An experimentally appearing inconsistency with hidden variable theories in quantum optical phenomena

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Abstract

We investigate the significant striking aspect of the most basic two quantum observables: (1) an observable that is obtained though a single projection operator and (2) an observable that is obtained though a single identity operator. Kochen and Specker show that some value assignment to elements of a set of projection operators and the identity operator should fail. Now, we greatly simplify the Kochen–Specker theorem. Our simplified version says that we cannot assign a preexistence value as +1 to both quantum operators (the projection operator and the identity operator) even though the assignment is independent from each other. And finally, we are able to use the weak assumption (the identity operator possesses a preexistence value as +1) instead of the strong assumption (the existence of the weak hidden variable theory) in order to verify the Kochen–Specker theorem.

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I. INTRODUCTION

The great success of quantum mechanics (QM) (cf. [1–8]) is recognized by the scientific community of physical theories. The incompleteness argument for quantum mechanics itself is discussed by Einstein, Podolsky, and Rosen [9]. A hidden variable interpretation of quantum mechanics is an interesting topic of research [3, 4] and the inconsistency with hidden variable theories is discussed by Bell, Kochen, and Specker [10, 11].

Multipartite inequality as test for the Kochen–Specker theorem is proposed by Nagata [12]. It turns out that the Kochen–Specker theorem becomes a quite strong theorem when the dimension of the multipartite state highly increases, regardless of entanglement properties. A strengthened Kochen–Specker theorem, i.e., the free will theorem is discussed by Conway and Kochen [13]. An inconsistency within the hidden variable theory proposed by Kochen and Specker, using two symmetric measurements, by Nagata, Diep, and Nakamura [14]. They are independent of the order of measurements themselves.

In this paper, we investigate the significant striking aspect of the most basic two quantum observables: (1) an observable that is obtained though a single projection operator and (2) an observable that is obtained though a single identity operator. Here, both of quantum observables are well studied by Kochen and Specker in terms of the inconsistency with the hidden variable theory. They show that some value assignment to elements of a set of projection operators and the identity operator should fail.

Now, we greatly simplify the Kochen–Specker theorem. We here formulate the most basic two quantum observables on the two-dimensional space this: $H(x) \equiv \begin{pmatrix} +1 & 0 \\ 0 & +x \end{pmatrix}, (x = 0, +1)$. We are in the inconsistency with hidden variable theories introduced by Kochen and Specker when the following physical situation happens twice: The preexistence value is $+1$ by measuring quantum observable $H(x)$ in the quantum state $|\uparrow\rangle$. It turns out that we cannot assign the preexistence value for quantum measurement outcome as $+1$ when measuring quantum observable $H(x)$ in the quantum state $|\uparrow\rangle$.

In the case that $x = 0$, Projection operator does not have a counterpart in physical reality. In the case that $x = +1$, Identity operator does not have a counterpart in physical reality. Our simplified version says that we cannot assign a preexistence value as $+1$ to both quantum operators (the projection operator and the identity operator) even though the assignment is independent from each other. They do not have a counterpart in physical reality in the sense that the inconsistency with hidden variable theories is proposed by Kochen and Specker.

This paper is organized as follows:

In Sec. II, we review the Kochen–Specker theorem and we clarify the assumptions which are used in the theorem.

In Sec. III, we review the difference between probability and truth value approaches.

In Sec. IV, we review the mutual relation between several assumptions for the hidden variable theory.

In Sec. V, we modify the Kochen–Specker theorem presented in Sec. II into the statistical form so as to relax the assumptions which are required in the previous form.

In Sec. VI, we discuss the statistical Kochen–Specker assumption and we propose the commuting observable assignment rule.

In Sec. VII, we study the inconsistency with hidden variable theories using the identity operator or the projection operator.

In Sec. VIII, we also propose the statistical Kochen–Specker theorem based on an identity operator on the two-dimensional space.

In Sec. IX, we study the statistical Kochen–Specker theorem with the incompleteness in a real experiment.

In Sec. X, we also study the statistical and experimental accessible Kochen–Specker theorem based on an identity operator on the two-dimensional space.

In Sec. XI, we present the main theorem. We are able to use the weak assumption (the identity operator possesses a preexistence value as $+1$) instead of the strong assumption (the existence of the weak hidden variable theory) in order to verify the Kochen–Specker theorem.

Section XII deals with conclusions and discussions in this paper.

II. THE KOCHEN–SPECKER THEOREM

Let us mention the theorem proved [11] by Kochen and Specker along with Redhead's argument [4]. Throughout this paper, we assume von Neumann's projective measurements and we confine ourselves to the finite-dimensional and the discrete spectrum case. Let $\mathbf{R}$ denote the reals where $\pm\infty \notin \mathbf{R}$. We assume every eigenvalue in this paper lies in $\mathbf{R}$. Further, we assume that every Hermitian operator is associated with a unique observable because we do not need to distinguish them in this paper. Namely, we assume the correspondence rule holds:

Definition: (*The corresponding rule*).

There is a 1:1 corresponding between the set of Hermitian operators and the set of observables.

In this paper, we assume the validity of QM and we would like to investigate if the deterministic hidden variable interpretation of it is possible. The problem of hidden variables for QM may be interpreted in a similar fashion as introducing a set of hidden states.

Definition: (*The deterministic hidden variable interpretation of QM* (HV)).

The results of quantum measurement are deterministically related to the hidden state of the system emerging from the source. (It is assumed as the ontological determinism which says that a given hidden state at one time issues in a unique hidden state at a later time. However, it does not imply that the prediction of future hidden states can be effectively computed.)

We shall make the hidden state be mirrored in the interpretation of QM by introducing a classical probability space, which we call the set of hidden states.

