

Computing the Singular Value Decomposition (SVD) with Fixed Point CORDIC Operations with Application to MIMO-OFDM

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Abstract

In this paper, the computation of the Singular Value Decomposition (SVD) of complex matrices will be presented using fixed point arithmetic. The application of CORDIC operations for fixed point implementations of the SVD of complex matrices will be introduced. SVD plays a major role in Closed Loop MIMO OFDM systems. The impact of fixed point implementation of SVD in a Closed Loop MIMO-OFDM system is examined. The ratio of Maximum to Minimum Singular Value (MMSVR) is computed for both fixed point (CORDIC) and floating point operations (using the LAPACK library). The fixed point implementation closely tracks the floating point results over fading channel models. It is shown that for highly ill-conditioned sub carriers the fixed point implementation deviates from the floating point MMSVR. This leads to noise enhancement and degradation of performance. By adding transmit diversity in Closed Loop MIMO-OFDM the MMSVR can be reduced and performance substantially enhanced for the fixed point implementation. It is also shown how SVD can be used in Open Loop MIMO-OFDM systems. This paper is an important introduction to the algorithms implemented in the GitHub repository for MIMO-OFDM: <https://github.com/silicondsp/mimo-ofdm-release>

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Chapter 1

Introduction

In this paper, the computation of the Singular Value Decomposition (SVD) of complex matrices will be presented. In particular, the use of CORDIC operations for fixed point implementations of the SVD of complex matrices will be introduced. This paper is an important introduction to the algorithms implemented in the GitHub repository for MIMO-OFDM:

<https://github.com/silicondsp/mimo-ofdm-release>

The C Code written for the Capsim[®] Block Diagram Modeling and Simulation tool were written in 2006 through 2016 which also served the basis for the videos on MIMO OFDM in [4] and the MIMO-OFDM Tutorials developed in 2007 which were published in 2021 on YouTube [3].

The table below shows various MIMO OFDM Configurations. The beamforming Closed Loop MIMO-OFDM systems with SVD show improved performance over Open Loop MIMO-OFDM systems. Therefore, the fixed point, CORDIC based computation for the SVD of complex matrices, is the architecture that can drive the adoption of the superior Closed Loop SVD systems in Silicon [2].

| Configuration | Tx | Rx | CSI | FFTs | SVD | QR | Rate | Application |
|------------------|----|----|-----|------|-----|-----|------|---|
| 1x1 SISO | 1 | 1 | No | 1 | No | No | 1x | Lowest Data Rate and Power |
| 1x2 MRC | 1 | 2 | No | 2 | No | No | 1x | Longer Range More Power. |
| 2x2 Open Loop | 2 | 2 | No | 2 | 2x2 | No | 2x | Medium Data Rate |
| 2x2 Beam Forming | 2 | 2 | Yes | 2 | 2x2 | Yes | 2x | Medium Date Rate More Reliable than Open Loop |
| 2x3 Open Loop | 2 | 3 | No | 3 | 2x2 | Yes | 2x | Reliable Medium Data Rate |
| 4x2 Beam Forming | 4 | 2 | Yes | 2 | 2x2 | No | 2x | High Down Link Data Rate, Reliable Up-link, Low Power Video |
| 3x4 Open Loop | 3 | 4 | No | 4 | 3x3 | Yes | 3x | High Data Rate |
| 4x4 Beam Forming | 4 | 4 | Yes | 4 | 4x4 | No | 4x | Very High Data Rate |
| 1x4 Beam Forming | 1 | 4 | Yes | 4 | 4x4 | No | 1x | Very Long Range |

In [8] the performance of various MIMO diversity schemes are compared as shown in Figure 1.

The superior performance of the 4x2 Closed Loop SVD MIMO OFDM over the 2x2 OpenLoop MIMO-OFDM [11] is illustrated in Figure 2.

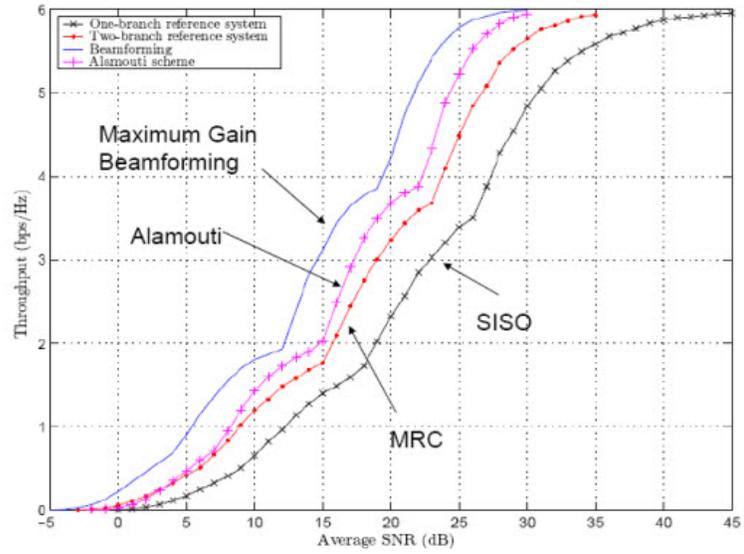


Figure 1: Diversity Schemes MIMO [8]

