
(*Nonzero component Vp_(α,α-1/2)^(a-1,b),(a,b) = *)
comp[7, 1, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] := (-(g[qq, pp, aj, bj] (((aj + bj + 1) / 2 - α)))^1/2
    KroneckerDelta[β - (α - 1 / 2)] KroneckerDelta[(aj - 1) - ai] KroneckerDelta[bj - bi]);
```

In[99]:=
```
(*x = 7 indicates matrix X = Vp. *)
(*n = 2 indicates Lower Off-diagonal blocks:*)
(*(aj,bj) = (ai+1,bi); (ai,bi) = (aj-1,bj) *)
axnMin[7, 2, qq_, pp_] := 0;
axnMax[7, 2, qq_, pp_] := qq;
bxnMin[7, 2, qq_, pp_] := 1;
bxnMax[7, 2, qq_, pp_] := pp;
aixn[7, 2, qq_, pp_, ai_, bi_, aj_, bj_] := ai
bixn[7, 2, qq_, pp_, ai_, bi_, aj_, bj_] := bi
ajxn[7, 2, qq_, pp_, ai_, bi_, aj_, bj_] := ai
bjxn[7, 2, qq_, pp_, ai_, bi_, aj_, bj_] := bi - 1
iROW[7, 2, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
jCOL[7, 2, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi - 1];
(*Nonzero component Vp_(α,α-1/2)^(a,b),(a,b-1) = *)
comp[7, 2, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] :=
  ((h[qq, pp, ai, bi] ((ai + bi) / 2 + α))^1/2 KroneckerDelta[β - (α - 1 / 2)] KroneckerDelta[aj - ai] KroneckerDelta[bj - (bi - 1)]);
```

In[110]:=
```
(*x = 8 indicates matrix X = Vm. *)
(*n = 1 indicates Upper Off-diagonal blocks:*)
(*(aj,bj) = (ai,bi+1) *)
(*(axn,bxn) = (aj,bj) for (x,n) = (6,1)*)
axnMin[8, 1, qq_, pp_] := 0;
axnMax[8, 1, qq_, pp_] := qq;
bxnMin[8, 1, qq_, pp_] := 1;
bxnMax[8, 1, qq_, pp_] := pp;
aixn[8, 1, qq_, pp_, ai_, bi_, aj_, bj_] := aj
bixn[8, 1, qq_, pp_, ai_, bi_, aj_, bj_] := bj - 1
ajxn[8, 1, qq_, pp_, ai_, bi_, aj_, bj_] := aj
bjxn[8, 1, qq_, pp_, ai_, bi_, aj_, bj_] := bj
jCOL[8, 1, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, aj, bj];
iROW[8, 1, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, aj, bj - 1];
(*Nonzero component Vm_(α,α+1/2)^(a,b-1),(a,b) = *)
comp[8, 1, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] := ((h[qq, pp, aj, bj] (((aj + bj + 1) / 2 + α)))^1/2
    KroneckerDelta[β - (α + 1 / 2)] KroneckerDelta[aj - ai] KroneckerDelta[(bj - 1) - bi]);
```

In[120]:=

```
(*x = 8 indicates matrix X = Vm. *)
(*n = 2 indicates Lower Off-diagonal blocks:*)
(*(aj,bj) = (ai-1,bi); (ai,bi) = (aj-1,bj) *)
(*(axn,bxn) = (ai,bi) for (x,n) = (6,2)*)
axnMin[8, 2, qq_, pp_] := 1;
axnMax[8, 2, qq_, pp_] := qq;
bxnMin[8, 2, qq_, pp_] := 0;
bxnMax[8, 2, qq_, pp_] := pp;
aixn[8, 2, qq_, pp_, ai_, bi_, aj_, bj_] := ai
bixn[8, 2, qq_, pp_, ai_, bi_, aj_, bj_] := bi
ajxn[8, 2, qq_, pp_, ai_, bi_, aj_, bj_] := ai - 1
bjxn[8, 2, qq_, pp_, ai_, bi_, aj_, bj_] := bi
iROW[8, 2, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai, bi];
jCOL[8, 2, qq_, pp_, ai_, bi_, aj_, bj_] := kab[qq, pp, ai - 1, bi];
(*Nonzero component Vm_(α,α+1/2)^(a,b),(a-1,b) = *)
comp[8, 2, qq_, pp_, ai_, bi_, aj_, bj_, α_, β_] :=
  (-(g[qq, pp, ai, bi] ((ai + bi) / 2 - α))^(1/2) KroneckerDelta[β - (α + 1 / 2)] KroneckerDelta[aj - (ai - 1)] KroneckerDelta[bj - bi]);
```

4. Verification of the TYUV basis of the sl(3,C) Lie algebra, Casimir

In[131]:=

```
(*
Check the Lie algebra. This section substitutes the above component formulas
 into the commutation relations (CRs) of the sl(3,C) Lie algebra. Mathematica
 checks whether each CR is satisfied, True, or not satisfied, False.
*)
```

In[132]:=

```
(* Section Glossary *)
(* lhs, rhs: Left- and right-sides of an equation. *)
(* termNOT0, termsNOT0: One or more terms that are
   expected to be nonzero are displayed. A CR check resulting in 0 =
 0 should be flagged and checked for typos. It may be a false positive.  *)
```

CRs with T,Y matrices only:

In[133]:=

```
(* [T⁺,T⁻] = 2T³*)
(*Tp_(α,α-1)^(a,b),(a,b) Tm_(α-1,α)^(a,b),(a,b)  -  Tm_(α,α+1)^(a,b),(a,b) Tp_(α+1,α)^(a,b),(a,b)      *)
(*= 2T3_(α,α)^(a,b),(a,b)*)
lhs = comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[4, 0, qq, pp, a, b, a, b, α - 1, α] -
   comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[3, 0, qq, pp, a, b, a, b, α + 1, α];
rhs = 2 comp[1, 0, qq, pp, a, b, a, b, α, α];
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, -(a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁺,T⁻] = 2T³: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

```
The formulas satisfy [T⁺,T⁻] = 2T³: True
```

In[137]:=

```
(* [T^3,T^+] = +T^+*)
(*T3_(α,α)^(a,b),(a,b) Tp_(α,α-1)^(a,b),(a,b) - Tp_(α,α-1)^(a,b),(a,b) T3_(α-1,α-1)^(a,b),(a,b) *)
(*= +Tp_(α,α-1)^(a,b),(a,b)*)
lhs = comp[1, 0, qq, pp, a, b, a, b, α, α] × comp[3, 0, qq, pp, a, b, a, b, α, α - 1] -
   comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[1, 0, qq, pp, a, b, a, b, α - 1, α - 1];
rhs = comp[3, 0, qq, pp, a, b, a, b, α, α - 1];
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T^3,T^+] = +T^+: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[T^3,T^+] = +T^+$: True

In[141]:=

```
(* [T^3,T^-] = -T^-*)
(*T3_(α,α)^(a,b),(a,b) Tm_(α,α+1)^(a,b),(a,b) - Tm_(α,α+1)^(a,b),(a,b) T3_(α+1,α+1)^(a,b),(a,b) *)
(*= +Tm_(α,α+1)^(a,b),(a,b)*)
lhs = comp[1, 0, qq, pp, a, b, a, b, α, α] × comp[4, 0, qq, pp, a, b, a, b, α, α + 1] -
   comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[1, 0, qq, pp, a, b, a, b, α + 1, α + 1];
rhs = - comp[4, 0, qq, pp, a, b, a, b, α, α + 1];
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T^3,T^-] = -T^-: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[T^3,T^-] = -T^-$: True

In[145]:=

```
(* [T^+,Y] = 0 *)
(*Tp_(α,α-1)^(a,b),(a,b) Y_(α-1,α-1)^(a,b),(a,b) - Y_(α,α)^(a,b),(a,b) Tp_(α,α-1)^(a,b),(a,b) *)
(*= 0*)
termNOT0 = comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[2, 0, qq, pp, a, b, a, b, α - 1, α - 1];
lhs = comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[2, 0, qq, pp, a, b, a, b, α - 1, α - 1] -
   comp[2, 0, qq, pp, a, b, a, b, α, α] × comp[3, 0, qq, pp, a, b, a, b, α, α - 1];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T^+,Y] = 0: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[T^+,Y] = 0$: True

In[150]:=

```
(* [T⁻,Y] = 0 *)
(*Tm_(α,α+1)^((a,b),(a,b)) Y_(α+1,α+1)^((a,b),(a,b)) - Y_(α,α)^((a,b),(a,b)) Tm_(α,α+1)^((a,b),(a,b)) *)
(*= 0*)
termNOT0 = comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[2, 0, qq, pp, a, b, a, b, α + 1, α + 1];
lhs = comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[2, 0, qq, pp, a, b, a, b, α + 1, α + 1] -
   comp[2, 0, qq, pp, a, b, a, b, α, α] × comp[4, 0, qq, pp, a, b, a, b, α, α + 1];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁻,Y] = 0: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy [T⁻,Y] = 0: True

In[155]:=

```
(* [T^3,Y] = 0*)
(*T3_(α,α)^((a,b),(a,b)) Y_(α,α)^((a,b),(a,b)) - Y_(α,α)^((a,b),(a,b)) T3_(α,α)^((a,b),(a,b)) *)
(*= 0*)
termNOT0 = comp[1, 0, qq, pp, a, b, a, b, α, α] × comp[2, 0, qq, pp, a, b, a, b, α, α];
lhs = comp[1, 0, qq, pp, a, b, a, b, α, α] × comp[2, 0, qq, pp, a, b, a, b, α, α] -
   comp[2, 0, qq, pp, a, b, a, b, α, α] × comp[1, 0, qq, pp, a, b, a, b, α, α];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T^3,Y] = 0: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[T^3,Y] = 0$: True

In[160]:=

```
(* CRs linear in U,V matrices: *)
```

In[161]:=

```
(* [T^3,U^+] = -1/2 U^+*)
(* [T^3,U_Upper^+]+[T^3,U_Lower^+] = -1/2 U_Upper^+ - 1/2 U_Lower^+ *)

(*T3_(α,α)^((a,b),(a,b)) Up_(α,α+1/2)^((a,b),(a+1,b)) - Up_(α,α+1/2)^((a,b),(a+1,b)) T3_(α+1/2,α+1/2)^((a+1,b),(a+1,b)) +
 T3_(α,α)^((a,b),(a,b)) Up_(α,α+1/2)^((a,b),(a,b-1)) - Up_(α,α+1/2)^((a,b),(a,b-1)) T3_(α+1/2,α+1/2)^((a,b-1),(a,b-1)) *)
(*= -1/2 Up_(α,α+1/2)^((a,b),(a+1,b)) - 1/2 Up_(α,α+1/2)^((a,b),(a,b-1)) *)
```

In[162]:=

```
lhs =
  Simplify[comp[1, 0, qq, pp, a, b, a, b, α, α] (comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2]) -
     (comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2])
      comp[1, 0, qq, pp, a + 1, b, a + 1, b, α + 1 / 2, α + 1 / 2] +
     comp[1, 0, qq, pp, a, b, a, b, α, α] (comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2]) -
     (comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2])
      comp[1, 0, qq, pp, a, b - 1, a, b - 1, α + 1 / 2, α + 1 / 2]];
rhs = -1/2 (comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2]) -
   1/2 (comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2]);
FullSimplify[lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T^3,U^+] = -1/2U^+: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[T^3,U^+] = -\frac{1}{2}U^+$: True

In[166]:=

(* $[T^3,U^-] = +\frac{1}{2}U^-$ *)

(* $[T^3,U_{Upper}{}^-]+[T^3,U_{Lower}{}^-] = +\frac{1}{2}U_{Upper}{}^- + \frac{1}{2}U_{Lower}{}^-$ *)

(* $T3_{(\alpha,\alpha)}{}^{(a,b),(a,b)}\,Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a,b+1)} - Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a,b+1)}\,T3_{(\alpha-1/2,\alpha-1/2)}{}^{(a,b+1),(a,b+1)} + T3_{(\alpha,\alpha)}{}^{(a,b),(a,b)}\,Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a-1,b)} - Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a-1,b)}\,T3_{(\alpha-1/2,\alpha-1/2)}{}^{(a-1,b),(a-1,b)}$ *)