Let $\mathcal{O}$ be the space of Hermitian operators described in a finite-dimensional Hilbert space, and $\mathcal{X}$ be the space of density operators described in the Hilbert space. Namely, $\mathcal{X}=\{\psi|\psi\in\mathcal{O}\wedge\psi\geq 0\wedge Tr[\psi]=1\}$. Moreover, let us consider a classical probability space (Ω,Σ,μ_ψ), where Ω is a nonempty sample space, Σ is a σ-algebra of subsets of Ω, and μ_ψ is a σ-additive normalized measure on Σ such that $\mu_\psi(\Omega)=1$. The subscript ψ expresses that the probability measure is determined uniquely when a state ψ is specified.

A quantum state ψ is described by a probability measure μ_ψ on the space Ω, so that, for each measurable subset Γ of Ω, $\mu_\psi(\Gamma)$ is the probability that the system is in a hidden state lying in Γ. The sample space can be reinterpreted as a set of hidden states:

Definition: (*The set of hidden states* (Ω)).

$$\Omega:=\{\omega|\omega:\mathcal{O}\to\mathbf{R}\}. \tag{1}$$

We assume that each Hermitian operator O is associated with a measurable real-valued function $f_O:\Omega\mapsto\mathbf{R}$. Later, we shall define what $f_O(\omega)$ is.

Throughout this paper, and only in this paper, so-called hidden variable theory implies quantum theory which accepts HV. That is,

Definition: (*The hidden variable theory in the KS type of approach*).

$$\text{The hidden variable theory} := \text{QM satisfying HV}. \tag{2}$$

We might define "hidden variable theory" by a different manner when we want to know what theory is possible in order to explain raw experimental data. For example, we already perform various experiments to check if Bell's nonlocal feature [10] appears in experimental data. On this purpose, generally, we do not decide what theory is possible before the experiment is done. The definition (2) is not consistent with motivation of such experiments. However, we assume that our attention is to investigate if QM can be imbedded into deterministic theory supplemented by hidden states. To look for Bell's nonlocal feature in front of raw data is not our aim. On this motivation, we assume that QM holds unconditionally in this paper. We may call this approach the Kochen–Specker (KS) type of approach.

At this stage, we do not yet define how each quantum mechanical expected value is expressed by hidden variable theory. We shall discuss this point in Sec. IV.

If we accept HV, then there should exist a preexistence value possessed by each Hermitian operator, irrespective of measurement. For a Hermitian operator O on a finite-dimensional Hilbert space, let us denote the preexisting value of O in a quantum state ψ and in a hidden state ω by the symbol $\omega(O)$. Define strictly the function f_O using the preexistence value $\omega(O)$ by

$$f_O(\omega):=\omega(O). \tag{3}$$

Now, we mention the principle of faithful measurement for later use (see the proof of theorem (22) in Sec. III):

Definition: (*The principle of faithful measurement* (FM)).

The result of quantum measurement of an operator in a state ψ is numerically equal to the preexistence value possessed by the operator immediately prior to the measurement in the state ψ.

FM implies that the preexisting value should coincide with the stochastic prediction of QM for the state ψ. Now, let us mention the elemental basic of probability. Even though the probability for an event is zero, there is a case where the event occurs. Only the condition $\mu(N)=0$ does not imply $N=\phi$, where μ and N represent a probability measure and an event, respectively. Of course, the converse proposition that $N=\phi$ implies $\mu(N)=0$ is true. Similarly, the condition $\mu(N)=1$ does not imply $N=\Omega$. This implies that FM does not provide any constraint on value assignments for the preexistence values in a hidden state. Thus, in this section, we do not particularly suppose that FM holds. However, FM connects QM to HV. (We will employ FM in Sec. III.) Instead of FM, we assume the value rule holds:

Definition: (*The value rule* (VR)).

If the quantum mechanical probability for λ turning up on measurement of an operator O is zero in a state ψ then λ cannot be the preexisting value $\omega(O)$ in the state ψ.

Unlike FM, VR provides a significant constraint on value assignments for the preexistence values in a hidden state. In this section, we confine our attention only to the case where HV and VR hold. We derive two important rules from VR as follows:

Definition: (*The spectrum rule*).

For a Hermitian operator O with a discrete spectrum, possible preexistence values for $\omega(O)$ are confined to the eigenvalues of the associated operator O.

Theorem.

$$\text{HV} \wedge \text{VR} \Rightarrow \text{Spectrum rule.} \tag{4}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. Assume we accept VR. Then the Born statistical formula (see (18) in Sec. III) says that the quantum mechanical probability for λ turning up on measurement of an operator O is zero in any state ψ if λ is not an eigenvalue of the associated operator O. Therefore, if λ is not an eigenvalue of the associated operator O, λ cannot be the preexisting value $\omega(O)$ in the state ψ. Hence, possible preexistence values for $\omega(O)$ are confined to the eigenvalues of the associated operator O. QED.

Then, the preexisting value of the identity operator for an arbitrary finite-dimensional space necessarily takes one in spite of any state. Another consequence of VR is the eigenvector rule:

Definition: (*The eigenvector rule*).

If the state $|\psi\rangle$ in question is an eigenvector satisfying $O|\psi\rangle = o|\psi\rangle$ where o is some real number known as an eigenvalue of O, then the preexisting value $\omega(O)$ in the state $|\psi\rangle$ is o.

Theorem.

$$\text{HV} \wedge \text{VR} \Rightarrow \text{Eigenvector rule.} \tag{5}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. Assume we accept VR. Suppose that the state $|\psi\rangle$ in question is an eigenvector satisfying $O|\psi\rangle = o|\psi\rangle$ where o is some real number known as an eigenvalue of O. Then the Born statistical formula (see (18) in Sec. III) says that the quantum mechanical probability for λ turning up on measurement of an operator O is zero in the state $|\psi\rangle$ if λ is not the real number o. Therefore, if λ is not the real number o, λ cannot be the preexisting value $\omega(O)$ in the state $|\psi\rangle$. Hence, the possible preexisting value $\omega(O)$ in the state $|\psi\rangle$ is confined to o. QED.