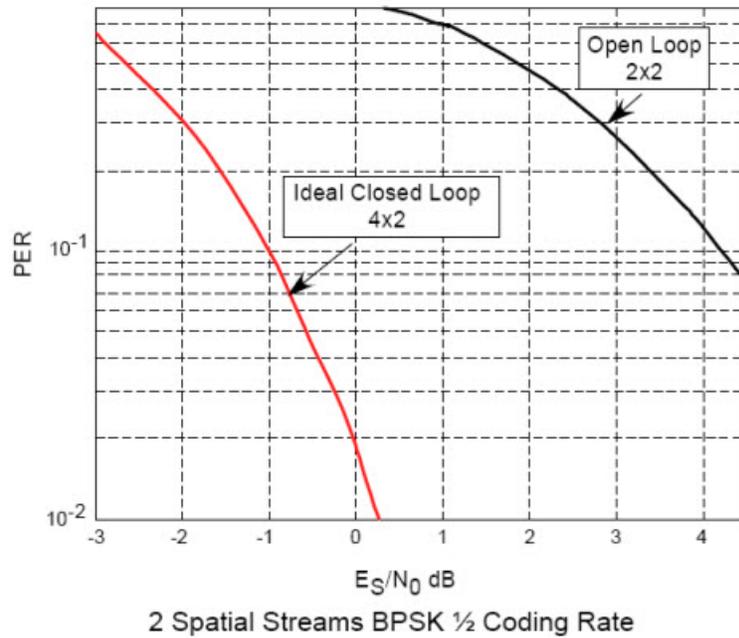


Figure 2: Closed Loop SVD 4x2 versus 2x2 Open Loop MIMO OFDM Performance Comparison [11]

Chapter 2

Brief Overview of MIMO OFDM Communication Receiver

A MIMO communication channel is shown in Figure 3. The Figure shows the complex frequency response for a single carrier in a MIMO OFDM system from each transmit antenna to each receive antenna. The complex elements h_{jk} ($j=1, 2, 3$) and ($k=1, 2$) form a complex matrix H . To implement a MIMO OFDM system, the receiver needs to estimate the channel per carrier in order to perform equalization and separate the multiplexed streams (or to generate a single stream in diversity schemes). To do this, each transmitted packet contains high throughput training symbols. The receiver processes these training symbols and estimates the channel. In an open loop zero forcing MIMO system, in order to separate and equalize the received channels, the Pseudo-inverse [12] of the channel is computed for each carrier's H channel matrix.

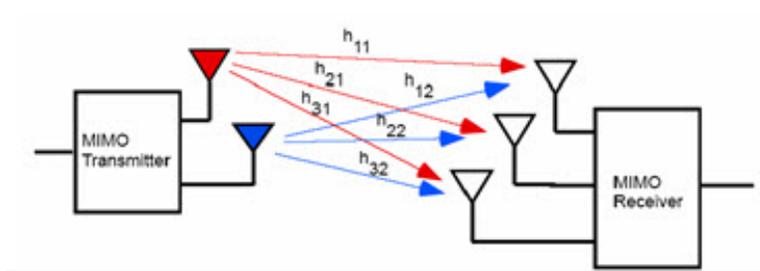


Figure 3: 2x3 MIMO Per Carrier Channel Model

For a comprehensive video tutorial on MIMO-OFDM check [\[3\]](#).

Chapter 3

Implementing the Pseudo-inverse for Open Loop MIMO with an SVD of the H Matrix

3.1 Introduction

We will show that a basic 2x2 MIMO open loop equalizer can be derived from a basic 2x2 Singular Value Decomposition (SVD) of the H matrix. That is, an open loop MIMO system can be implemented with SVD. The key benefit is that for beamforming we use SVD and the SVD computation engine can be used for Open Loop systems.