(*= $+\frac{1}{2}Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a,b+1)} + \frac{1}{2}Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a-1,b)}$ *)

In[167]:=

```
lhs =
  Simplify[comp[1, 0, qq, pp, a, b, a, b, α, α] (comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2]) -
     (comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2])
      comp[1, 0, qq, pp, a, b + 1, a, b + 1, α - 1 / 2, α - 1 / 2] +
     comp[1, 0, qq, pp, a, b, a, b, α, α] (comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2]) -
     (comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2])
      comp[1, 0, qq, pp, a - 1, b, a - 1, b, α - 1 / 2, α - 1 / 2]];
rhs = +1/2 (comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2]) +
   1/2 (comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2]);
FullSimplify[lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T^3,U^-] = +1/2U^-: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[T^3,U^-] = +\frac{1}{2}U^-$: True

In[171]:=

```
(* [T^3,V^+] = +1/2 V^+ *)
(* [T^3,V_Upper^+]+[T^3,V_Lower^+] = +1/2 V_Upper^+ + 1/2 V_Lower^+ *)

(*T3_(α,α)^(a,b),(a,b) Vp_(α,α-1/2)^(a,b),(a+1,b) - Vp_(α,α-1/2)^(a,b),(a+1,b) T3_(α-1/2,α-1/2)^(a+1,b),(a+1,b) +
 T3_(α,α)^(a,b),(a,b) Vp_(α,α-1/2)^(a,b),(a,b-1) - Vp_(α,α-1/2)^(a,b),(a,b-1) T3_(α-1/2,α-1/2)^(a,b-1),(a,b-1) *)
(*= +1/2 Vp_(α,α-1/2)^(a,b),(a+1,b) + 1/2 Vp_(α,α-1/2)^(a,b),(a,b-1) *)
lhs =
  Simplify[comp[1, 0, qq, pp, a, b, a, b, α, α] (comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2]) -
     (comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2])
      comp[1, 0, qq, pp, a + 1, b, a + 1, b, α - 1 / 2, α - 1 / 2] +
     comp[1, 0, qq, pp, a, b, a, b, α, α] (comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2]) -
     (comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2])
      comp[1, 0, qq, pp, a, b - 1, a, b - 1, α - 1 / 2, α - 1 / 2]];
rhs = + 1/2 (comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2]) +
   1/2 (comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2]);
Simplify[lhs - rhs, {a ≥ 0, b ≥ 1}];
Print["The formulas satisfy [T^3,V^+] = + 1/2 V^+: ", 0 == Simplify[lhs - rhs, {a ≥ 0, b ≥ 1}]]
```

The formulas satisfy $[T^3,V^+] = +\frac{1}{2}V^+$: True

In[175]:=

```
(* [T^3,V^-] = -1/2 V^- *)
(* [T^3,V_Upper^-]+[T^3,V_Lower^-] = -1/2 V_Upper^- - 1/2 V_Lower^- *)

(*T3_(α,α)^(a,b),(a,b) Vm_(α,α+1/2)^(a,b),(a,b+1) - Vm_(α,α+1/2)^(a,b),(a,b+1) T3_(α+1/2,α+1/2)^(a,b+1),(a,b+1) +
 T3_(α,α)^(a,b),(a,b) Vm_(α,α+1/2)^(a,b),(a-1,b) - Vm_(α,α+1/2)^(a,b),(a-1,b) T3_(α+1/2,α+1/2)^((a-1,b),(a-1,b) *)
(*= -1/2 Vm_(α,α+1/2)^(a,b),(a,b+1) - 1/2 Vm_(α,α+1/2)^(a,b),(a-1,b) *)
```

In[176]:=

```
lhs =
  Simplify[comp[1, 0, qq, pp, a, b, a, b, α, α] (comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2]) -
    (comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2])
     comp[1, 0, qq, pp, a, b + 1, a, b + 1, α + 1 / 2, α + 1 / 2] +
    comp[1, 0, qq, pp, a, b, a, b, α, α] (comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2]) -
    (comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2])
     comp[1, 0, qq, pp, a - 1, b, a - 1, b, α + 1 / 2, α + 1 / 2]];
rhs = -1/2 (comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2]) -
   1/2 (comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2]);
Simplify[lhs - rhs, {a ≥ 1, b ≥ 0}];
Print["The formulas satisfy [T^3,V^-] = -1/2 V^-: ", 0 == Simplify[lhs - rhs, {a ≥ 1, b ≥ 0}]]
```

The formulas satisfy $[T^3, V^-] = -\frac{1}{2} V^-$: True

In[180]:=

```
(* [Y,U^+] = +U^+ *)
(* [Y,U_Upper^+]+[Y,U_Lower^+] = +U_Upper^+ +U_Lower^+ *)
```

$(* Y_{(\alpha,\alpha)}{}^{(a,b),(a,b)} Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a+1,b)} - Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a+1,b)} Y_{(\alpha+1/2,\alpha+1/2)}{}^{(a+1,b),(a+1,b)} + Y_{(\alpha,\alpha)}{}^{(a,b),(a,b)} Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a,b-1)} - Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a,b-1)} Y_{(\alpha+1/2,\alpha+1/2)}{}^{(a,b-1),(a,b-1)} *)$

$(*= +Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a+1,b)} + Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a,b-1)} *)$

In[181]:=

```
lhs =
  Simplify[comp[2, 0, qq, pp, a, b, a, b, α, α] (comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2]) -
    (comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2])
     comp[2, 0, qq, pp, a + 1, b, a + 1, b, α + 1 / 2, α + 1 / 2] +
    comp[2, 0, qq, pp, a, b, a, b, α, α] (comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2]) -
    (comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2])
     comp[2, 0, qq, pp, a, b - 1, a, b - 1, α + 1 / 2, α + 1 / 2]];
rhs = (comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2]) +
   (comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2]);
Simplify[lhs - rhs, {a ≥ 0, b ≥ 1}];
Print["The formulas satisfy [Y,U^+] = +U^+: ", 0 == Simplify[lhs - rhs, {a ≥ 0, b ≥ 1}]]
```

The formulas satisfy $[Y, U^+] = +U^+$: True

In[185]:=

```
(* [Y,U⁻] = -U⁻*)
(* [Y,U_Upper⁻]+[Y,U_Lower⁻] = -U_Upper⁻-U_Lower⁻ *)

(*Y_(α,α)^(a,b),(a,b) Um_(α,α-1/2)^(a,b),(a,b+1) - Um_(α,α-1/2)^(a,b),(a,b+1) Y_(α-1/2,α-1/2)^(a,b+1),(a,b+1) +
 Y_(α,α)^(a,b),(a,b) Um_(α,α-1/2)^(a,b),(a-1,b) - Um_(α,α-1/2)^(a,b),(a-1,b) Y_(α-1/2,α-1/2)^(a-1,b),(a-1,b) *)
(*= +Um_(α,α-1/2)^(a,b),(a,b+1) +Um_(α,α-1/2)^(a,b),(a-1,b) *)
lhs =
  Simplify[comp[2, 0, qq, pp, a, b, a, b, α, α] (comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2]) -
    (comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2])
     comp[2, 0, qq, pp, a, b + 1, a, b + 1, α - 1 / 2, α - 1 / 2] +
    comp[2, 0, qq, pp, a, b, a, b, α, α] (comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2]) -
    (comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2])
     comp[2, 0, qq, pp, a - 1, b, a - 1, b, α - 1 / 2, α - 1 / 2]];
rhs = - (comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2]) -
   (comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2]);
Simplify[lhs - rhs, {a ≥ 1, b ≥ 0}];
Print["The formulas satisfy [Y,U⁻] = -U⁻: ", 0 == Simplify[lhs - rhs, {a ≥ 0, b ≥ 1}]]
```

```
The formulas satisfy [Y,U⁻] = -U⁻: True
```

In[189]:=

```
(* [Y,V⁺] = +V⁺*)
(* [Y,V_Upper⁺]+[Y,V_Lower⁺] = +V_Upper⁺+V_Lower⁺ *)

(*Y_(α,α)^(a,b),(a,b) Vp_(α,α-1/2)^(a,b),(a+1,b) - Vp_(α,α-1/2)^(a,b),(a+1,b) Y_(α-1/2,α-1/2)^(a+1,b),(a+1,b) +
 Y_(α,α)^(a,b),(a,b) Vp_(α,α-1/2)^(a,b),(a,b-1) - Vp_(α,α-1/2)^(a,b),(a,b-1) Y_(α-1/2,α-1/2)^(a,b-1),(a,b-1) *)
(*= +Vp_(α,α-1/2)^(a,b),(a+1,b) +Vp_(α,α-1/2)^(a,b),(a,b-1) *)
```

In[190]:=

```
lhs =
  Simplify[comp[2, 0, qq, pp, a, b, a, b, α, α] (comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2]) -
    (comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2])
     comp[2, 0, qq, pp, a + 1, b, a + 1, b, α - 1 / 2, α - 1 / 2] +
    comp[2, 0, qq, pp, a, b, a, b, α, α] (comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2]) -
    (comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2])
     comp[2, 0, qq, pp, a, b - 1, a, b - 1, α - 1 / 2, α - 1 / 2]];
rhs = + (comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2]) +
   (comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2]);
Simplify[lhs - rhs, {a ≥ 0, b ≥ 1}];
Print["The formulas satisfy [Y,V⁺] = +V⁺: ", 0 == Simplify[lhs - rhs, {a ≥ 0, b ≥ 1}]]
```

```
The formulas satisfy [Y,V⁺] = +V⁺: True
```

In[194]:=

```
(* [Y,V⁻] = -V⁻*)
(* [Y,V_Upper⁻]+[Y,V_Lower⁻] = -V_Upper⁻-V_Lower⁻ *)

(*Y_(α,α)^(a,b),(a,b) Vm_(α,α+1/2)^(a,b),(a,b+1) - Vm_(α,α+1/2)^(a,b),(a,b+1) Y_(α+1/2,α+1/2)^(a,b+1),(a,b+1) +
 Y_(α,α)^(a,b),(a,b) Vm_(α,α+1/2)^(a,b),(a-1,b) - Vm_(α,α+1/2)^(a,b),(a-1,b) Y_(α+1/2,α+1/2)^(a-1,b),(a-1,b) *)
(*= -Vm_(α,α+1/2)^(a,b),(a,b+1) -Vm_(α,α+1/2)^(a,b),(a-1,b) *)
lhs =
  Simplify[comp[2, 0, qq, pp, a, b, a, b, α, α] (comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2]) -
    (comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2])
     comp[2, 0, qq, pp, a, b + 1, a, b + 1, α + 1 / 2, α + 1 / 2] +
    comp[2, 0, qq, pp, a, b, a, b, α, α] (comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2]) -
    (comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2])
     comp[2, 0, qq, pp, a - 1, b, a - 1, b, α + 1 / 2, α + 1 / 2]];
rhs = - (comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2]) -
   (comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2]);
Simplify[lhs - rhs, {a ≥ 0, b ≥ 1}];
Print["The formulas satisfy [Y,V⁻] = -V⁻: ", 0 == Simplify[lhs - rhs, {a ≥ 0, b ≥ 1}]]
```

The formulas satisfy [Y,V⁻] = -V⁻: True

In[198]:=

```
(* [T⁻,U⁺] = 0*)
(* [T⁻,U_Upper⁺]+[T⁻,U_Lower⁺] = 0 *)

(*Tm_(α,α+1)^(a,b),(a,b) Up_(α+1,α+3/2)^(a,b),(a+1,b) - Up_(α,α+1/2)^(a,b),(a+1,b) Tm_(α+1/2,α+3/2)^(a+1,b),(a+1,b) +
 Tm_(α,α+1)^(a,b),(a,b) Up_(α+1,α+3/2)^(a,b),(a,b-1) - Up_(α,α+1/2)^(a,b),(a,b-1) Tm_(α+1/2,α+3/2)^(a,b-1),(a,b-1) *)
(*= 0*)
termsNOT0 = FullSimplify[
   {comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[5, 1, qq, pp, a, b, a + 1, b, α + 1, α + 3 / 2],
    comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[5, 2, qq, pp, a, b, a, b - 1, α + 1, α + 3 / 2]},
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
lhs = comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[5, 1, qq, pp, a, b, a + 1, b, α + 1, α + 3 / 2] -
   comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] ×
    comp[4, 0, qq, pp, a + 1, b, a + 1, b, α + 1 / 2, α + 3 / 2] +
   comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[5, 2, qq, pp, a, b, a, b - 1, α + 1, α + 3 / 2] -
   comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] ×
    comp[4, 0, qq, pp, a, b - 1, a, b - 1, α + 1 / 2, α + 3 / 2];
rhs = 0;
out1 = FullSimplify[PowerExpand[lhs - rhs],
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁻,U⁺] = 0: ", 0 == FullSimplify[PowerExpand[lhs - rhs],
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy [T⁻,U⁺] = 0: True