Definition: (*The functional calculus rule*).

Assume that a Hermitian operator A is written by $A = \sum_i a_i P_i$ where $\{P_i\}$ is a basis. Define the Hermitian operator $g(A)$ by

$$g(A) := \sum_i g(a_i) P_i. \tag{6}$$

Here, g stands a function $g : \mathbf{R} \mapsto \mathbf{R}$.

Kochen and Specker introduce the following assumption for the preexisting values:

Definition: (*The functional rule* (FUNC)).

$$\omega(g(O)) = g(\omega(O)) (\Leftrightarrow f_{g(O)}(\omega) = g(f_O(\omega))), \tag{7}$$

for an arbitrary state ψ.

We refer to Eq. (7) as the functional rule (FUNC). This means that the algebraic structure of quantum mechanical operators should be mirrored in the algebraic structure of the preexisting values of the operators. Since $g(O)|\psi\rangle = g(o)|\psi\rangle$, FUNC holds by appealing to the eigenvector rule (see (5)) when the state $|\psi\rangle$ in question is an eigenvector of O. We begin by deriving a result from FUNC.

Definition: (*The product rule* (PROD)).

If Hermitian operators A and B commute, then

$$\omega(AB) = \omega(A) \cdot \omega(B), \tag{8}$$

for an arbitrary state ψ.

Lemma.

$$\text{HV} \wedge \text{FUNC (7)} \Rightarrow \text{HV} \wedge \text{PROD (8).} \tag{9}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. Suppose now that A and B are two commuting Hermitian operators. Since A and B commute they can be diagonalized simultaneously. This means that there exists a basis $\{P_i\}$ by which we can expand $A = \sum_i a_i P_i$, and such that B can also be expanded in the form $B = \sum_i b_i P_i$. Now construct a Hermitian operator $O := \sum_i o_i P_i$ with real values o_i, which are all different. Here O is assumed to be nondegenerate by construction. Let us define functions j and k by $j(o_i) := a_i$ and $k(o_i) := b_i$, respectively. Then we can see that if A and B commute, there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $AB = h(O)$ where $h := j \cdot k$. So we have

$$\begin{aligned} &\omega(AB) = \omega(h(O)) = h(\omega(O)) = j(\omega(O)) \cdot k(\omega(O)) \\ &= \omega(j(O)) \cdot \omega(k(O)) = \omega(A) \cdot \omega(B), \end{aligned} \tag{10}$$

where we use FUNC (7). QED.

A simple version of the KS theorem proves that we *cannot to ascribe values for all Hermitian operators of a set when HV, VR, and PROD (8) hold.* To see this, we review the simplified the KS theorem.

We shall follow the KS theorem proposed by Peres [15] and refined by Mermin [16] for two spin-1/2 systems. Let us try an attempt to assign values to the nine operators $\sigma_x^1\sigma_y^2, \sigma_y^1\sigma_x^2, \sigma_x^1\sigma_x^2, \sigma_y^1\sigma_y^2, \sigma_z^1\sigma_z^2, \sigma_x^1, \sigma_y^1, \sigma_x^2, \sigma_y^2$ for an arbitrary state ψ. Precisely speaking, for example, σ_x^1 means $\sigma_x^1 \otimes I^2$ and so on. Omitting the identity operator on the two-dimensional space, we abbreviate those as above. Let $\omega(O)$ denote the value of an operator O for the state ψ in question. Then, one can see that

$$\begin{aligned} [X]^\psi &:= \omega(\sigma_x^1\sigma_x^2)\omega(\sigma_y^1\sigma_y^2)\omega(\sigma_z^1\sigma_z^2) \\ &= \omega(\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2\sigma_z^1\sigma_z^2) = \omega(-I) = -1, \end{aligned} \tag{11}$$

where I represents the identity operator for the four-dimensional space. By the way we can factorize two of the terms as $\omega(\sigma_x^1\sigma_x^2) = \omega(\sigma_x^1)\omega(\sigma_x^2)$ and $\omega(\sigma_y^1\sigma_y^2) = \omega(\sigma_y^1)\omega(\sigma_y^2)$. Further, we have $\omega(\sigma_x^1\sigma_y^2) = \omega(\sigma_x^1)\omega(\sigma_y^2)$ and $\omega(\sigma_y^1\sigma_x^2) = \omega(\sigma_y^1)\omega(\sigma_x^2)$. Hence we get $\omega(\sigma_x^1\sigma_x^2)\omega(\sigma_y^1\sigma_y^2) = \omega(\sigma_x^1\sigma_y^2)\omega(\sigma_y^1\sigma_x^2)$ and

$$\begin{aligned} [X]^\psi &= \omega(\sigma_x^1\sigma_x^2)\omega(\sigma_y^1\sigma_y^2)\omega(\sigma_z^1\sigma_z^2) \\ &= \omega(\sigma_x^1\sigma_y^2)\omega(\sigma_y^1\sigma_x^2)\omega(\sigma_z^1\sigma_z^2) \\ &= \omega(\sigma_x^1\sigma_y^2\sigma_y^1\sigma_x^2\sigma_z^1\sigma_z^2) = \omega(I) = 1. \end{aligned} \tag{12}$$

Noting the relation (11), we see that an attempt to assign values to the above nine operators for an arbitrary state ψ should fail.