3.2 Derivation of SVD Based Computation of ZF Equalizer

The MIMO Zero forcing equalizer is computed from the estimated MIMO channel using the Pseudo Inverse.

$$W = H^+ = (H^\dagger H)^{-1} H^\dagger \quad (1)$$

The streams are equalized from the received streams \mathbf{y} .

$$\hat{\mathbf{x}} = W\mathbf{y} = \mathbf{x} + W\mathbf{n} \quad (2)$$

Compute the SVD of H and substitute for H .

$$H = U\Sigma V^\dagger \quad (3)$$

Define,

$$Q = V\Sigma^\dagger \quad (4)$$

$$W = (Q^\dagger)^{-1} Q^{-1} Q U^\dagger \quad (5)$$

$$(AB)^{-1} = B^{-1}A^{-1} \quad (6)$$

$$W = H^+ = (V\Sigma^\dagger\Sigma V^\dagger)^{-1}V\Sigma^\dagger U^\dagger \quad (7)$$

$$W = (V\Sigma^\dagger\Sigma V^\dagger)^{-1}V\Sigma^\dagger U^\dagger \quad (8)$$

$$W = (QQ^\dagger)^{-1}QU^\dagger \quad (9)$$

$$V^\dagger V = I \quad (10)$$

$$W = (\Sigma V^\dagger)^{-1}U^\dagger = (V^\dagger)^{-1}\Sigma^{-1}U^\dagger \quad (11)$$

So we can express the Pseudoinverse W as:

$$W = V\Sigma^{-1}U^\dagger \quad (12)$$

Therefore,, using an SVD 2x2 kernel we can perform both beamforming and open loop MIMO equalization.

For a 2x3 Open Loop MIMO we use Σ^+ instead of Σ^- .

3.3 Numerical Example for 2x3 MIMO Using Mathematica[®]

$$H = \begin{pmatrix} -0.189106 - 0.0969304i & 0.216669 - 0.649563i \\ 0.640475 - 0.011601i & -0.378494 + 0.156319i \\ -0.944997 + 0.753418i & -0.911438 + 0.156696i \end{pmatrix} \quad (13)$$

$$H^\dagger = \begin{pmatrix} -0.189106 + 0.0969304i & 0.640475 + 0.011601i & -0.944997 - 0.753418i \\ 0.216669 + 0.649563i & -0.378494 - 0.156319i & -0.911438 - 0.156696i \end{pmatrix} \quad (14)$$

$$A = H^\dagger H = \begin{pmatrix} 1.91616 + 0.i & 0.757123 + 0.778182i \\ 0.757123 - 0.778182i & 1.49184 + 0.i \end{pmatrix} \quad (15)$$

$$A^{-1} = (H^\dagger H)^{-1} = \begin{pmatrix} 0.888105 + 2.7755575615628914i & -0.450721 - 0.463257i \\ -0.450721 + 0.463257i & 1.1407 + 0.i \end{pmatrix} \quad (16)$$

$$AA^{-1} = \begin{pmatrix} 1. + 1.0869519530951558i & 0. + 1.1102230246251565i \\ 1.1102230246251565i - 1.1102230246251565i & 1. - 5.551115123125783i \end{pmatrix} \quad (17)$$

$$W = (H^\dagger H)^{-1} H^\dagger = A^{-1} H^\dagger \quad (18)$$

$$W = \begin{pmatrix} 0.0353115 - 0.30706i & 0.666989 + 0.256099i & -0.501043 - 0.176258i \\ 0.287485 + 0.609664i & -0.725799 + 0.113163i & -0.264724 - 0.276939i \end{pmatrix} \quad (19)$$

$$H = U \Sigma V^\dagger \quad (20)$$

$$\Sigma = \begin{pmatrix} 1.6763831774230495 & 0 \\ 0 & 0.7731362818695853 \\ 0 & 0 \end{pmatrix} \quad (21)$$

$$V = \begin{pmatrix} -0.460479 + 3.8702414787441454i & 0.822232 - 6.713442749671403i & 0. + 0.i \\ -0.264437 + 0.271793i & -0.696262 + 0.715628i & 0. + 0.i \end{pmatrix} \quad (22)$$

$$U = \begin{pmatrix} 0.20633 + 0.275292i & 0.158498 + 0.527621i & -0.320077 - 0.689701i \\ -0.237323 - 0.138866i & 0.678284 - 0.389239i & 0.354321 - 0.432752i \\ 0.63358 - 0.636091i & -0.254543 - 0.141868i & 0.142906 - 0.297697i \end{pmatrix} \quad (23)$$

$$\Sigma^+ = PseudoInverse[\Sigma] = \begin{pmatrix} 0.5965223306148947 & 0 & 0 \\ 0 & 1.2934330252640798 & 0 \end{pmatrix}$$