In[203]:=

```
(* [T⁺,U⁻] = 0*)
(* [T⁺,U_Upper⁻]+[T⁺,U_Lower⁻] = 0 *)

(*Tp_(α,α-1)^(a,b),(a,b) Um_(α-1,α-3/2)^(a,b),(a,b+1) - Um_(α,α-1/2)^(a,b),(a,b+1) Tp_(α-1/2,α-3/2)^(a,b+1),(a,b+1) +
 Tp_(α,α-1)^(a,b),(a,b) Um_(α-1,α-3/2)^(a,b),(a-1,b) - Um_(α,α-1/2)^(a,b),(a-1,b) Tp_(α-1/2,α-3/2)^(a-1,b),(a-1,b) *)
(*= 0*)
```

In[204]:=

```
termsNOT0 = FullSimplify[
   {comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[6, 1, qq, pp, a, b, a, b + 1, α - 1, α - 3 / 2],
    comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[6, 2, qq, pp, a, b, a - 1, b, α - 1, α - 3 / 2]},
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
lhs = comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[6, 1, qq, pp, a, b, a, b + 1, α - 1, α - 3 / 2] -
   comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] ×
    comp[3, 0, qq, pp, a, b + 1, a, b + 1, α - 1 / 2, α - 3 / 2] +
   comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[6, 2, qq, pp, a, b, a - 1, b, α - 1, α - 3 / 2] -
   comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
    comp[3, 0, qq, pp, a - 1, b, a - 1, b, α - 1 / 2, α - 3 / 2];
rhs = 0;
out1 = FullSimplify[PowerExpand[lhs - rhs],
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁺,U⁻] = 0: ", 0 == FullSimplify[PowerExpand[lhs - rhs],
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[T^{+},U^{-}] = 0$: True

In[209]:=

```
(* [T⁺,V⁺] = 0*)
(* [T⁺,V_Upper⁺]+[T⁺,V_Lower⁺] = 0 *)

(*Tp_(α,α-1)^(a,b),(a,b) Vp_(α-1,α-3/2)^(a,b),(a+1,b) - Vp_(α,α-1/2)^(a,b),(a+1,b) Tp_(α-1/2,α-3/2)^(a+1,b),(a+1,b) +
 Tp_(α,α-1)^(a,b),(a,b) Vp_(α-1,α-3/2)^(a,b),(a,b-1) - Vp_(α,α-1/2)^(a,b),(a,b-1) Tp_(α-1/2,α-3/2)^(a,b-1),(a,b-1) *)
(*= 0*)
```

In[210]:=

```
termsNOT0 = FullSimplify[
   {comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[7, 1, qq, pp, a, b, a + 1, b, α - 1, α - 3 / 2],
    comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[7, 2, qq, pp, a, b, a, b - 1, α - 1, α - 3 / 2]},
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
lhs = comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[7, 1, qq, pp, a, b, a + 1, b, α - 1, α - 3 / 2] -
   comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2] ×
    comp[3, 0, qq, pp, a + 1, b, a + 1, b, α - 1 / 2, α - 3 / 2] +
   comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[7, 2, qq, pp, a, b, a, b - 1, α - 1, α - 3 / 2] -
   comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2] ×
    comp[3, 0, qq, pp, a, b - 1, a, b - 1, α - 1 / 2, α - 3 / 2];
rhs = 0;
out1 = FullSimplify[PowerExpand[lhs - rhs],
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁺,V⁺] = 0: ", 0 == FullSimplify[PowerExpand[lhs - rhs],
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[T^+,V^+] = 0$: True

In[215]:=

(* $[T^-,V^-] = 0$*)
(* $[T^-,V_{Upper}{}^-]+[T^-,V_{Lower}{}^-] = 0$ *)

(* $Tm_{(\alpha,\alpha+1)}{}^{(a,b),(a,b)} Vm_{(\alpha+1,\alpha+3/2)}{}^{(a,b),(a,b+1)} - Vm_{(\alpha,\alpha+1/2)}{}^{(a,b),(a,b+1)} Tm_{(\alpha+1/2,\alpha+3/2)}{}^{(a,b+1),(a,b+1)} + Tm_{(\alpha,\alpha+1)}{}^{(a,b),(a,b)} Vm_{(\alpha+1,\alpha+3/2)}{}^{(a,b),(a-1,b)} - Vm_{(\alpha,\alpha+1/2)}{}^{(a,b),(a-1,b)} Tm_{(\alpha+1/2,\alpha+3/2)}{}^{(a-1,b),(a-1,b)}$ *)
(*= 0*)

In[216]:=

```
termsNOT0 = FullSimplify[
   {comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[8, 1, qq, pp, a, b, a, b + 1, α + 1, α + 3 / 2],
    comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[8, 2, qq, pp, a, b, a - 1, b, α + 1, α + 3 / 2]},
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
lhs = comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[8, 1, qq, pp, a, b, a, b + 1, α + 1, α + 3 / 2] -
   comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2] ×
    comp[4, 0, qq, pp, a, b + 1, a, b + 1, α + 1 / 2, α + 3 / 2] +
   comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[8, 2, qq, pp, a, b, a - 1, b, α + 1, α + 3 / 2] -
   comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2] ×
    comp[4, 0, qq, pp, a - 1, b, a - 1, b, α + 1 / 2, α + 3 / 2];
rhs = 0;
out1 = FullSimplify[PowerExpand[lhs - rhs],
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁻,V⁻] = 0: ", 0 == FullSimplify[PowerExpand[lhs - rhs],
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[T^-,V^-] = 0$: True

In[221]:=

```
(* [T⁺,U⁺] = +V⁺*)
(* [T⁺,U_Upper⁺]+[T⁺,U_Lower⁺] = +V_Upper⁺+V_Lower⁺ *)
```

$$(*Tp_{(\alpha,\alpha-1)}{}^{(a,b),(a,b)}Up_{(\alpha-1,\alpha-1/2)}{}^{(a,b),(a+1,b)} - Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a+1,b)}Tp_{(\alpha+1/2,\alpha-1/2)}{}^{(a+1,b),(a+1,b)} + Tp_{(\alpha,\alpha-1)}{}^{(a,b),(a,b)}Up_{(\alpha-1,\alpha-1/2)}{}^{(a,b),(a,b-1)} - Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a,b-1)}Tp_{(\alpha+1/2,\alpha-1/2)}{}^{(a,b-1),(a,b-1)} *)$$

$$(*= +Vp_{(\alpha,\alpha-1/2)}{}^{(a,b),(a+1,b)}+Vp_{(\alpha,\alpha-1/2)}{}^{(a,b),(a,b-1)} *)$$

In[222]:=

```
lhs = comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[5, 1, qq, pp, a, b, a + 1, b, α - 1, α - 1 / 2] -
   comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] ×
    comp[3, 0, qq, pp, a + 1, b, a + 1, b, α + 1 / 2, α - 1 / 2] +
   comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[5, 2, qq, pp, a, b, a, b - 1, α - 1, α - 1 / 2] -
   comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] ×
    comp[3, 0, qq, pp, a, b - 1, a, b - 1, α + 1 / 2, α - 1 / 2];
rhs =
  +comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2] + comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2];
FullSimplify[lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁺,U⁺] = +V⁺: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]

The formulas satisfy [T⁺,U⁺] = +V⁺: True
```

In[226]:=

```
(* [T⁻,U⁻] = -V⁻*)
(* [T⁻,U_Upper⁻]+[T⁻,U_Lower⁻] = -V_Upper⁻-V_Lower⁻ *)
```

$$(*Tm_{(\alpha,\alpha+1)}{}^{(a,b),(a,b)}Um_{(\alpha+1,\alpha+1/2)}{}^{(a,b),(a,b+1)} - Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a,b+1)}Tm_{(\alpha-1/2,\alpha+1/2)}{}^{(a,b+1),(a,b+1)} + Tm_{(\alpha,\alpha+1)}{}^{(a,b),(a,b)}Um_{(\alpha+1,\alpha+1/2)}{}^{(a,b),(a-1,b)} - Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a-1,b)}Tm_{(\alpha-1/2,\alpha+1/2)}{}^{(a-1,b),(a-1,b)} *)$$

$$(*= -Vm_{(\alpha,\alpha+1/2)}{}^{(a,b),(a,b+1)}-Vm_{(\alpha,\alpha+1/2)}{}^{(a,b),(a-1,b)} *)$$

In[227]:=

```
lhs = comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[6, 1, qq, pp, a, b, a, b + 1, α + 1, α + 1 / 2] -
   comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] ×
    comp[4, 0, qq, pp, a, b + 1, a, b + 1, α - 1 / 2, α + 1 / 2] +
   comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[6, 2, qq, pp, a, b, a - 1, b, α + 1, α + 1 / 2] -
   comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
    comp[4, 0, qq, pp, a - 1, b, a - 1, b, α - 1 / 2, α + 1 / 2];
rhs =
  -comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2] - comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2];
FullSimplify[lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁻,U⁻] = -V⁻: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]

The formulas satisfy [T⁻,U⁻] = -V⁻: True
```

In[231]:=

```
(* [T⁻,V⁺] = +U⁺*)
(* [T⁻,V_Upper⁺]+[T⁻,V_Lower⁺] = +U_Upper⁺+U_Lower⁺ *)

(*Tm_(α,α+1)^(a,b),(a,b) Vp_(α+1,α+1/2)^(a,b),(a+1,b) - Vp_(α,α-1/2)^(a,b),(a+1,b) Tm_(α-1/2,α+1/2)^(a+1,b),(a+1,b) +
 Tm_(α,α+1)^(a,b),(a,b) Vp_(α+1,α+1/2)^(a,b),(a,b-1) - Vp_(α,α-1/2)^(a,b),(a,b-1) Tm_(α-1/2,α+1/2)^(a,b-1),(a,b-1) *)
(*= +Up_(α,α+1/2)^(a,b),(a+1,b) +Up_(α,α+1/2)^(a,b),(a,b-1) *)
```

In[232]:=

```
lhs = comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[7, 1, qq, pp, a, b, a + 1, b, α + 1, α + 1 / 2] -
   comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2] ×
    comp[4, 0, qq, pp, a + 1, b, a + 1, b, α - 1 / 2, α + 1 / 2] +
   comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[7, 2, qq, pp, a, b, a, b - 1, α + 1, α + 1 / 2] -
   comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2] ×
    comp[4, 0, qq, pp, a, b - 1, a, b - 1, α - 1 / 2, α + 1 / 2];
rhs =
  +comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] + comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2];
FullSimplify[lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁻,V⁺] = +U⁺: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy [T⁻,V⁺] = +U⁺: True

In[236]:=

```
(* [T⁺,V⁻] = -U⁻*)
(* [T⁺,V_Upper⁻]+[T⁺,V_Lower⁻] = -U_Upper⁻-U_Lower⁻ *)

(*Tp_(α,α-1)^(a,b),(a,b) Vm_(α-1,α-1/2)^(a,b),(a,b+1) - Vm_(α,α+1/2)^(a,b),(a,b+1) Tp_(α-1/2,α-1/2)^(a,b+1),(a,b+1) +
 Tp_(α,α-1)^(a,b),(a,b) Vm_(α-1,α-1/2)^(a,b),(a-1,b) - Vm_(α,α+1/2)^(a,b),(a-1,b) Tp_(α+1/2,α+1/2)^(a-1,b),(a-1,b) *)
(*= -Um_(α,α-1/2)^(a,b),(a,b+1) -Um_(α,α-1/2)^(a,b),(a-1,b) *)
lhs = comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[8, 1, qq, pp, a, b, a, b + 1, α - 1, α - 1 / 2] -
   comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2] ×
    comp[3, 0, qq, pp, a, b + 1, a, b + 1, α + 1 / 2, α - 1 / 2] +
   comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[8, 2, qq, pp, a, b, a - 1, b, α - 1, α - 1 / 2] -
   comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2] ×
    comp[3, 0, qq, pp, a - 1, b, a - 1, b, α + 1 / 2, α - 1 / 2];
rhs =
  -comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] - comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2];
FullSimplify[lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [T⁺,V⁻] = -U⁻: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy [T⁺,V⁻] = -U⁻: True

In[240]:=