We follow another version of the KS theorem presented in Ref. [16] for three spin-1/2 systems. Let us try an attempt to assign values to the ten operators $\sigma_x^1\sigma_y^2\sigma_y^3, \sigma_y^1\sigma_x^2\sigma_y^3, \sigma_y^1\sigma_y^2\sigma_x^3, \sigma_x^1\sigma_x^2\sigma_x^3, \sigma_x^1, \sigma_y^1, \sigma_x^2, \sigma_y^2, \sigma_x^3, \sigma_y^3$ for an arbitrary state ψ. Let $\omega(O)$ denote the value of an operator O for the state ψ in question. Then, one can see that

$$\begin{aligned} &[Y]^\psi := \omega(\sigma_x^1\sigma_y^2\sigma_y^3)\omega(\sigma_y^1\sigma_x^2\sigma_y^3)\omega(\sigma_y^1\sigma_y^2\sigma_x^3)\omega(\sigma_x^1\sigma_x^2\sigma_x^3) \\ &= \omega(\sigma_x^1\sigma_y^2\sigma_y^3\sigma_y^1\sigma_x^2\sigma_y^3\sigma_y^1\sigma_y^2\sigma_x^3\sigma_x^1\sigma_x^2\sigma_x^3) = \omega(-I) = -1, \end{aligned} \tag{13}$$

where I represents the identity operator for the eight-dimensional space. By the way, we can factorize each of the four terms as

$$\begin{aligned} \omega(\sigma_x^1\sigma_y^2\sigma_y^3) &= \omega(\sigma_x^1)\omega(\sigma_y^2)\omega(\sigma_y^3), \\ \omega(\sigma_y^1\sigma_x^2\sigma_y^3) &= \omega(\sigma_y^1)\omega(\sigma_x^2)\omega(\sigma_y^3), \\ \omega(\sigma_y^1\sigma_y^2\sigma_x^3) &= \omega(\sigma_y^1)\omega(\sigma_y^2)\omega(\sigma_x^3), \\ \omega(\sigma_x^1\sigma_x^2\sigma_x^3) &= \omega(\sigma_x^1)\omega(\sigma_x^2)\omega(\sigma_x^3). \end{aligned} \tag{14}$$

From PROD (8), we have $(\omega(\sigma_k^j))^2 = \omega(I), (j = 1, 2, 3, k = x, y)$. Here, σ_k^1 means $\sigma_k^1 \otimes I^2 \otimes I^3$ and so on. Omitting two identity operators on the two-dimensional space, we abbreviate those as above. Thus, we get

$$\begin{aligned} &[Y]^\psi = \omega(\sigma_x^1\sigma_y^2\sigma_y^3)\omega(\sigma_y^1\sigma_x^2\sigma_y^3)\omega(\sigma_y^1\sigma_y^2\sigma_x^3)\omega(\sigma_x^1\sigma_x^2\sigma_x^3) \\ &= (\omega(\sigma_x^1))^2(\omega(\sigma_y^1))^2(\omega(\sigma_x^2))^2(\omega(\sigma_y^2))^2(\omega(\sigma_x^3))^2(\omega(\sigma_y^3))^2 \\ &= \omega(I)\omega(I)\omega(I)\omega(I)\omega(I)\omega(I) = \omega(I) = 1. \end{aligned} \tag{15}$$

Noting (13), we see that an attempt to assign values to the above ten operators for an arbitrary state ψ should fail.

These two examples provide significantly simplified the KS theorem which says “all versus nothing” demolition of the deterministic hidden variable interpretation of QM (i.e., HV) under the assumption that VR and FUNC (7) hold. Further, they are of the state-independent form. The result provided in this section is a part of the following theorem:

Theorem: (*The Kochen-Specker theorem*).
For every quantum state described in a finite-dimensional Hilbert space ($\mathrm{Dim}(\mathcal{H}) \geq 3$),

$$\mathrm{HV} \wedge \mathrm{VR} \wedge \mathrm{FUNC}\ (7) \Rightarrow \perp, \tag{16}$$

where $\perp$ means Contradiction. That is, these three assumptions do not hold at the same time.

III. PROBABILITY AND TRUTH VALUE

In this section, we review the important difference between probability and truth value. (This section is a review of [4], p.131).
Definition: (*The characteristic function*).
The characteristic function $\chi_\Delta : \mathbf{R} \mapsto \{0, 1\}$ is defined by

$$\chi_\Delta(x) := \begin{cases} 1 & x \in \Delta \\ 0 & x \notin \Delta, \end{cases} \tag{17}$$

where Δ is any subset of the reals $\mathbf{R}$.
Now we introduce the notation θ which represents one result of quantum measurement. Suppose that the measurement of a Hermitian operator A for a system in the state ψ yields a value $\theta(A) \in \mathbf{R}$.
Using the characteristic function χ_Δ and θ, the Born statistical formula is expressed as follows:
Definition: (*The Born statistical formula* (BSF)).

$$Prob(\Delta)^\psi_{\theta(A)} := Tr[\psi\chi_\Delta(A)]. \tag{18}$$

The whole symbol $(\Delta)^\psi_{\theta(A)}$ is used to denote the proposition that the value of $\theta(A)$ lies in the set Δ in the quantum state ψ. And $Prob$ denotes the probability that the proposition holds.
We define the inverse image of a set B with respect to a function g.
Definition: (*The inverse image*).

$$g^{-1}(B) := \{x | g(x) \in B\}. \tag{19}$$

Definition: (*The statistical functional rule* (STAT FUNC)).

$$Prob(\Delta)^\psi_{\omega(g(A))} = Prob(g^{-1}(\Delta))^\psi_{\omega(A)}. \tag{20}$$

Lemma.