(24)

$$V\Sigma^+ = \begin{pmatrix} -0.460479 + 3.8702414787441454i & 0.822232 - 6.713442749671403i & 0. + 0.i \\ -0.264437 + 0.271793i & -0.696262 + 0.715628i & 0. + 0.i \end{pmatrix} \quad (25)$$

$$U^\dagger = \begin{pmatrix} 0.20633 - 0.275292i & -0.237323 + 0.138866i & 0.63358 + 0.636091i \\ 0.158498 - 0.527621i & 0.678284 + 0.389239i & -0.254543 + 0.141868i \\ -0.320077 + 0.689701i & 0.354321 + 0.432752i & 0.142906 + 0.297697i \end{pmatrix} \quad (26)$$

$$W = V\Sigma^+U^\dagger = \begin{pmatrix} 0.0353115 - 0.30706i & 0.666989 + 0.256099i & -0.501043 - 0.176258i \\ 0.287485 + 0.609664i & -0.725799 + 0.113163i & -0.264724 - 0.276939i \end{pmatrix} \quad (27)$$

This result for W matches:

$$W = (H^\dagger H)^{-1}H^\dagger = \begin{pmatrix} 0.0353115 - 0.30706i & 0.666989 + 0.256099i & -0.501043 - 0.176258i \\ 0.287485 + 0.609664i & -0.725799 + 0.113163i & -0.264724 - 0.276939i \end{pmatrix} \quad (28)$$

Chapter 4

Singular Value Decomposition of Arbitrary Complex 2x2 Matrices Using CORDIC Operations

This section contains the formula and matrix manipulations for the Singular Value Decomposition of arbitrary complex matrices. The key point is to compute the SVD such that CORDIC computation units, rotations and arctangents, can be used. The method is based on the work outlined in [10].

A 2x2 arbitrary complex matrix is used to illustrate the technique.
 Define the Matrix A ,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = U\Sigma V^\dagger \quad (29)$$

We start out with the Matrix A and use Mathematica to compute the SVD:

$$A = \begin{pmatrix} 2 + 3i & 5 + 7i \\ 9 - 3i & 8 + 6i \end{pmatrix} = U\Sigma V^\dagger \quad (30)$$

Where $\Sigma = [18.5769, 5.8221]$ and

$$U = \begin{pmatrix} -0.77513 - 0.22045i & -0.30045 + 0.51021i \\ -0.51724 + 0.28818i & -0.20187 - 0.78017i \end{pmatrix} \quad (31)$$

$$V = \begin{pmatrix} -0.71051 + 0i & 0.70369 + 0i \\ -0.23065 + 0.66482i & -0.23288 + 0.67126i \end{pmatrix} \quad (32)$$

The first step in computing the SVD is to convert the matrix A into Polar form. This is accomplished using the ArcTan Cordic function. Note that the Arctan Cordic function also computed the modulus. In "C" notation:

$$\theta = \text{ArcTanCordic}(x, y, \&r); \quad (33)$$

$$\begin{aligned} r_{11} &= \text{Abs}[a_{11}] \\ \theta_{11} &= \text{Arg}[a_{11}] \\ r_{12} &= \text{Abs}[a_{12}] \\ \theta_{12} &= \text{Arg}[a_{12}] \\ r_{21} &= \text{Abs}[a_{21}] \\ \theta_{21} &= \text{Arg}[a_{21}] \\ r_{22} &= \text{Abs}[a_{22}] \\ \theta_{22} &= \text{Arg}[a_{22}] \end{aligned} \quad (34)$$

$$AA = \begin{pmatrix} r_{11}e^{i\theta_{11}} & r_{12}e^{i\theta_{12}} \\ r_{21}e^{i\theta_{21}} & r_{22}e^{i\theta_{22}} \end{pmatrix} \quad (35)$$

Define the following angles:

$$\alpha = \frac{\text{Imag}[a_{22}] + \text{Imag}[a_{21}]}{2} \quad (36)$$

$$\beta = \alpha \quad (37)$$

$$\eta = \frac{\text{Imag}[a_{22}] - \text{Imag}[a_{21}]}{2} \quad (38)$$

$$\omega = -\eta \quad (39)$$

Define the left and right matrices U_1 and V_1 :

$$U_1 = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \quad (40)$$

$$V_1 = \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{i\omega} \end{pmatrix} \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} \quad (41)$$