```
(* CRs that are quadratic in U,V matrices*)
```

In[241]:=

```
(* [U⁺,U⁻] = +3Y/2-T^3 *)
(*1. [U_Upper⁺,U_Lower⁻]+[U_Lower⁺,U_Upper⁻] = 3Y/2-T^3 *)

(*Up_(α,α+1/2)^(a,b),(a+1,b) Um_(α+1/2,α)^(a+1,b),(a,b) - Um_(α,α-1/2)^(a,b),(a-1,b) Up_(α-1/2,α)^(a-1,b),(a,b) +
 Up_(α,α+1/2)^(a,b),(a,b-1) Um_(α+1/2,α)^(a,b-1),(a,b) - Um_(α,α-1/2)^(a,b),(a,b+1) Up_(α-1/2,α)^(a,b+1),(a,b) *)
lhs = Simplify[
   comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] × comp[6, 2, qq, pp, a + 1, b, a, b, α + 1 / 2, α] -
    comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] × comp[5, 1, qq, pp, a - 1, b, a, b, α - 1 / 2, α] +
    comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] × comp[6, 1, qq, pp, a, b - 1, a, b, α + 1 / 2, α] -
    comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] ×
     comp[5, 2, qq, pp, a, b + 1, a, b, α - 1 / 2, α]];
rhs = Expand[3 comp[2, 0, qq, pp, a, b, a, b, α, α] / 2 - comp[1, 0, qq, pp, a, b, a, b, α, α]];
FullSimplify[lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 0, -(a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U_Upper⁺,U_Lower⁻]+[U_Lower⁺,U_Upper⁻] = 3Y/2-T^3: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 0, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Upper}{}^{+},U_{Lower}{}^{-}]+[U_{Lower}{}^{+},U_{Upper}{}^{-}] = 3Y/2-T^3$: True

In[245]:=

```
(*2. [U_Upper⁺,U_Upper⁻] = 0 *)
(*Up_(α,α+1/2)^(a,b),(a+1,b) Um_(α+1/2,α)^(a+1,b),(a+1,b+1) -
 Um_(α,α-1/2)^(a,b),(a,b+1) Up_(α-1/2,α)^(a,b+1),(a+1,b+1) *)
termNOT0 = Simplify[comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] ×
    comp[6, 1, qq, pp, a + 1, b, a + 1, b + 1, α + 1 / 2, α], {a ≥ 0, b ≥ 0}];
lhs = Simplify[comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] × comp[6, 1, qq, pp, a + 1,
      b, a + 1, b + 1, α + 1 / 2, α] - comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] ×
     comp[5, 1, qq, pp, a, b + 1, a + 1, b + 1, α - 1 / 2, α], {a ≥ 0, b ≥ 0}];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, -(a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U_Upper⁺,U_Upper⁻] = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Upper}{}^{+},U_{Upper}{}^{-}] = 0$: True

In[250]:=

```
(*3. [U_Lower⁺,U_Lower⁻] = 0 *)
(*Up_(α,α+1/2)^(a,b),(a,b-1) Um_(α+1/2,α)^(a,b-1),(a-1,b-1) -
 Um_(α,α-1/2)^(a,b),(a-1,b) Up_(α-1/2,α)^(a-1,b),(a-1,b-1) *)
```

In[251]:=

```
termNOT0 = Simplify[comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] ×
    comp[6, 2, qq, pp, a, b - 1, a - 1, b - 1, α + 1 / 2, α], {a ≥ 1, b ≥ 1}];
lhs = Simplify[comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] × comp[6, 2, qq, pp, a,
      b - 1, a - 1, b - 1, α + 1 / 2, α] - comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
     comp[5, 2, qq, pp, a - 1, b, a - 1, b - 1, α - 1 / 2, α], {a ≥ 1, b ≥ 1}];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U_Lower^+,U_Lower^-] = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Lower}{}^{+},U_{Lower}{}^{-}]$ = 0: True

In[256]:=

```
(* [V^+,V^-] = +3Y/2+T^3 *)
(*1. [V_Upper^+,V_Lower^-]+[V_Lower^+,V_Upper^-] = 3Y/2+T^3 *)

(*Vp_(α,α-1/2)^(a,b),(a+1,b) Vm_(α-1/2,α)^(a+1,b),(a,b) - Vm_(α,α+1/2)^(a,b),(a-1,b) Vp_(α+1/2,α)^(a-1,b),(a,b) +
 Vp_(α,α-1/2)^(a,b),(a,b-1) Vm_(α-1/2,α)^(a,b-1),(a,b) - Vm_(α,α+1/2)^(a,b),(a,b+1) Vp_(α+1/2,α)^(a,b+1),(a,b)    *)
lhs = Simplify[
   comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2] × comp[8, 2, qq, pp, a + 1, b, a, b, α - 1 / 2, α] -
    comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2] × comp[7, 1, qq, pp, a - 1, b, a, b, α + 1 / 2, α] +
    comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2] × comp[8, 1, qq, pp, a, b - 1, a, b, α - 1 / 2, α] -
    comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2] ×
     comp[7, 2, qq, pp, a, b + 1, a, b, α + 1 / 2, α]];
rhs = Expand[3 comp[2, 0, qq, pp, a, b, a, b, α, α] / 2 + comp[1, 0, qq, pp, a, b, a, b, α, α]];
FullSimplify[lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [V_Upper^+,V_Lower^-]+[V_Lower^+,V_Upper^-] = 3Y/2+T^3: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[V_{Upper}{}^{+},V_{Lower}{}^{-}]+[V_{Lower}{}^{+},V_{Upper}{}^{-}] = 3Y/2+T^3$: True

In[260]:=

```
(*2. [V_Upper^+,V_Upper^-] = 0 *)
(*Vp_(α,α-1/2)^(a,b),(a+1,b) Vm_(α-1/2,α)^(a+1,b),(a+1,b+1) -
 Vm_(α,α+1/2)^(a,b),(a,b+1) Vp_(α+1/2,α)^(a,b+1),(a+1,b+1)    *)
```

In[261]:=

```
termNOT0 = Simplify[comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2] ×
   comp[8, 1, qq, pp, a + 1, b, a + 1, b + 1, α - 1 / 2, α], {a ≥ 0, b ≥ 0}];
lhs = Simplify[comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2] × comp[8, 1, qq, pp, a + 1,
     b, a + 1, b + 1, α - 1 / 2, α] - comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2] ×
    comp[7, 1, qq, pp, a, b + 1, a + 1, b + 1, α + 1 / 2, α], {a ≥ 0, b ≥ 0}];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [V_Upper^+,V_Upper^-] = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

```
The formulas satisfy [V_Upper^+,V_Upper^-] = 0: True
```

In[266]:=

```
(*3. [V_Lower^+,V_Lower^-] = 0 *)

(*Vp_(α,α-1/2)^(a,b),(a,b-1) Vm_(α-1/2,α)^(a,b-1),(a-1,b-1) -
 Vm_(α,α+1/2)^(a,b),(a-1,b) Vp_(α+1/2,α)^(a-1,b),(a-1,b-1)     *)
termNOT0 = Simplify[comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2] ×
   comp[8, 2, qq, pp, a, b - 1, a - 1, b - 1, α - 1 / 2, α], {a ≥ 1, b ≥ 1}];
lhs = Simplify[comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2] × comp[8, 2, qq, pp, a,
     b - 1, a - 1, b - 1, α - 1 / 2, α] - comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2] ×
    comp[7, 2, qq, pp, a - 1, b, a - 1, b - 1, α + 1 / 2, α], {a ≥ 1, b ≥ 1}];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [V_Lower^+,V_Lower^-] = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

```
The formulas satisfy [V_Lower^+,V_Lower^-] = 0: True
```

In[271]:=

```
(* [U^+,V^-] = T^- *)
(*1. [U_Upper^+,V_Lower^-] + [U_Lower^+,V_Upper^-] = T^- *)

(*Up_(α,α+1/2)^(a,b),(a+1,b) Vm_(α+1/2,α+1)^(a+1,b),(a,b) - Vm_(α,α+1/2)^(a,b),(a-1,b) Up_(α+1/2,α+1)^(a-1,b),(a,b) +
 Up_(α,α+1/2)^(a,b),(a,b-1) Vm_(α+1/2,α+1)^(a,b-1),(a,b) - Vm_(α,α+1/2)^(a,b),(a,b+1) Up_(α+1/2,α+1)^(a,b+1),(a,b)    *)
(*= Tm_(α,α+1)^(a,b),(a,b) *)
```

In[272]:=

```
lhs = Simplify[(comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2])
     (comp[8, 2, qq, pp, a + 1, b, a, b, α + 1 / 2, α + 1]) -
    (comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2])
     (comp[5, 1, qq, pp, a - 1, b, a, b, α + 1 / 2, α + 1]) +
    (comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2])
     (comp[8, 1, qq, pp, a, b - 1, a, b, α + 1 / 2, α + 1]) -
    (comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2])
     (comp[5, 2, qq, pp, a, b + 1, a, b, α + 1 / 2, α + 1])];
rhs = Expand[comp[4, 0, qq, pp, a, b, a, b, α, α + 1]];
Simplify[lhs - rhs, {a ≥ 0, b ≥ 0}];
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U⁺,V⁻] = T⁻: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy [U^+,V^-] = T^-: True

In[277]:=

```
(* [U⁺,V⁻] = T⁻ *)
(*2. [U_Upper⁺,V_Upper⁻] = 0 *)

(*Up_(α,α+1/2)^(a,b),(a+1,b) Vm_(α+1/2,α+1)^(a+1,b),(a+1,b+1) -
 Vm_(α,α+1/2)^(a,b),(a,b+1) Up_(α+1/2,α+1)^(a,b+1),(a+1,b+1)     *)
termNOT0 = Simplify[comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] ×
    comp[8, 1, qq, pp, a + 1, b, a + 1, b + 1, α + 1 / 2, α + 1], {a ≥ 0, b ≥ 0}];
lhs =
  Simplify[comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] × comp[8, 1, qq, pp, a + 1, b, a + 1, b + 1,
      α + 1 / 2, α + 1] - comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2] ×
     comp[5, 1, qq, pp, a, b + 1, a + 1, b + 1, α + 1 / 2, α + 1], {a ≥ 0, b ≥ 0}];
rhs = Expand[comp[4, 0, qq, pp, a, b, a + 1, b + 1, α, α + 1]];
Print["The formulas satisfy [U_Upper⁺,V_Upper⁻] = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy [U_{Upper}^+,V_{Upper}^-] = 0: True

In[281]:=

```
(* [U⁺,V⁻] = T⁻ *)
(*3. [U_Lower⁺,V_Lower⁻] = 0 *)

(*Up_(α,α+1/2)^(a,b),(a,b-1) Vm_(α+1/2,α+1)^(a,b-1),(a-1,b-1) -
 Vm_(α,α+1/2)^(a,b),(a-1,b) Up_(α+1/2,α+1)^(a-1,b),(a-1,b-1) *)
termNOT0 = Simplify[comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] ×
    comp[8, 2, qq, pp, a, b - 1, a - 1, b - 1, α + 1 / 2, α + 1], {a ≥ 1, b ≥ 1}];
lhs =
  Simplify[comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] × comp[8, 2, qq, pp, a, b - 1, a - 1, b - 1,
      α + 1 / 2, α + 1] - comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2] ×
     comp[5, 2, qq, pp, a - 1, b, a - 1, b - 1, α + 1 / 2, α + 1], {a ≥ 1, b ≥ 1}];
rhs = Expand[comp[4, 0, qq, pp, a, b, a - 1, b - 1, α, α + 1]];
Print["The formulas satisfy [U_Lower⁺,V_Lower⁻] = 0: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], qq ≥ a ≥ 1, pp ≥ b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Lower}{}^{+},V_{Lower}{}^{-}] = 0$: True