$$\chi_\Delta(g(x)) = \chi_{g^{-1}(\Delta)}(x), (x \in \mathbf{R}). \tag{21}$$

Proof. If $g(x) \in \Delta$, LHS=1 and then since $x \in g^{-1}(\Delta)$, RHS=1. On the other hand, if $g(x) \notin \Delta$, LHS=0 and then since $x \notin g^{-1}(\Delta)$, RHS=0. So, for all values of x, the two functions on the left and right hand side of Eq. (21) have the same value. Hence the two functions are equal. QED.
Theorem.

$$\mathrm{HV} \wedge \mathrm{FM} \Rightarrow \mathrm{STAT\ FUNC}\ (20). \tag{22}$$

Proof. By BSF (18), we have

$$\begin{aligned} &Prob(\Delta)^\psi_{\theta(g(A))} = Tr[\psi\chi_\Delta(g(A))] \\ &= Tr[\psi\chi_{g^{-1}(\Delta)}(A)] = Prob(g^{-1}(\Delta))^\psi_{\theta(A)}. \end{aligned} \tag{23}$$

Let us assume HV and FM (see Sec. II) hold. Suppose that the system is in a hidden state ω. Then we can substitute $\omega(A)$ for $\theta(A)$ and we obtain STAT FUNC (20). QED.
STAT FUNC is a theorem of QM under the assumption that HV and FM hold. We begin by considering the relation between FUNC (7) and STAT FUNC (20) under the assumption that HV and FM hold.
Let us reinterpret STAT FUNC as a statement about long-run frequencies for preexistence values of a Hermitian operator to lie in the indicated ranges in the given quantum state ψ. We cannot replace (20) by

$$Val(\Delta)^\psi_{\omega(g(A))} = Val(g^{-1}(\Delta))^\psi_{\omega(A)}, \tag{24}$$

where Val denotes the truth valuation ascribed to propositions such as $(\Delta)^{\psi}_{\omega(A)}$.

Formally, $Val : \{\mathbf{P}\} \mapsto \{0,1\}$ is a function that maps the set of propositions of the form $(\Delta)^{\psi}_{\omega(A)}$ onto the two-element set of truth values, i.e., True and False. The function Val should be contrasted with the function $Prob : \{\mathbf{P}\} \mapsto [0,1]$, which maps the set of propositions $\{\mathbf{P}\}$ onto real numbers in the interval [0,1].

The relation (24) says $\omega(A)$ is a value in $g^{-1}(\Delta)$ if and only if $\omega(g(A))$ is a value in Δ. So the long-run frequencies, and hence the probabilities, must match because the outcomes match on each occasion. It is clear, however, that the converse implication does not hold. We cannot deduce (24) from (20). The match in long-run frequencies specified by (20) in no way guarantees a match in outcome on each particular occasion. On the other hand, (24) is equivalent to FUNC (7).

Theorem.

$$\mathrm{HV} \wedge (24) \Leftrightarrow \mathrm{HV} \wedge \mathrm{FUNC}\ (7). \tag{25}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. We first notice that FUNC implies (24).

$$\omega(g(A)) \in \Delta \Leftrightarrow g(\omega(A)) \in \Delta \Leftrightarrow \omega(A) \in g^{-1}(\Delta), \tag{26}$$

which is another way of expressing the content of (24). Conversely, (24) implies FUNC. If (24) holds, we have

$$\begin{aligned} &\omega(A) \in \Delta \Rightarrow \omega(A) \in g^{-1}(g(\Delta)) \\ &\Leftrightarrow \omega(g(A)) \in g(\Delta), \end{aligned} \tag{27}$$

where we have used (24) taking $g(\Delta)$ as Δ. Now take $\Delta = \{\lambda\}$, which is a singleton set. Then we have

$$\omega(A) \in \{\lambda\} \Leftrightarrow \omega(A) = \lambda, \tag{28}$$

and, from $g(\{\lambda\}) = \{g(x) | x \in \{\lambda\}\} = \{g(\lambda)\}$ and (27),

$$\begin{aligned} &\omega(g(A)) \in g(\{\lambda\}) (= \{g(\omega(A))\}) \\ &\Leftrightarrow \omega(g(A)) = g(\omega(A)), \end{aligned} \tag{29}$$

which is FUNC. QED.

Now we summarize the inclusion relation provided in this section as follows:

$$\text{STAT FUNC } (20) \Leftarrow (24) \Leftrightarrow \text{FUNC } (7). \tag{30}$$

Therefore, even though we get the refutation of the validity of FUNC (7), it does not imply the refutation of STAT FUNC (20).

Roughly speaking, the problem is as follows: STAT FUNC can be consistent with the deterministic hidden variable interpretation of QM (i.e., HV). However, if we introduce an additional assumption (i.e., JD (40) holds), such an interpretation is not possible. Let us write STAT FUNC (20), simply, as $P(U) = P(V)$ where $U := (\Delta)^{\psi}_{\omega(g(A))}$ and $V := (g^{-1}(\Delta))^{\psi}_{\omega(A)}$. The assumption is related to the joint probability that the two propositions U and V hold simultaneously, i.e., $P(U \cap V)$. If we want to interpret QM by HV, we cannot assume that

$$P(U) = P(V) = P(U \cap V). \tag{31}$$

This cannot be consistent with QM. However we can derive (31) from HV under the condition of JD (40). (It follows from theorem (45) that $P(\overline{U} \cap V) = P(U \cap \overline{V}) = 0$ which is equivalent to (31). Here, the upper bar is for complements.) We shall review this argument in Sec. IV and Sec. V.

IV. RELATIONS BETWEEN VARIOUS RULES

In this section, we review the mutual relation between several assumptions about hidden variable theory along with Fine [17–19].

Definition: (*Almost everywhere*).