The first steps are to use the U_1 and V_1 matrices to transform the matrix A into an upper triangle matrix R_{lower} where

$$R_{lower} = RL = U_1 \times A \times V_1 \quad (42)$$

Convert the Upper Triangular Matrix RL to Polar Coordinates:

$$\begin{aligned} r_{11} &= \text{Abs}[rl_{11}] \\ \theta_{11} &= \text{Arg}[rl_{11}] \\ r_{12} &= \text{Abs}[rl_{12}] \\ \theta_{12} &= \text{Arg}[rl_{12}] \\ r_{21} &= \text{Abs}[rl_{21}] \\ \theta_{21} &= \text{Arg}[rl_{21}] \\ r_{22} &= \text{Abs}[rl_{22}] \\ \theta_{22} &= \text{Arg}[rl_{22}] \end{aligned} \quad (43)$$

Now we need to transform RL into a real matrix R . We define the angles,

$$\alpha = \frac{\theta_{11} + \theta_{12}}{2} \quad (44)$$

$$\eta = \frac{\theta_{11} - \theta_{12}}{2} \quad (45)$$

$$\beta = \eta \quad (46)$$

$$\omega = -\eta \quad (47)$$

Define,

$$\phi p \psi = -\text{ArcTan}(r_{12}, r_{22} + r_{11}) \quad (48)$$

$$\phi m \psi = \text{ArcTan}(r_{12}, r_{22} + r_{11}) \quad (49)$$

$$\phi = \frac{\phi p \psi + \phi m \psi}{2} \quad (50)$$

$$\psi = \frac{\phi p \psi - \phi m \psi}{2} \quad (51)$$

$$U_2 = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \quad (52)$$

$$V_2 = \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{i\omega} \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix} \quad (53)$$

Compute,

$$R = U_2 \times RL \times V_2 \quad (54)$$

The Matrix R is a real matrix.

The next step is to use Jacobi Rotation to diagonalize R to obtain the Singular Values.

For Jacobe Rotations see [7] p. 249 Algorithm 5.12.

We have,

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \quad (55)$$

Let,

$$x = r_{11} - r_{22} \quad (56)$$

$$y = 2r_{12} \quad (57)$$

In "C" notation compute θ :

$$\theta = \text{ArcTanCordic}(x, y, \&r); \quad (58)$$

Now compute $\cos(\theta)$ and $\sin(\theta)$ using the Cordic Rotation module. In "C" notation, with $x = 1, y = 0$,

$$\text{CordicRotate}(\&x, \&y, \theta); \quad (59)$$

This function returns, $x = \cos(\theta)$ and $y = \sin(\theta)$.

$$\Sigma = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (60)$$

Computation of U Matrix:

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \times U_2 \times U_1 \quad (61)$$

Computation of V Matrix:

$$V = V_1 \times V_2 \times \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (62)$$

Chapter 5

Computing the SVD of Rectangular Matrix with Square SVD Algorithms

The following is from [6] and is concerned with computing the SVD of a rectangular matrix using the SVD of a square matrix.

Assume that we are interested in the SVD of an $m \times n$ matrix R . Let $m \geq n$. If not, Transpose the matrix if $m < n$. The Rectangular matrix R is decomposed into the product $Q \times S$ of an $m \times n$ matrix. Q satisfying $Q^H Q = I_n$ S is square of order n . Then the SVD of S is computed as:

$$S = UDV^H \quad (63)$$

With $U^H U = V^H V = I_n$.
 D is real diagonal. The SVD of R .

$$R = (QU)DV^H \quad (64)$$

The implementation of the decomposition $R = QS$ is based on the Givens method in which plane rotations are applied to the rows of R in a specific order. For complex, an appropriate rotation is applied.

To illustrate we quote Example 3.6 provided in [7].

Example 3.6. We illustrate two intermediate steps in computing the QR decomposition of a 5-to-4 matrix using Givens rotations.

$$\begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{pmatrix} \quad (65)$$

to

$$\begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{pmatrix} \quad (66)$$

we multiply

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & c & -s \\ & & s & c \end{pmatrix} \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{pmatrix} = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} \quad (67)$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & c' & -s' \\ & & s' & c' \\ & & & 1 \end{pmatrix} \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{pmatrix} \quad (68)$$

Chapter 6

Comparison of the CORDIC Based Fixed Point with Floating Point Beamforming MIMO OFDM

6.1 Introduction

We have implemented the 2x2 SVD (Beamforming) MIMO OFDM System using the Fixed Point CORDIC based 2x2 SVD kernel. In this section we compare the fixed point SVD implementation with the floating point SVD based on the LAPACK[1] library. In the simulations the 2x2 Beamforming MIMO OFDM system shown in Figure 4 (using Capsim[®] and IEEE 802.11a 54 Mbps streams) is used where the receiver is either based on the floating point LAPACK library, or the fixed point CORDIC based SVD.