In[285]:=

```
(* [U⁻,V⁺] = -T⁺ *)
(*1. [U_Upper⁻,V_Lower⁺] + [U_Lower⁻,V_Upper⁺] = -T⁺ *)

(*Um_(α,α-1/2)^(a,b),(a,b+1) Vp_(α-1/2,α-1)^(a,b+1),(a,b) - Vp_(α,α-1/2)^(a,b),(a,b-1) Um_(α-1/2,α-1)^(a,b-1),(a,b) +
 Um_(α,α-1/2)^(a,b),(a-1,b) Vp_(α-1/2,α-1)^(a-1,b),(a,b) - Vp_(α,α-1/2)^(a,b),(a+1,b) Um_(α-1/2,α-1)^(a+1,b),(a,b) *)
(*= Tp_(α,α-1)^(a,b),(a,b) *)
lhs = Simplify[(comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2])
     (comp[7, 2, qq, pp, a, b + 1, a, b, α - 1 / 2, α - 1]) -
    (comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2])
     (comp[6, 1, qq, pp, a, b - 1, a, b, α - 1 / 2, α - 1]) +
    (comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2])
     (comp[7, 1, qq, pp, a - 1, b, a, b, α - 1 / 2, α - 1]) -
    (comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2])
     (comp[6, 2, qq, pp, a + 1, b, a, b, α - 1 / 2, α - 1])];
rhs = -Expand[comp[3, 0, qq, pp, a, b, a, b, α, α - 1]];
Simplify[lhs - rhs, {a ≥ 1, b ≥ 1}];
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U⁻,V⁺] = -T⁺: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U^{-},V^{+}] = -T^{+}$: True

In[290]:=

```
(* [U_Upper^-,V_Upper^+] = -T^+ *)
(*2. [U_Upper^-,V_Upper^+] = 0 *)

(*Um_(α,α-1/2)^(a,b),(a,b+1) Vp_(α-1/2,α-1)^(a,b+1),(a+1,b+1) -
 Vp_(α,α-1/2)^(a,b),(a+1,b) Um_(α-1/2,α-1)^(a+1,b),(a+1,b+1)    *)
termNOT0 = Simplify[comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] ×
    comp[7, 1, qq, pp, a, b + 1, a + 1, b + 1, α - 1 / 2, α - 1], {a ≥ 0, b ≥ 0}];
lhs =
  Simplify[comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] × comp[7, 1, qq, pp, a, b + 1, a + 1, b + 1,
      α - 1 / 2, α - 1] - comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2] ×
     comp[6, 1, qq, pp, a + 1, b, a + 1, b + 1, α - 1 / 2, α - 1], {a ≥ 0, b ≥ 0}];
rhs = -Expand[comp[3, 0, qq, pp, a, b, a + 1, b + 1, α, α - 1]];
Print["The formulas satisfy [U_Upper^-,V_Upper^+] = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Upper}{}^{-},V_{Upper}{}^{+}]$ = 0: True

In[294]:=

```
(* [U^-,V^+] = -T^+ *)
(*3. [U_Lower^-,V_Lower^+] = 0 *)

(*Um_(α,α-1/2)^(a,b),(a-1,b) Vp_(α-1/2,α-1)^(a-1,b),(a-1,b-1) -
 Vp_(α,α-1/2)^(a,b),(a,b-1) Um_(α-1/2,α-1)^(a,b-1),(a-1,b-1)    *)
termNOT0 = Simplify[comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
    comp[7, 2, qq, pp, a - 1, b, a - 1, b - 1, α - 1 / 2, α - 1], {a ≥ 1, b ≥ 1}];
lhs = Simplify[comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
     comp[7, 2, qq, pp, a - 1, b, a - 1, b - 1, α - 1 / 2, α - 1] -
    comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2] × comp[6, 2, qq, pp, a, b - 1,
      a - 1, b - 1, α - 1 / 2, α - 1], {pp ≥ 0, qq ≥ 0, qq ≥ a ≥ 1, pp ≥ b ≥ 1}];
rhs = -Expand[comp[3, 0, qq, pp, a, b, a - 1, b - 1, α, α - 1]];
Print["The formulas satisfy [U_Lower^-,V_Lower^+] = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Lower}{}^{-},V_{Lower}{}^{+}]$ = 0: True

In[298]:=

```
(* [U⁺,V⁺] = 0 *)
(*1. [U_Upper⁺,V_Lower⁺]+[U_Lower⁺,V_Upper⁺] = 0 *)

(*Up_(α,α+1/2)^(a,b),(a+1,b) Vp_(α+1/2,α)^(a+1,b),(a+1,b-1) -
 Vp_(α,α-1/2)^(a,b),(a,b-1) Up_(α-1/2,α)^(a,b-1),(a+1,b-1) + Up_(α,α+1/2)^(a,b),(a,b-1) Vp_(α+1/2,α)^(a,b-1),(a+1,b-1) -
 Vp_(α,α-1/2)^(a,b),(a+1,b) Up_(α-1/2,α)^(a+1,b),(a+1,b-1)     *)
termsNOT0 =
  Simplify[{comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] × comp[7, 2, qq, pp, a + 1, b, a + 1,
      b - 1, α + 1 / 2, α], +comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] ×
     comp[7, 1, qq, pp, a, b - 1, a + 1, b - 1, α + 1 / 2, α]}];
lhs = Simplify[comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] ×
     comp[7, 2, qq, pp, a + 1, b, a + 1, b - 1, α + 1 / 2, α] -
    comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2] × comp[5, 1, qq, pp, a, b - 1,
      a + 1, b - 1, α - 1 / 2, α] + comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] ×
     comp[7, 1, qq, pp, a, b - 1, a + 1, b - 1, α + 1 / 2, α] - comp[7, 1, qq, pp, a, b,
      a + 1, b, α, α - 1 / 2] × comp[5, 2, qq, pp, a + 1, b, a + 1, b - 1, α - 1 / 2, α]];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U_Upper⁺,V_Lower⁺]+[U_Lower⁺,V_Upper⁺] = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Upper}{}^{+},V_{Lower}{}^{+}]+[U_{Lower}{}^{+},V_{Upper}{}^{+}] = 0$: True

In[303]:=

```
(* [U⁺,V⁺] = 0 *)
(*2. [U_Upper⁺,V_Upper⁺] = 0 *)

(*Up_(α,α+1/2)^(a,b),(a+1,b) Vp_(α+1/2,α)^(a+1,b),(a+2,b) -
  Vp_(α,α-1/2)^(a,b),(a+1,b) Up_(α-1/2,α)^(a+1,b),(a+2,b) = 0 *)
termNOT0 = Simplify[(comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2])
    (comp[7, 1, qq, pp, a + 1, b, a + 2, b, α + 1 / 2, α])];
lhs = Simplify[(comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2])
     (comp[7, 1, qq, pp, a + 1, b, a + 2, b, α + 1 / 2, α]) -
    (comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2])
     (comp[5, 1, qq, pp, a + 1, b, a + 2, b, α - 1 / 2, α])];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, -(a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U_Upper⁺,V_Upper⁺] = 0 : ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Upper}{}^{+},V_{Upper}{}^{+}] = 0$: True

In[308]:=

```
(* [U⁺,V⁺] = 0 *)
(*3. [U_Lower⁺,V_Lower⁺] = 0 *)
(*Up_(α,α+1/2)^(a,b),(a,b-1) Vp_(α+1/2,α)^(a,b-1),(a,b-2) -
  Vp_(α,α-1/2)^(a,b),(a,b-1) Up_(α-1/2,α)^(a,b-1),(a,b-2) = 0    *)
Simplify[comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] ×
   comp[7, 2, qq, pp, a, b - 1, a, b - 2, α + 1 / 2, α], {a ≥ 1, b ≥ 1}];
lhs = Simplify[comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] ×
     comp[7, 2, qq, pp, a, b - 1, a, b - 2, α + 1 / 2, α] - comp[7, 2, qq, pp, a, b, a, b - 1,
      α, α - 1 / 2] × comp[5, 2, qq, pp, a, b - 1, a, b - 2, α - 1 / 2, α], {a ≥ 0, b ≥ 2}];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 2, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U_Lower⁺,V_Lower⁺] = 0 : ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 2, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Lower}{}^{+},V_{Lower}{}^{+}]$ = 0 : True

In[313]:=

```
(* [U⁻,V⁻] = 0 *)
(*1. [U_Upper⁻,V_Lower⁻] + [U_Lower⁻,V_Upper⁻] = 0 *)

(*Um_(α,α-1/2)^(a,b),(a,b+1) Vm_(α-1/2,α)^(a,b+1),(a-1,b+1) -
 Vm_(α,α+1/2)^(a,b),(a-1,b) Um_(α+1/2,α)^(a-1,b),(a-1,b+1) + Um_(α,α-1/2)^(a,b),(a-1,b) Vm_(α-1/2,α)^(a-1,b),(a-1,b+1) -
 Vm_(α,α+1/2)^(a,b),(a,b+1) Um_(α+1/2,α)^(a,b+1),(a-1,b+1)     *)
termsNOT0 =
  Simplify[{comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] × comp[8, 2, qq, pp, a, b + 1, a - 1,
      b + 1, α - 1 / 2, α], +comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
     comp[8, 1, qq, pp, a - 1, b, a - 1, b + 1, α - 1 / 2, α]}];
lhs = Simplify[comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] ×
     comp[8, 2, qq, pp, a, b + 1, a - 1, b + 1, α - 1 / 2, α] -
    comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2] × comp[6, 1, qq, pp, a - 1, b,
      a - 1, b + 1, α + 1 / 2, α] + comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
     comp[8, 1, qq, pp, a - 1, b, a - 1, b + 1, α - 1 / 2, α] - comp[8, 1, qq, pp, a, b,
      a, b + 1, α, α + 1 / 2] × comp[6, 2, qq, pp, a, b + 1, a - 1, b + 1, α + 1 / 2, α]];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U_Upper⁻,V_Lower⁻] + [U_Lower⁻,V_Upper⁻] = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Upper}{}^{-},V_{Lower}{}^{-}]+[U_{Lower}{}^{-},V_{Upper}{}^{-}]$ = 0: True

In[318]:=

```
(* [U⁻,V⁻] = 0 *)
(*2. [U_Upper⁻,V_Upper⁻] = 0 *)

(*Um_(α,α-1/2)^(a,b),(a,b+1) Vm_(α-1/2,α)^(a,b+1),(a,b+2) -
  Vm_(α,α+1/2)^(a,b),(a,b+1) Um_(α+1/2,α)^(a,b+1),(a,b+2) = 0 *)
termNOT0 = Simplify[(comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2])
    (comp[8, 1, qq, pp, a, b + 1, a, b + 2, α - 1 / 2, α])];
lhs = Simplify[(comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2])
     (comp[8, 1, qq, pp, a, b + 1, a, b + 2, α - 1 / 2, α]) -
    (comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2])
     (comp[6, 1, qq, pp, a, b + 1, a, b + 2, α + 1 / 2, α])];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U_Upper⁻,V_Upper⁻] = 0 : ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 0, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Upper}^{-},V_{Upper}^{-}]$ = 0 : True

In[323]:=

```
(* [U⁻,V⁻] = 0 *)
(*3. [U_Lower⁻,V_Lower⁻] = 0 *)
(*Um_(α,α-1/2)^(a,b),(a-1,b) Vm_(α-1/2,α)^(a-1,b),(a-2,b) -
  Vm_(α,α+1/2)^(a,b),(a-1,b) Um_(α+1/2,α)^(a-1,b),(a-2,b) = 0     *)
Simplify[comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
   comp[8, 2, qq, pp, a - 1, b, a - 2, b, α - 1 / 2, α], {a ≥ 2, b ≥ 0}];
lhs = Simplify[comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
     comp[8, 2, qq, pp, a - 1, b, a - 2, b, α - 1 / 2, α] - comp[8, 2, qq, pp, a, b, a - 1, b,
      α, α + 1 / 2] × comp[6, 2, qq, pp, a - 1, b, a - 2, b, α + 1 / 2, α], {a ≥ 0, b ≥ 2}];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 2, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy [U_Lower⁻,V_Lower⁻] = 0 : ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 2, b ≥ 0, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $[U_{Lower}^{-},V_{Lower}^{-}]$ = 0 : True

In[328]:=