Let (Ω, Σ, μ) be a measure space and $P(\omega)$ be a proposition related to $\omega \in \Omega$. If there exists N such that

$$\begin{aligned} &\{\omega | P(\omega) \text{ does not hold.}\} \subset N \wedge N \in \Sigma \\ &\wedge \mu(N) = 0, \end{aligned} \tag{32}$$

then we say that $P(\omega)$ holds almost everywhere with respect to μ in Ω. Then we abbreviate this as follows:

$$P(\omega), (\mu - a.e.). \tag{33}$$

In other words, (33) implies that $P(\omega)$ must hold except all measurable subsets of Ω that have the probability measure of zero.

Definition: (*The probability distribution rule* (D)).

$$\mu_\psi(f_A^{-1}(\Delta)) := Prob(\Delta)^\psi_{\theta(A)}. \tag{34}$$

We call hidden variable theory satisfying D (34) weak hidden variable theory. Form this definition (34), it is required that the value of $\omega(I)$ is $+1$ almost everywhere, i.e., $f_I(\omega) = +1, (\mu_\psi - a.e.)$. Here, I represents the identity operator for an arbitrary finite-dimensional space.

Let S_A stand for the spectrum of the Hermitian operator A.

Theorem. If

$$\begin{aligned} Tr[\psi A] &= \sum_{y\in S_A} Prob(\{y\})^\psi_{\theta(A)} y, \\ E_\psi(A) &:= \int_{\omega\in\Omega} \mu_\psi(d\omega) f_A(\omega), \end{aligned} \tag{35}$$

then

$$\text{HV} \wedge \text{D (34)} \Rightarrow Tr[\psi A] = E_\psi(A). \tag{36}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. Note

$$\begin{aligned} &\omega \in f_A^{-1}(\{y\}) \Leftrightarrow f_A(\omega) \in \{y\} \Leftrightarrow y = f_A(\omega), \\ &\int_{\omega\in f_A^{-1}(\{y\})} \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} = 1, \\ &y \neq y' \Rightarrow f_A^{-1}(\{y\}) \cap f_A^{-1}(\{y'\}) = \phi. \end{aligned} \tag{37}$$

Hence we have

$$\begin{aligned} Tr[\psi A] &= \sum_{y\in S_A} Prob(\{y\})^\psi_{\theta(A)} y \\ &= \sum_{y\in \mathbf{R}} Prob(\{y\})^\psi_{\theta(A)} y \\ &= \sum_{y\in \mathbf{R}} \mu_\psi(f_A^{-1}(\{y\})) y \\ &= \sum_{y\in \mathbf{R}} \mu_\psi(f_A^{-1}(\{y\})) y \\ &\times \int_{\omega\in f_A^{-1}(\{y\})} \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} \\ &= \sum_{y\in \mathbf{R}} \int_{\omega\in f_A^{-1}(\{y\})} \mu_\psi(f_A^{-1}(\{y\})) \\ &\times \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} f_A(\omega) \\ &= \int_{\omega\in\Omega} \mu_\psi(d\omega) f_A(\omega) = E_\psi(A). \end{aligned} \tag{38}$$

QED.

Definition: (*The quantum mechanical joint probability distribution for commuting observables* (QJD)).

$$Prob(\Delta, \Delta')^\psi_{\theta(A),\theta(B)} := Tr[\psi \chi_\Delta(A) \chi_{\Delta'}(B)], \tag{39}$$

where the notation on the LHS of (39) is a generalization of the symbol $(\Delta)^{\psi}_{\theta(A)}$ to express the proposition that measurement results of A and B will lie in the sets Δ and Δ', respectively. We shall use Eq. (39) to define the quantum-mechanical joint probability distribution for commuting observables.

Definition: (*The joint probability distribution rule* (JD)).

$$\mu_{\psi}(f_A^{-1}(\Delta)\cap f_B^{-1}(\Delta')) := Prob(\Delta,\Delta')^{\psi}_{\theta(A),\theta(B)}, \tag{40}$$

for all commuting pairs A, B in $\mathcal{O}$.

We shall use Eq. (40) to define the hidden variable theoretical joint probability distribution for commuting observables.

Theorem.

$$\begin{aligned}&\text{QJD (39)} \Rightarrow \text{BSF (18)},\\ &\text{HV} \wedge \text{JD (40)} \Rightarrow \text{HV} \wedge \text{D (34)}.\end{aligned} \tag{41}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. Let Δ' be $\mathbf{R}$ in QJD (39) and JD (40). Then we have

$$\begin{aligned}&\mu_{\psi}(f_A^{-1}(\Delta)\cap f_B^{-1}(\mathbf{R})) = Prob(\Delta,\mathbf{R})^{\psi}_{\theta(A),\theta(B)}\\ &= Tr[\psi\chi_{\Delta}(A)\chi_{\mathbf{R}}(B)]\\ &\Leftrightarrow \mu_{\psi}(f_A^{-1}(\Delta)) = Prob(\Delta)^{\psi}_{\theta(A)} = Tr[\psi\chi_{\Delta}(A)].\end{aligned} \tag{42}$$

QED.