6.2 Comparison Between Floating Point and Fixed Point CORDIC

The comparison between floating point and CORDIC fixed point is key to exploring the degradation introduced by the fixed point implementation of the 2x2 SVD and to demonstrate that the fixed point 2x2 SVD performs well over a wide range of MIMO channels.

As show in Figures 5 and 6 the fixed point CORDIC based 2x2 SVD MIMO OFDM system performs well compared to the floating point LAPACK based 2x2 SVD. The slight apparent degradation is due to the finite precision in the fixed point CORDIC 2x2 SVD implementation.

A key comparison is the case where channel noise is added. In this case, we expect that the finite precision fixed point CORDIC SVD will enhance noise and degrade performance compared to the floating point LAPACK implementation. This is shown in Figures 7 and 8 where noise variance of $1e-5$ (10^{-5}) was added to each receive chain.

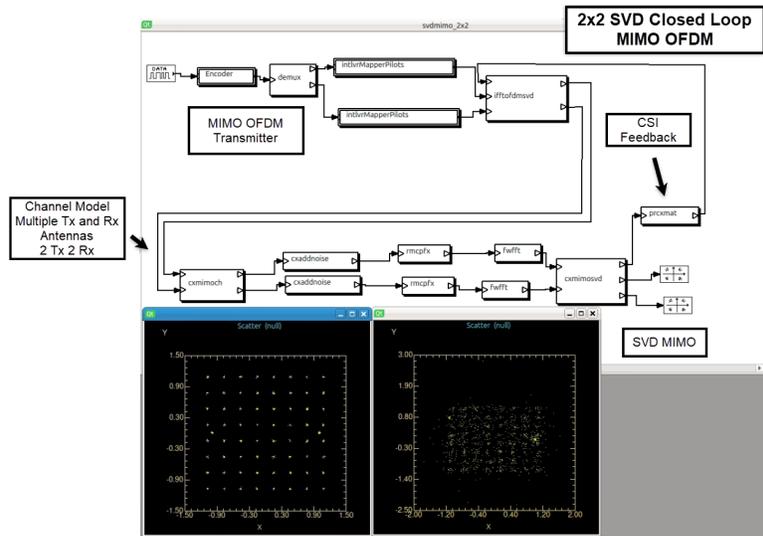


Figure 4: Capsim[®] MIMO OFDM 2x2 Closed Loop with Channel Modeling and Noise Specification

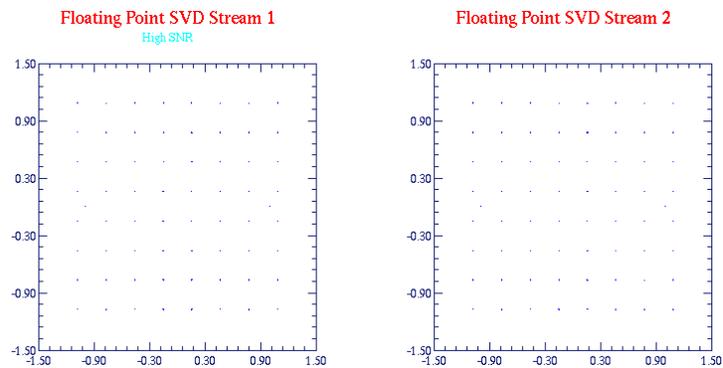


Figure 5: 2x2 Beamforming MIMO OFDM 64 QAM Constellations for Each Stream LAPACK Floating Point Library, High SNR

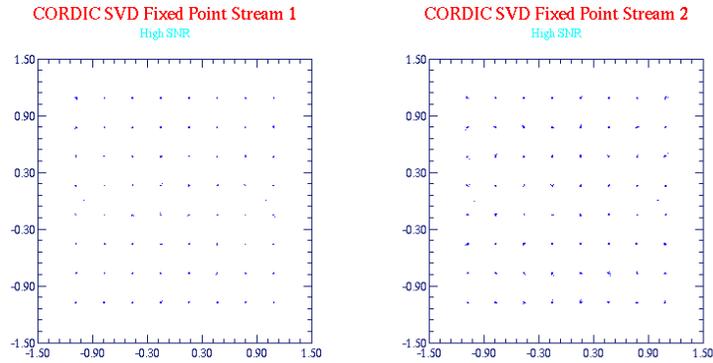


Figure 6: 2x2 Beam Forming MIMO OFDM 64 QAM Constellations for Each Stream CORDIC Fixed Point 2x2 SVD , High SNR