```
(* Casimir *)
(*1. Casimir,
```

$\frac{1}{2}(\{T^+,T^-\}+\{U_{Upper}{}^+,U_{Lower}{}^-\}+\{U_{Lower}{}^+,U_{Upper}{}^-\}+\{V_{Upper}{}^+,V_{Lower}{}^-\}+\{V_{Lower}{}^+,V_{Upper}{}^-\})+\left(T^3\right)^2+\frac{3}{4}Y^2 = \frac{1}{3}\left(p^2+pq+q^2+3p+3q\right)$ IdentityMatrix*)

(* $\frac{1}{2}\left(Tp_{(\alpha,\alpha-1)}{}^{(a,b),(a,b)}Tm_{(\alpha-1,\alpha)}{}^{(a,b),(a,b)}+Tm_{(\alpha,\alpha+1)}{}^{(a,b),(a,b)}Tp_{(\alpha+1,\alpha)}{}^{(a,b),(a,b)}+\right.$
$Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a+1,b)}Um_{(\alpha+1/2,\alpha)}{}^{(a+1,b),(a,b)}+Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a-1,b)}Up_{(\alpha-1/2,\alpha)}{}^{(a-1,b),(a,b)}+$
$Up_{(\alpha,\alpha+1/2)}{}^{(a,b),(a,b-1)}Um_{(\alpha+1/2,\alpha)}{}^{(a,b-1),(a,b)}+Um_{(\alpha,\alpha-1/2)}{}^{(a,b),(a,b+1)}Up_{(\alpha-1/2,\alpha)}{}^{(a,b+1),(a,b)}+$
$Vp_{(\alpha,\alpha-1/2)}{}^{(a,b),(a+1,b)}Vm_{(\alpha-1/2,\alpha)}{}^{(a+1,b),(a,b)}-Vm_{(\alpha,\alpha+1/2)}{}^{(a,b),(a-1,b)}Vp_{(\alpha+1/2,\alpha)}{}^{(a-1,b),(a,b)}+$
$\left.Vp_{(\alpha,\alpha-1/2)}{}^{(a,b),(a,b-1)}Vm_{(\alpha-1/2,\alpha)}{}^{(a,b-1),(a,b)}-Vm_{(\alpha,\alpha+1/2)}{}^{(a,b),(a,b+1)}Vp_{(\alpha+1/2,\alpha)}{}^{(a,b+1),(a,b)}\right)+$
$T3_{(\alpha,\alpha)}{}^{(a,b),(a,b)}T3_{(\alpha,\alpha)}{}^{(a,b),(a,b)}+\frac{3}{4}Y_{(\alpha,\alpha)}{}^{(a,b),(a,b)}Y_{(\alpha,\alpha)}{}^{(a,b),(a,b)} = \frac{1}{3}$
$\left(pp^2+pp\ qq+qq^2+3pp+3qq\right)$
*)

In[329]:=

```
lhs = Expand[
  Simplify[1/2 (comp[3, 0, qq, pp, a, b, a, b, α, α - 1] × comp[4, 0, qq, pp, a, b, a, b, α - 1, α] +
       comp[4, 0, qq, pp, a, b, a, b, α, α + 1] × comp[3, 0, qq, pp, a, b, a, b, α + 1, α] +
       comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] × comp[6, 2, qq, pp, a + 1,
         b, a, b, α + 1 / 2, α] + comp[6, 2, qq, pp, a, b, a - 1, b, α, α - 1 / 2] ×
        comp[5, 1, qq, pp, a - 1, b, a, b, α - 1 / 2, α] + comp[5, 2, qq, pp, a, b,
         a, b - 1, α, α + 1 / 2] × comp[6, 1, qq, pp, a, b - 1, a, b, α + 1 / 2, α] +
       comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] × comp[5, 2, qq, pp, a, b + 1,
         a, b, α - 1 / 2, α] + comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2] ×
        comp[8, 2, qq, pp, a + 1, b, a, b, α - 1 / 2, α] + comp[8, 2, qq, pp, a, b,
         a - 1, b, α, α + 1 / 2] × comp[7, 1, qq, pp, a - 1, b, a, b, α + 1 / 2, α] +
       comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2] × comp[8, 1, qq, pp, a, b - 1,
         a, b, α - 1 / 2, α] + comp[8, 1, qq, pp, a, b, a, b + 1, α, α + 1 / 2] ×
        comp[7, 2, qq, pp, a, b + 1, a, b, α + 1 / 2, α]) +
    comp[1, 0, qq, pp, a, b, a, b, α, α]^2 + 3/4 comp[2, 0, qq, pp, a, b, a, b, α, α]^2
   ]];
```

In[330]:=

```
rhs = Simplify[(1/3 (pp^2 + pp qq + qq^2 + 3 pp + 3 qq))];
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy
   1/2 ({T^+,T^-}+{U_Upper^+,U_Lower^-}+{U_Lower^+,U_Upper^-}+{V_Upper^+,V_Lower^-}+{V_Lower^+,V_Upper^-})+(T^3)^2+3/4 Y^2
   = 1/3 (p^2+pq+q^2+3p+3q) IdentityMatrix: ", 0 == FullSimplify[lhs - rhs,
   {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy

$$\frac{1}{2}(\{T^+,T^-\}+\{U_{Upper}{}^+,U_{Lower}{}^-\}+\{U_{Lower}{}^+,U_{Upper}{}^-\}+\{V_{Upper}{}^+,V_{Lower}{}^-\}+\{V_{Lower}{}^+,V_{Upper}{}^-\})+(T^3)^2+\frac{3}{4}Y^2$$

$$=\frac{1}{3}(p^2+pq+q^2+3p+3q)\text{ IdentityMatrix: True}$$

In[333]:=

```
(* Casimir *)
(*2. Casimir, {U_Upper^+,U_Upper^-}+{V_Upper^+,V_Upper^-} = 0*)
(*Up_(α,α+1/2)^(a,b),(a+1,b) Um_(α+1/2,α)^(a+1,b),(a+1,b+1) +
  Um_(α,α-1/2)^(a,b),(a,b+1) Up_(α-1/2,α)^(a,b+1),(a+1,b+1) +Vp_(α,α-1/2)^(a,b),(a+1,b) Vm_(α-1/2,α)^(a+1,b),(a+1,b+1) +
  Vm_(α,α+1/2)^(a,b),(a,b+1) Vp_(α+1/2,α)^(a,b+1),(a+1,b+1) = 0
*)
lhs = Simplify[comp[5, 1, qq, pp, a, b, a + 1, b, α, α + 1 / 2] ×
     comp[6, 1, qq, pp, a + 1, b, a + 1, b + 1, α + 1 / 2, α] +
    comp[6, 1, qq, pp, a, b, a, b + 1, α, α - 1 / 2] × comp[5, 1, qq, pp, a, b + 1,
      a + 1, b + 1, α - 1 / 2, α] + comp[7, 1, qq, pp, a, b, a + 1, b, α, α - 1 / 2] ×
     comp[8, 1, qq, pp, a + 1, b, a + 1, b + 1, α - 1 / 2, α] + comp[8, 1, qq, pp, a, b,
      a, b + 1, α, α + 1 / 2] × comp[7, 1, qq, pp, a, b + 1, a + 1, b + 1, α + 1 / 2, α]
  ];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy {U_Upper^+,U_Upper^-}+{V_Upper^+,V_Upper^-} = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, -(a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $\{U_{Upper}{}^+,U_{Upper}{}^-\}+\{V_{Upper}{}^+,V_{Upper}{}^-\} = 0$: True

In[337]:=

```
(* Casimir *)
(*3. Casimir, {ULower+,ULower-}+{VLower+,VLower-} = 0*)
(*Up(α,α+1/2) (a,b),(a,b-1) Um(α+1/2,α) (a,b-1),(a-1,b-1) +
  Um(α,α-1/2) (a,b),(a-1,b) Up(α-1/2,α) (a-1,b),(a-1,b-1) +Vp(α,α-1/2) (a,b),(a,b-1) Vm(α-1/2,α) (a,b-1),(a-1,b-1) +
  Vm(α,α+1/2) (a,b),(a-1,b) Vp(α+1/2,α) (a-1,b),(a-1,b-1) = 0
*)
termNOT0 =
  Simplify[{comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] × comp[6, 2, qq, pp, a, b - 1, a - 1,
      b - 1, α + 1 / 2, α], comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2] ×
     comp[7, 2, qq, pp, a - 1, b, a - 1, b - 1, α + 1 / 2, α]}, {a ≥ 1, b ≥ 1}];
lhs = Simplify[comp[5, 2, qq, pp, a, b, a, b - 1, α, α + 1 / 2] ×
     comp[6, 2, qq, pp, a, b - 1, a - 1, b - 1, α + 1 / 2, α] + comp[6, 2, qq, pp, a, b,
      a - 1, b, α, α - 1 / 2] × comp[5, 2, qq, pp, a - 1, b, a - 1, b - 1, α - 1 / 2, α] +
    comp[7, 2, qq, pp, a, b, a, b - 1, α, α - 1 / 2] × comp[8, 2, qq, pp, a, b - 1,
      a - 1, b - 1, α - 1 / 2, α] + comp[8, 2, qq, pp, a, b, a - 1, b, α, α + 1 / 2] ×
     comp[7, 2, qq, pp, a - 1, b, a - 1, b - 1, α + 1 / 2, α], {a ≥ 1, b ≥ 1}];
rhs = 0;
FullSimplify[lhs - rhs,
  {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}];
Print["The formulas satisfy {ULower+,ULower-}+{VLower+,VLower-} = 0: ", 0 == FullSimplify[
   lhs - rhs, {Element[{a, b, α}, Reals], a ≥ 1, b ≥ 1, - (a + b) / 2 ≤ α ≤ (a + b) / 2}]]
```

The formulas satisfy $\{U_{Lower}{}^{+},U_{Lower}{}^{-}\}+\{V_{Lower}{}^{+},V_{Lower}{}^{-}\}$ = 0: True

In[342]:=

5. User Selected (p,q): TYUV basis calculated, checked, and saved

In[343]:=

```
(* User:  Please set the values of p and q here. *)
p = 2; q = 1;
```

In[344]:=

```
Print["This is the (p,q) = (", p, ",", q, ")-irrep."];
```

This is the (p,q) = (2,1)-irrep.

In[345]:=

In[346]:=

```
(*
          Section 4 Glossary

 t:spin of an su(2)-irrep. Background: The diagonal block matrices for the
    T-matrices T3,Tp,Tm are su(2)-irreps, each with a spin t. su(2) is a subalgebra of
      u(2) which is a subalgebra of sl(3,C)



  numTspins: the number of su(2) irreps = the number of spins t.
   numTspins: the number of rows and columns in the square array of blocks in a matrix.
              Each matrix is a square array of blocks,
with each block itself a matrix.
              See ii. Samples of Matrices at the start of the notebook.

  dimREP:    dimension of the irrep. At the component level, the matrices are square and
              each row and column has dimREP components.

    matrixX[x]: The matrix for the generator x.
    matrixX1 is a dummy quantity, temporary and reassigned values as needed.

    x:   ID for the generators. Key: x=1,2,3,4,5,6,7,8 for T3,Y,Tp,Tm,Up,Um,Vp,Vm, resp.

   n:  (i,j)-block type. n = 0,1,2 for diagonal, upper, lower, respectively.
   An n=0 diagonal block has i=j,
an n=1 upper block has i<j, and an n=2 lower block has i>j.