Definition: (*The functional rule holding almost everywhere* (FUNC A.E.)).

$$f_{g(A)}(\omega) = g(f_A(\omega)), (\mu_{\psi} - a.e.). \tag{43}$$

That is

$$\mu_{\psi}(\{\omega | f_{g(A)}(\omega) \neq g(f_A(\omega))\}) = 0. \tag{44}$$

Theorem.

$$\text{HV} \wedge \text{JD (40)} \Rightarrow \text{HV} \wedge \text{D (34)} \wedge \text{FUNC A.E. (43)}. \tag{45}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. Suppose we have JD (40) for all commuting pairs A, B in $\mathcal{O}$. Let y be any real number, and let $S := \{\omega | f_{g(A)}(\omega) = y\}$ and $T := \{\omega | g(f_A(\omega)) = y\}$. We want $\mu_{\psi}(\overline{S}\cap T) = \mu_{\psi}(S\cap\overline{T}) = 0$. This will follow if we have $\mu_{\psi}(S) = \mu_{\psi}(T) = \mu_{\psi}(S\cap T)$ since

$$\begin{aligned}&\mu_{\psi}(S\cap\overline{T}) + \mu_{\psi}(S\cap T) = \mu_{\psi}(S),\\ &\mu_{\psi}(\overline{S}\cap T) + \mu_{\psi}(S\cap T) = \mu_{\psi}(T).\end{aligned} \tag{46}$$

Note

$$\omega \in f_{g(A)}^{-1}(\{y\}) \Leftrightarrow f_{g(A)}(\omega) \in \{y\} \Leftrightarrow y = f_{g(A)}(\omega), \tag{47}$$

and

$$\begin{aligned}&\omega \in f_A^{-1}(g^{-1}(\{y\})) \Leftrightarrow f_A(\omega) \in g^{-1}(\{y\})\\ &\Leftrightarrow g(f_A(\omega)) \in \{y\} \Leftrightarrow y = g(f_A(\omega)).\end{aligned} \tag{48}$$

Theorem (41) says that JD (40) yields D (34). Then, from (23), we have

$$\begin{aligned}&\mu_{\psi}(T) = \mu_{\psi}(\{\omega | \omega \in f_A^{-1}(g^{-1}(\{y\}))\})\\ &= Prob(g^{-1}(\{y\}))^{\psi}_{\theta(A)} = Prob(\{y\})^{\psi}_{\theta(g(A))}\\ &= \mu_{\psi}(\{\omega | \omega \in f_{g(A)}^{-1}(\{y\})\}) = \mu_{\psi}(S).\end{aligned} \tag{49}$$

Using the spectral representation of A, it follows that $\chi_{\Delta}(A)\chi_{g(\Delta)}(g(A)) = \chi_{\Delta}(A)$ for any set Δ, where $g(\Delta) = \{g(x) | x \in \Delta\}$. Because, $\chi_{\Delta}(z) = 1 \Leftrightarrow z \in \Delta \Rightarrow g(z) \in g(\Delta) \Leftrightarrow \chi_{g(\Delta)}(g(z)) = 1$ holds ($z \in \mathbf{R}$). Hence,

$$\begin{aligned}&Prob(\Delta, g(\Delta))^{\psi}_{\theta(A),\theta(g(A))}\\ &= Tr[\psi\chi_{\Delta}(A)\chi_{g(\Delta)}(g(A))] = Tr[\psi\chi_{\Delta}(A)]\\ &= Prob(\Delta)^{\psi}_{\theta(A)}.\end{aligned} \tag{50}$$

On the other hand, we have $g(g^{-1}(\Delta)) = \Delta$ because $g(g^{-1}(\Delta)) = \{g(x)|x \in g^{-1}(\Delta)\} = \{g(x)|g(x) \in \Delta\} = \Delta$. Therefore, taking $g^{-1}(\{y\})$ as Δ, we have

$$\begin{aligned}&Prob(g^{-1}(\{y\}),\{y\})^{\psi}_{\theta(A),\theta(g(A))}\\&= Prob(g^{-1}(\{y\}))^{\psi}_{\theta(A)} = \mu_{\psi}(T).\end{aligned} \tag{51}$$

But, from JD (40) we have

$$\begin{aligned}&Prob(g^{-1}(\{y\}),\{y\})^{\psi}_{\theta(A),\theta(g(A))}\\&= \mu_{\psi}(f_A^{-1}(g^{-1}(\{y\})) \cap f_{g(A)}^{-1}(\{y\}))\\&= \mu_{\psi}(T \cap S).\end{aligned} \tag{52}$$

QED.

Definition: (*The product rule holding almost everywhere* (PROD A.E.)).

If Hermitian operators A and B commute, then

$$f_{AB}(\omega) = f_A(\omega) \cdot f_B(\omega), (\mu_{\psi} - a.e.). \tag{53}$$

Theorem.

$$\text{HV} \wedge \text{FUNC A.E. (43)} \Rightarrow \text{HV} \wedge \text{PROD A.E. (53)}. \tag{54}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. Suppose now that A and B are two commuting Hermitian operators. In Sec. II, we discuss that we can define functions j and k and there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $AB = h(O)$ where $h := j \cdot k$. So we have the following:

$$\begin{aligned}&f_{AB}(\omega) = f_{h(O)}(\omega) = h(f_O(\omega)) = j(f_O(\omega)) \cdot k(f_O(\omega))\\&= f_{j(O)}(\omega) \cdot f_{k(O)}(\omega) = f_A(\omega) \cdot f_B(\omega), (\mu_{\psi} - a.e.),\end{aligned} \tag{55}$$

where FUNC A.E. (43) is used. QED.