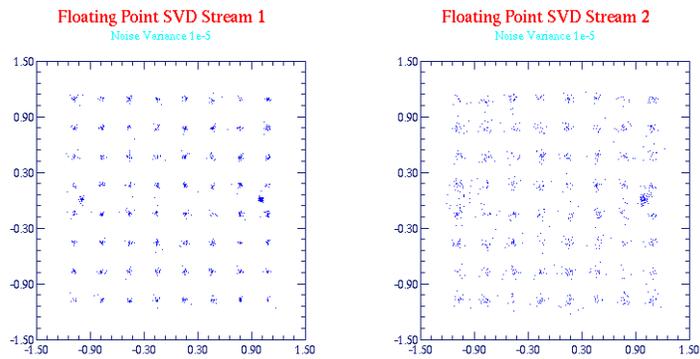


Figure 7: 2x2 Beam Forming MIMO OFDM 64 QAM Constellations for Each Stream LAPACK Floating Point Library, Medium SNR

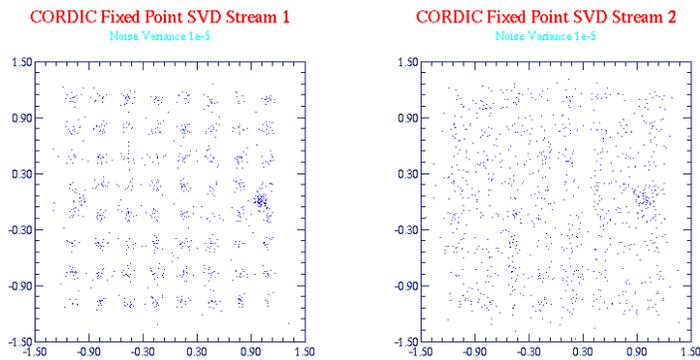


Figure 8: 2x2 Beam Forming MIMO OFDM 64 QAM Constellations for Each Stream CORDIC Fixed Point 2x2 SVD , Medium SNR

6.3 SVD Singular Value Ratio Comparing Floating Point LAPACK versus Fixed Point CORDIC 2x2 SVD

To show that the fixed point CORDIC 2x2 SVD tracks the floating point SVD in a 2x2 MIMO OFDM system we show the plot of the ratio of Singular Values for various carriers (52 for IEEE 802.11a streams) in Figure 9 with additive noise (variance $1e-5$). Note that the ratios track very well over the 52 carriers. The deviation is at the high Singular Value Ratio which corresponds to an ill conditioned channel at that carrier frequency. The enhancement of roundoff noise is caused at this point and other ill conditioned channel conditions.

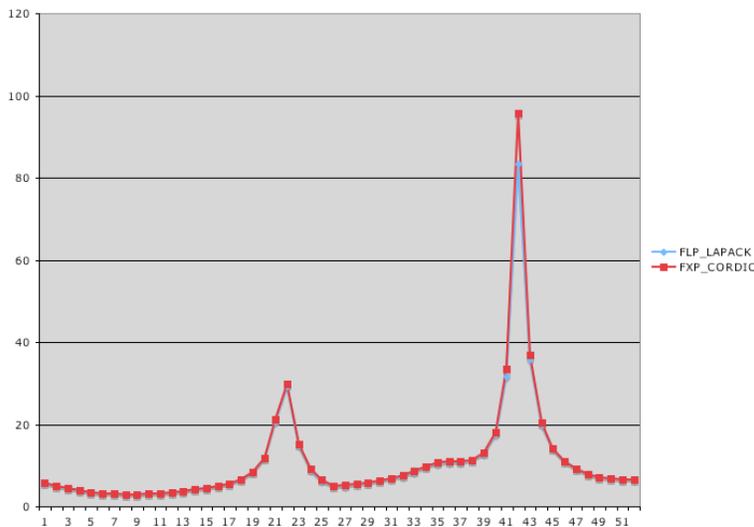


Figure 9: Singular Value Ratio: Floating Point versus Fixed Point 2x2 Closed Loop for Each Carrier(52)

6.4 On the Performance of Open Loop MIMO Systems

In [9] the Maximum to Minimum Singular Value Ratio (MMSVR) is defined. It is shown that it is a key predictor of the performance of Open-Loop MIMO OFDM systems over many channel realizations. In Figure 10, it is shown that for 3x4 MIMO OFDM the probability of the MMSVR is shifted to the left (lower values, less ill conditioned situations) compared to the 3x3 Open Loop system. Track the peak value to see this result. Since we can implement an Open Loop MIMO system with an SVD computation engine, and we have shown the performance of fixed point CORDIC SVD with noise and degradation due to high MMSVR, adding an extra receive antenna (3x4 versus 3x3) can greatly improve the performance of the fixed point CORDIC implementation.