    (t2i,t2j): (2*ti,2*tj), double the spin t parameters, t2 = 2*t.

   checkCRs: Logic variable. 'True' means that the generators satisfy the 2 CRs of the
     su(3) Lie algebra and the quadratic Casimir equation.

      checkTRACE: Logic variable. 'True' means that the trace of the generators vanishes.
*)
```

In[347]:=

```
(*Calculate metrics of the (p,q) irrep.*)
Tspins = Flatten[Table[(a + b) / 2, {b, 0, p}, {a, 0, q}]];
numTspins = (p + 1) (q + 1);
dimREP = (1 / 2) (p + 1) (q + 1) (p + q + 2);
```

In[350]:=

```
(*Calculate T3,Y,Tp,Tm, the block diagonal generators of u(2) subalgebra*)
(*n=0 indicates diagonal blocks *)
n = 0;
For[x = 1, x ≤ 4, x++,
  matrixX1 = Table[0, {i, dimREP}, {j, dimREP}];
  For[b = bxnMin[x, n, q, p], b ≤ bxnMax[x, n, q, p] , b++,
```

```
    For[a = axnMin[x, n, q, p], a ≤ axnMax[x, n, q, p] , a++, t2 = a + b;
     Table[matrixX1⟦(*r*) (t2 / 2 - α + 1) + ((a + b) (a + b + 1) + q b (2 + b + q)) / 2,
        (*c*) (t2 / 2 - β + 1) + ((a + b) (a + b + 1) + q b (2 + b + q)) / 2⟧ =
       comp[x, n, q, p, a, b, a, b, α, β], {α, t2 / 2, -t2 / 2, -1}, {β, t2 / 2, -t2 / 2, -1}]]];
  matrixX[x] = matrixX1
 ];
Clear[n, a, b, matrixX1];
(*Calculate Up,Um,Vp,Vm*)
For[x = 5, x ≤ 8, x++,
  matrixX1 = Table[0, {i, dimREP}, {j, dimREP}];
  (*n=1 indicates Upper Blocks*)
  n = 1;
  For[bj = bxnMin[x, n, q, p], bj ≤ bxnMax[x, n, q, p] , bj++,
   For[aj = axnMin[x, n, q, p],
    aj ≤ axnMax[x, n, q, p] , aj++, ai = aixn[x, n, qq, pp, ai, bi, aj, bj];
    bi = bixn[x, n, qq, pp, ai, bi, aj, bj];
    t2i = ai + bi;
    t2j = aj + bj;
    Table[matrixX1⟦(*r*) (t2i / 2 - α + 1) + ((ai + bi) (ai + bi + 1) + q bi (2 + bi + q)) / 2, (*c*)
        (t2j / 2 - β + 1) + ((aj + bj) (aj + bj + 1) + q bj (2 + bj + q)) / 2⟧ = comp[x, n, q, p,
        ai, bi, aj, bj, α, β], {α, t2i / 2, -t2i / 2, -1}, {β, t2j / 2, -t2j / 2, -1}]]];
  matrixX2 = Table[0, {i, dimREP}, {j, dimREP}];
  (*n=2 indicates Lower Blocks*)
  n = 2;
  For[bi = bxnMin[x, n, q, p], bi ≤ bxnMax[x, n, q, p] , bi++,
   For[ai = axnMin[x, n, q, p],
    ai ≤ axnMax[x, n, q, p] , ai++, aj = ajxn[x, n, qq, pp, ai, bi, aj, bj];
    bj = bjxn[x, n, qq, pp, ai, bi, aj, bj];
    t2i = ai + bi;
    t2j = aj + bj;
    Table[matrixX2⟦(*r*) (t2i / 2 - α + 1) + ((ai + bi) (ai + bi + 1) + q bi (2 + bi + q)) / 2, (*c*)
        (t2j / 2 - β + 1) + ((aj + bj) (aj + bj + 1) + q bj (2 + bj + q)) / 2⟧ = comp[x, n, q, p,
        ai, bi, aj, bj, α, β], {α, t2i / 2, -t2i / 2, -1}, {β, t2j / 2, -t2j / 2, -1}]]];
  matrixX[x] = matrixX1 + matrixX2
 ];
Clear[x, n, ai, bi, aj, bj, matrixX1, matrixX2];
(* Recall ID: x=1,2,3,4,5,6,7,8 for ID = T3,Y,Tp,Tn,Up,Um,Vp,Vm, respectively. *)
T3 = matrixX[1]; Y = matrixX[2]; Tp = matrixX[3];
Tm = matrixX[4]; Up =  matrixX[5]; Um =  matrixX[6];
Vp =  matrixX[7]; Vm =  matrixX[8];
checkCRs =
  ({0} == Union[Flatten[Simplify[{T3.Tp - Tp.T3 - Tp, T3.Tm - Tm.T3 - (-Tm), Tp.Y - Y.Tp,
        T3.Y - Y.T3, Tm.Y - Y.Tm, T3.Up - Up.T3 - (- (1 / 2) Up), T3.Um - Um.T3 - (+ (1 / 2) Um),
        T3.Vp - Vp.T3 - (1 / 2) Vp, T3.Vm - Vm.T3 - (- (1 / 2) Vm), Tp.Tm - Tm.Tp - 2 T3,
        Tp.Up - Up.Tp - Vp, Tp.Um - Um.Tp, Tp.Vp - Vp.Tp, Tp.Vm - Vm.Tp - (-Um),
        Tm.Up - Up.Tm, Tm.Um - Um.Tm - (-Vm), Tm.Vp - Vp.Tm - Up, Tm.Vm - Vm.Tm,
        Y.Up - Up.Y - (+ Up), Y.Um - Um.Y - (- Um), Y.Vp - Vp.Y - (+ Vp), Y.Vm - Vm.Y - (-Vm),
```

```
      Up.Um - Um.Up - (3 Y / 2 - T3), Up.Vp - Vp.Up, Up.Vm - Vm.Up - (+ Tm),
      Um.Vp - Vp.Um - (- Tp), Um.Vm - Vm.Um, Vp.Vm - Vm.Vp - (3 Y / 2 + T3),
      (1 / 2) (Tp.Tm + Tm.Tp) + T3.T3 + (1 / 2) (Vp.Vm + Vm.Vp) + (1 / 2) (Up.Um + Um.Up) +
       (3 / 4) Y.Y - (((p^2 + p * q + q^2 + 3 p + 3 q) / 3) IdentityMatrix[dimREP])}]]]);
checkTRACE = ({0} == Union[Table[Sum[matrixX[i][[j, j]], {j, dimREP}], {i, 8}]] );
If[checkCRs && checkTRACE,
  Print["The generators satisfy the Lie algebra and they have zero trace."],
  Print["ERROR: The generators fail to satisfy the Lie algebra or
     they don't have zero trace or something else has happened."]];
```

The generators satisfy the Lie algebra and they have zero trace.

In[361]:=

```
(*Upload the generators to a data file as output.*)
SetDirectory[NotebookDirectory[]];
fileName = StringJoin["TYUVbasisForp", ToString[p], "q", ToString[q], "irrep.dat"];
If[checkCRs && checkTRACE,
 Put[{{p, q}, T3, Y, Tp, Tm, Up, Um, Vp, Vm}, fileName], Print["Either the check
    for the CRs or the check for null trace fails,so there is NO OUTPUT."]]
```

In[363]:=

```
NotebookDirectory[]
```

Out[363]=

```
C:\Users\momen\Dropbox\HOME_DESKTOP-0MRE5OJ\SendXXX_CJP_CEJPetc\SENDxxx\2009_08
  Formulas4SU3Matrices\2025SU3short\viXra\
```

In[364]:=

```
(*Download the generators from the file*)
fileName = StringJoin["TYUVbasisForp", ToString[p], "q", ToString[q], "irrep.dat"];
(*For the purpose of checking the downloaded generators, they are given new names.*)
{{p1, q1}, T3a, Ya, Tpa, Tma, Upa, Uma, Vpa, Vma} = Get[fileName];
```

In[366]:=

```
checkCRsa =
  ({0} == Union[Flatten[Simplify[{T3a.Tpa - Tpa.T3a - Tpa, T3a.Tma - Tma.T3a - (-Tma), Tpa.Ya -
        Ya.Tpa, T3a.Ya - Ya.T3a, Tma.Ya - Ya.Tma, T3a.Upa - Upa.T3a - (- (1 / 2) Upa),
      T3a.Uma - Uma.T3a - (+ (1 / 2) Uma), T3a.Vpa - Vpa.T3a - (1 / 2) Vpa,
      T3a.Vma - Vma.T3a - (- (1 / 2) Vma), Tpa.Tma - Tma.Tpa - 2 T3a, Tpa.Upa - Upa.Tpa - Vpa,
      Tpa.Uma - Uma.Tpa, Tpa.Vpa - Vpa.Tpa, Tpa.Vma - Vma.Tpa - (-Uma), Tma.Upa - Upa.Tma,
      Tma.Uma - Uma.Tma - (-Vma), Tma.Vpa - Vpa.Tma - Upa, Tma.Vma - Vma.Tma,
      Ya.Upa - Upa.Ya - (+ Upa), Ya.Uma - Uma.Ya - (- Uma), Ya.Vpa - Vpa.Ya - (+ Vpa),
      Ya.Vma - Vma.Ya - (-Vma), Upa.Uma - Uma.Upa - (3 Ya / 2 - T3a), Upa.Vpa - Vpa.Upa,
      Upa.Vma - Vma.Upa - (+ Tma), Uma.Vpa - Vpa.Uma - (- Tpa), Uma.Vma - Vma.Uma,
      Vpa.Vma - Vma.Vpa - (3 Ya / 2 + T3a), (1 / 2) (Tpa.Tma + Tma.Tpa) + T3a.T3a +
       (1 / 2) (Vpa.Vma + Vma.Vpa) + (1 / 2) (Upa.Uma + Uma.Upa) + (3 / 4) Ya.Ya -
       (((p1^2 + p1 * q1 + q1^2 + 3 p1 + 3 q1) / 3) IdentityMatrix[dimREP])}]]]);
Print["The downloaded generators for the {p,q} = ",
 {p, q}, " irrep satisfy the Lie algebra: ", checkCRsa, "."]
```

The downloaded generators for the {p,q} = {2, 1} irrep satisfy the Lie algebra: True.

6. For the (p,q) irrep, the su(3) basis calculated, checked, saved, and retrieved

The transformation from the TYUV basis of sl(3,C) to the Fj basis of su(3):

F1 = (Tp+Tm)/2 , F2 = -i(Tp-Tm)/2 , F3 = T3 , F4 = (Vp+Vm)/2 ,
F5 = -i(Vp-Vm)/2 , F6 = (Up+Um)/2 , F7 = -i(Up-Um)/2 , F8 = $\sqrt{3}$ Y/2

In[368]:=

```
(*The F^i basis matrices for the numerical example *)
Fi =
  {1/2 (Tp + Tm), -i/2 (Tp - Tm), T3, 1/2 (Vp + Vm), -i/2 (Vp - Vm), 1/2 (Up + Um), -i/2 (Up - Um), Sqrt[3]/2 Y};
```

The following cells are copied from another notebook, GellManndijkCasimir.nb

In[369]:=

```
(*Gell-Mann matrices, Ref. [2]*)
λ[1] = {{0,1,0},{1,0,0},{0,0,0}}; λ[2] = {{0,-i,0},{i,0,0},{0,0,0}}; λ[3] = {{1,0,0},{0,-1,0},{0,0,0}}; λ[4] = {{0,0,1},{0,0,0},{1,0,0}};
λ[5] = {{0,0,-i},{0,0,0},{i,0,0}}; λ[6] = {{0,0,0},{0,0,1},{0,1,0}}; λ[7] = {{0,0,0},{0,0,-i},{0,i,0}}; λ[8] = 1/3^(1/2) {{1,0,0},{0,1,0},{0,0,-2}};
Table[{i, λ[i] // MatrixForm}, {i, 8}];
```

In[372]:=

```
(*Normalization:  tr(λ[j]λ[k]) = 2 δjk *)
Print["Normalization. tr(λ[j]λ[k]) = 2 δjk: ", {0} == Union[
   Flatten[Table[Sum[(λ[j].λ[k])[[i, i]], {i, 3}], {j, 3}, {k, 3}] - 2 IdentityMatrix[3]]]]
```

Normalization. tr(λ[j]λ[k]) = 2 δ_{jk}: True

In[373]:=

```
(*Calculate the structure constants fijk of the su(3) Lie algebra in the Fj basis.*)
(*fijk - Structure constants [Fi,Fj] = ⅈ Σkfijk Fk ,
where Fi = λi/2, Fi FUNDAMENTAL REP*)
fijk[i_, j_, k_] := (-ⅈ / 4) Sum[((λ[i].λ[j] - λ[j].λ[i]).λ[k])[[n, n]], {n, 3}]
fTable = {};
For[i = 1, i ≤ 8, i++, For[j = i, j ≤ 8, j++,
  For[k = j, k ≤ 8, k++, If[(fijk[i, j, k] > 0.00001) || (fijk[i, j, k] < -0.00001),
    AppendTo[fTable, {i, j, k, fijk[i, j, k]}]]]]]
Partition[Flatten[{{"i", "j", "k", "fijk"}, fTable}], 4] // Grid
```

Out[376]=

i	j	k	fijk
1	2	3	1
1	4	7	$\frac{1}{2}$
1	5	6	$-\frac{1}{2}$
2	4	6	$\frac{1}{2}$
2	5	7	$\frac{1}{2}$
3	4	5	$\frac{1}{2}$
3	6	7	$-\frac{1}{2}$
4	5	8	$\frac{\sqrt{3}}{2}$
6	7	8	$\frac{\sqrt{3}}{2}$

In[377]:=