For example a 4x2 Closed Loop MIMO-OFDM should decrease the MMSVR and is recommended for fixed point implementation.

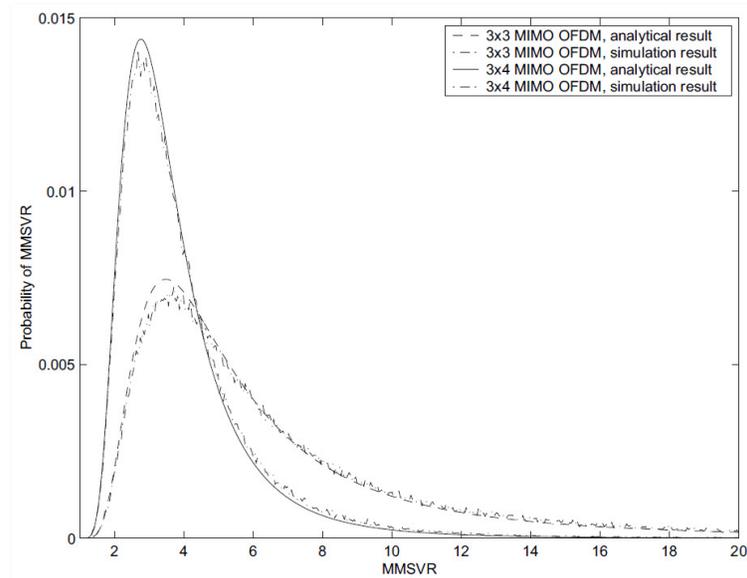


Figure 10: Analytical and simulated probability density of MMSVR for 3×3 and 3×4 MIMO OFDM configurations [9].

The throughput comparison of MIMO-OFDM systems at 20MHz for 3x3 and 3x4 systems is shown in Figure 11 due to A. Maltsev and A. Davydov(see reference [19] in [9]. Note the performance advantage of 3x4 over 3x3 MIMO OFDM as shown in the Figure are predicted analytically by the MMSVR in [9].

For a demonstration of adding a receive chain to a 2x2 Open Loop MIMO-OFDM showing performance enhancement, see the video on Open Loop 2x2 and 2x3 MIMO-OFDM Systems [4].

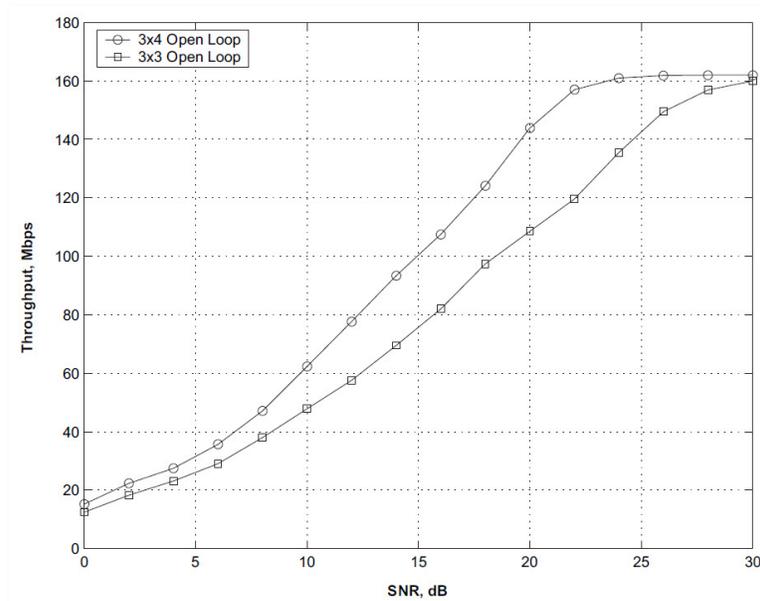


Figure 11: Throughput comparison of MIMO-OFDM systems at 20MHz Reference 19 in[9]

Chapter 7

MIMO OFDM Repository

A full implementation of the fixed point SVD computation in MIMO-OFDM systems is provided in the Capsim[®] based Block Diagram Modeling and Simulation environment at the GitHub Repository[5]:

<https://github.com/silicondsp/mimo-ofdm-release>

The Capsim[®] simulation includes both Open Loop MIMO-OFDM systems and Closed Loop MIMO-OFDM systems with CSI feedback. Floating point

implementations using LAPACK for matrix operations are also included for comparison with fixed point CORDIC operation implementations.

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Chapter 8

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