```
Print["Check the CRs for the fundamental rep: λ[i].λ[j]-λ[j].λ[i] = 2ⅈ fijk λ[k]: ",
 {0} == Union[Flatten[Table[
     (λ[i].λ[j] - λ[j].λ[i]) - (2 ⅈ Sum[fijk[i, j, k] × λ[k], {k, 3}]), {i, 3}, {j, 3}]]]]
Print["For i fixed: fijk = -fikj: ",
 {0} == Union[Flatten[Table[fijk[i, j, k] + fijk[i, k, j], {i, 3}, {j, 3}, {k, 3}]]]]
Print["For j fixed: fijk = -fkji: ",
 {0} == Union[Flatten[Table[fijk[i, j, k] + fijk[k, j, i], {i, 3}, {j, 3}, {k, 3}]]]]
Print["For k fixed: fijk = -fjik: ",
 {0} == Union[Flatten[Table[fijk[i, j, k] + fijk[j, i, k], {i, 3}, {j, 3}, {k, 3}]]]]
```

```
Check the CRs for the fundamental rep: λ[i].λ[j]-λ[j].λ[i] = 2ⅈ fijk λ[k]: True
For i fixed: fijk = -fikj: True
For j fixed: fijk = -fkji: True
For k fixed: fijk = -fjik: True
```

In[381]:=

```
(*dijk - symmetric coefficients*)
dijk[i_, j_, k_] := (1 / 4) Sum[((λ[i].λ[j] + λ[j].λ[i]).λ[k])[[n, n]], {n, 3}]
dTable = {};
For[i = 1, i ≤ 8, i++, For[j = i, j ≤ 8, j++,
  For[k = j, k ≤ 8, k++, If[(dijk[i, j, k] > 0.00001) || (dijk[i, j, k] < -0.00001),
    AppendTo[dTable, {i, j, k, dijk[i, j, k]}]]]]]
Partition[Flatten[{{"i", "j", "k", "dijk"}, dTable}], 4] // Grid
(*dijk[1,1,1]
  Partition[
   Flatten[{{"i","j","k",dijk},Table[{i,j,k,dijk[i,j,k]},{i,8},{j,8},{k,8}]}],4]//Grid*)
```

Out[384]=

i	j	k	dijk
1	1	8	$\frac{1}{\sqrt{3}}$
1	4	6	$\frac{1}{2}$
1	5	7	$\frac{1}{2}$
2	2	8	$\frac{1}{\sqrt{3}}$
2	4	7	$-\frac{1}{2}$
2	5	6	$\frac{1}{2}$
3	3	8	$\frac{1}{\sqrt{3}}$
3	4	4	$\frac{1}{2}$
3	5	5	$\frac{1}{2}$
3	6	6	$-\frac{1}{2}$
3	7	7	$-\frac{1}{2}$
4	4	8	$-\frac{1}{2\sqrt{3}}$
5	5	8	$-\frac{1}{2\sqrt{3}}$
6	6	8	$-\frac{1}{2\sqrt{3}}$
7	7	8	$-\frac{1}{2\sqrt{3}}$
8	8	8	$-\frac{1}{\sqrt{3}}$

In[385]:=

```
Print["Check the anticommutator equation for the fundamental rep:
   λ[i].λ[j]+λ[j].λ[i] = 4δij/3 + 2 dijk λ[k]: ", {0} == Union[Flatten[
    Table[(λ[i].λ[j] + λ[j].λ[i]) - (4 IdentityMatrix[3] × IdentityMatrix[8][[i, j]] / 3 +
        2 Sum[dijk[i, j, k] × λ[k], {k, 8}]), {i, 8}, {j, 8}]]]]
Print["For i fixed, dijk = +dikj: ",
 {0} == Union[Flatten[Table[dijk[i, j, k] - dijk[i, k, j], {i, 8}, {j, 8}, {k, 8}]]]]
Print["For j fixed, dijk = +dkji: ",
 {0} == Union[Flatten[Table[dijk[i, j, k] - dijk[k, j, i], {i, 8}, {j, 8}, {k, 8}]]]]
Print["For k fixed, dijk = +djik: ",
 {0} == Union[Flatten[Table[dijk[i, j, k] - dijk[j, i, k], {i, 8}, {j, 8}, {k, 8}]]]]
```

Check the anticommutator equation for the
fundamental rep: λ[i].λ[j]+λ[j].λ[i] = 4δ_{ij}/3 + 2 dijk λ[k]: True

For i fixed, dijk = +dikj: True

For j fixed, dijk = +dkji: True

For k fixed, dijk = +djik: True

In[389]:=

```
(*Practice the CR for one ij before trying all ij:*)
i = 1; j = 2;
Fi⟦i⟧.Fi⟦j⟧ - Fi⟦j⟧.Fi⟦i⟧ // MatrixForm;
ⅈ fijk[i, j, 3] × Fi⟦3⟧ // MatrixForm;
Fi⟦i⟧.Fi⟦j⟧ - Fi⟦j⟧.Fi⟦i⟧ - ⅈ fijk[i, j, 3] × Fi⟦3⟧ // MatrixForm;
fijk[i, j, 3];
Clear[i, j]
```

In[395]:=

```
Print["For (p,q) = ", {p, q},
 "-irrep, the generators Fi satisfy the su(3) Lie algebra, [Fi,Fj] = ⅈfijk Fk: ",
 {0} == Union[Flatten[
    Table[Fi⟦i⟧.Fi⟦j⟧ - Fi⟦j⟧.Fi⟦i⟧ - ⅈ Sum[fijk[i, j, k] × Fi⟦k⟧, {k, 8}], {i, 8}, {j, 8}]]]]
```

For (p,q) = {2, 1}
-irrep, the generators Fi satisfy the su(3) Lie algebra, [Fi,Fj] = ⅈfijk Fk: True

In[396]:=

```
(*Upload the generators to a data file as output.*)
SetDirectory[NotebookDirectory[]];
fileName = StringJoin["FjbasisForp", ToString[p], "q", ToString[q], "irrep.dat"];
If[checkCRs && checkTRACE,
 Put[{{p, q}, Fi⟦1⟧, Fi⟦2⟧, Fi⟦3⟧, Fi⟦4⟧, Fi⟦5⟧, Fi⟦6⟧, Fi⟦7⟧, Fi⟦8⟧}, fileName];
 Print["The Fj for the {p,q} = ", {p, q}, " su(3) irrep are saved in the file '",
  fileName, "'."], Print["Either the check for the CRs
    or the check for null trace fails,so there is NO OUTPUT."]]
```

The Fj for the {p,q} = {2, 1} su(3) irrep are saved in the file 'FjbasisForp2q1irrep.dat'.

In[398]:=

```
NotebookDirectory[]
```

Out[398]=

C:\Users\momen\Dropbox\HOME_DESKTOP-0MRE5OJ\SendXXX_CJP_CEJPetc\SENDxxx\2009_08
Formulas4SU3Matrices\2025SU3short\viXra\

In[399]:=

```
(*Download the generators from the file*)
(*USER:
 If you want to download generators from a previously saved file,
change the values of p and q here.*)
(*p = ; q = ;*)
fileName = StringJoin["FjbasisForp", ToString[p], "q", ToString[q], "irrep.dat"];
(*For the purpose of checking the saving and retrieving operations,
the downloaded quantities are given new names.*)
(*{{p1,q1},Fia[[1]],Fia[[2]],Fia[[3]],Fia[[4]],Fia[[5]],Fia[[6]],Fia[[7]],Fia[[8]]}=Get[fileName];*)
{{p1, q1}, F1a, F2a, F3a, F4a, F5a, F6a, F7a, F8a} = Get[fileName];
Fia = {F1a, F2a, F3a, F4a, F5a, F6a, F7a, F8a};
```

In[402]:=

```
Print["For (p,q) = ", {p, q},
 "-irrep, the downloaded generators Fia satisfy the su(3) Lie algebra: ",
 {0} == Union[Flatten[Table[
     Fia[[i]].Fia[[j]] - Fia[[j]].Fia[[i]] - i Sum[fijk[i, j, k] × Fia[[k]], {k, 8}], {i, 8}, {j, 8}]]]]
```

```
For (p,q) = {2, 1}-irrep, the downloaded generators Fia satisfy the su(3) Lie algebra: True
```

7. For the (p,q) irrep, checks of the two Casimir operators.

In[403]:=

```
(*Quadratic Casimir equation: C1 = ΣkFkFk = (p^2+q^2+3p+3q+p q)/3*)
C1 = Sum[Fi[[k]].Fi[[k]], {k, 8}];
Print["C1 commutes with all eight Fj generators: ",
 {0} == Union[Flatten[Table[ C1.Fi[[i]] - Fi[[i]].C1, {i, 8}]]]]
Print["Check that C1 = ΣkFkFk = [(p^2+q^2+3p+3q+p q)/3]I: ", {0} ==
  Union[
   Flatten[Sum[Fi[[k]].Fi[[k]], {k, 8}] - IdentityMatrix[dimREP] (p^2 + q^2 + 3 p + 3 q + p q) / 3]]]
Print["where I is the identity matrix"]
```

```
C1 commutes with all eight Fj generators: True
Check that C1 = ΣkFkFk = [(p^2+q^2+3p+3q+p q)/3]I: True
where I is the identity matrix
```

In[407]:=

```
(*Cubic Casimir: C2 = ΣijkdijkFiFjFk = (p-q)(3+p+2q)(3+q+2p)/18*)
C2 = Simplify[Sum[dijk[i, j, k] × Fi[[i]].Fi[[j]].Fi[[k]], {i, 8}, {j, 8}, {k, 8}]];
Print["C2 commutes with all eight Fj generators: ",
 {0} == Union[Flatten[Table[ C2.Fi[[i]] - Fi[[i]].C2, {i, 8}]]]]
Print["Check that C2 = ΣijkdijkFiFjFk = [(p-q)(3+p+2q)(3+q+2p)/18]I: ", {0} ==
  Union[Flatten[Simplify[Sum[dijk[i, j, k] × Fi[[i]].Fi[[j]].Fi[[k]], {i, 8}, {j, 8}, {k, 8}]] -
     IdentityMatrix[dimREP] (p - q) (3 + p + 2 q) (3 + q + 2 p) / 18]]]
Print["where I is the identity matrix"]
```

C2 commutes with all eight Fj generators: True

Check that C2 $= \Sigma_{ijk} d_{ijk} F_i F_j F_k = [(p-q)(3+p+2q)(3+q+2p)/18]$ I: True

where I is the identity matrix

8. FYI

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References

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[2] See Walter Greiner and Berndt Müller, *Quantum Mechanics, Symmetries*, 2nd revised edition (Springer-Verlag, Berlin, 1994), Sec. 7.3.
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[4] } Walter Pfeifer, *The Lie Algebras su(N)* , 1st ed.; Birkhauser Verlag: Basel, Switzerland, 2003.
[5] Richard Shurtleff, *Formulas for SU(3) Matrix Generators* , viXra:2409.0060, the omnibus presentation (2025).
[6] Wolfram Repository
<iframe src=”https://www.wolframcloud.com/obj/shurtleffr/Published/2025SU3shortviXra.nb” width=”600” height=”800”></iframe>
[7] DropBox Access
<iframe src=”https://www.dropbox.com/scl/fi/4zq9pnn5kvhq3wkb4vgim/2025SU3shortviXra.nb?r-lkey=tsfnmvqr3x5dhegoaci49joxy&dl=0“ width=”600” height=”800”></iframe>
[8] Wolfram Research, Inc., Mathematica, Version 14.2, Champaign, IL (2024)

In[411]:=
```
MaxMemoryUsed[]
```
Out[411]=
```
211 401 872
```