Lemma. If

$$\begin{aligned}&\mu_{\psi}(\overline{S} \cap S') = \mu_{\psi}(\overline{S'} \cap S)\\&= \mu_{\psi}(\overline{T} \cap T') = \mu_{\psi}(\overline{T'} \cap T) = 0,\end{aligned} \tag{56}$$

then

$$\mu_{\psi}(S \cap T) = \mu_{\psi}(S' \cap T'). \tag{57}$$

Proof. Note

$$\begin{aligned}&\mu_{\psi}(\overline{S} \cap S' \cap T) + \mu_{\psi}(S \cap S' \cap T) = \mu_{\psi}(S' \cap T),\\&\mu_{\psi}(\overline{S'} \cap S \cap T) + \mu_{\psi}(S \cap S' \cap T) = \mu_{\psi}(S \cap T).\end{aligned} \tag{58}$$

If the following relation holds

$$\mu_{\psi}(\overline{S} \cap S') = \mu_{\psi}(\overline{S'} \cap S) = 0, \tag{59}$$

then

$$\mu_{\psi}(\overline{S} \cap S' \cap T) = \mu_{\psi}(\overline{S'} \cap S \cap T) = 0. \tag{60}$$

Therefore, from (58), we have

$$\mu_{\psi}(S \cap S' \cap T) = \mu_{\psi}(S' \cap T) = \mu_{\psi}(S \cap T). \tag{61}$$

Similar to the argument by changing S to T, S' to T', and T to S', we get

$$\mu_{\psi}(T \cap T' \cap S') = \mu_{\psi}(T' \cap S') = \mu_{\psi}(T \cap S'). \tag{62}$$

From the relations (61) and (62), we conclude

$$\mu_{\psi}(T \cap S) = \mu_{\psi}(T' \cap S'). \tag{63}$$

QED.

Lemma.

$$\begin{aligned}&\text{HV} \land \text{PROD A.E. (53)}\\ &\Rightarrow f_{\chi_\Delta(A)}(\omega) \in \{0,1\}, (\mu_\psi - a.e.).\end{aligned} \tag{64}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. By PROD A.E. (53), we have

$$\begin{aligned}&f_{\chi_\Delta(A)}(\omega) \cdot f_{\chi_\Delta(A)}(\omega) = f_{\chi_\Delta(A)}(\omega), (\mu_\psi - a.e.)\\ \Leftrightarrow\ &f_{\chi_\Delta(A)}(\omega) \in \{0,1\}, (\mu_\psi - a.e.).\end{aligned} \tag{65}$$

QED.

Theorem.

$$\begin{aligned}&\text{HV} \land \text{D (34)} \land \text{PROD A.E. (53)}\\ &\Rightarrow \text{HV} \land \text{JD (40)}\end{aligned} \tag{66}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. Suppose $[A, B] = \mathbf{0}$ holds. It follows from QJD (39), BSF (18), and D (34) that

$$\begin{aligned}&Prob(\Delta, \Delta')^\psi_{\theta(A),\theta(B)}\\ &= Tr[\psi\chi_\Delta(A)\chi_{\Delta'}(B)](see(39))\\ &= Tr[\psi\chi_{\{1\}}(\chi_\Delta(A)\chi_{\Delta'}(B))]\\ &= Prob(\{1\})^\psi_{\theta(\chi_\Delta(A)\chi_{\Delta'}(B))}(see(18))\\ &= \mu_\psi(f^{-1}_{\chi_\Delta(A)\chi_{\Delta'}(B)}(\{1\}))(see(34)).\end{aligned} \tag{67}$$

PROD A.E. (53) and the lemma (64) say that

$$\begin{aligned}&(67) = \mu_\psi(\{\omega|\omega \in f^{-1}_{\chi_\Delta(A)\chi_{\Delta'}(B)}(\{1\})\})\\ &= \mu_\psi(\{\omega|f_{\chi_\Delta(A)\chi_{\Delta'}(B)}(\omega) = 1\})\\ &= \mu_\psi(\{\omega|f_{\chi_\Delta(A)}(\omega) \cdot f_{\chi_{\Delta'}(B)}(\omega) = 1\})(see(53))\\ &= \mu_\psi(\{\omega|f_{\chi_\Delta(A)}(\omega) = f_{\chi_{\Delta'}(B)}(\omega) = 1\})(see(64))\\ &= \mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\}) \cap f^{-1}_{\chi_{\Delta'}(B)}(\{1\})).\end{aligned} \tag{68}$$

On the other hand, we have

$$\begin{aligned}&\mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\}) \cap f^{-1}_A(\Delta))\\ &= \mu_\psi(\{\omega|f_{\chi_\Delta(A)}(\omega) = 1 \land f_A(\omega) \in \Delta\})\\ &= \mu_\psi(\{\omega|f_{\chi_\Delta(A)}(\omega) \cdot f_A(\omega) \in \Delta\})\\ &= \mu_\psi(\{\omega|f_{\chi_\Delta(A)\cdot A}(\omega) \in \Delta\})(see(53))\\ &= \mu_\psi(f^{-1}_{\chi_\Delta(A)\cdot A}(\Delta))\\ &= Prob(\Delta)^\psi_{\theta(\chi_\Delta(A)\cdot A)}(see(34))\\ &= Tr[\psi\chi_\Delta(\chi_\Delta(A) \cdot A)](see(18))\\ &= Tr[\psi\chi_\Delta(A)].\end{aligned} \tag{69}$$

We also obtain

$$\begin{aligned}&\mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\})) = Tr[\psi\chi_{\{1\}}(\chi_\Delta(A))]\\ &= Tr[\psi\chi_\Delta(A)] = \mu_\psi(f^{-1}_A(\Delta)).\end{aligned} \tag{70}$$

Note, (see (46))

$$\begin{aligned}&\mu_\psi(S \cap T) = \mu_\psi(S) = \mu_\psi(T)\\ &\Leftrightarrow \mu_\psi(S \cap \overline{T}) = \mu_\psi(\overline{S} \cap T) = 0.\end{aligned} \tag{71}$$

Therefore, from Eq. (69) and Eq. (70), we have

$$\begin{aligned}&\mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\})\cap\overline{f^{-1}_A(\Delta)})\\&=\mu_\psi(\overline{f^{-1}_{\chi_\Delta(A)}(\{1\})}\cap f^{-1}_A(\Delta))=0.\end{aligned}\tag{72}$$

Similarly we can get

$$\begin{aligned}&\mu_\psi(f^{-1}_{\chi_{\Delta'}(B)}(\{1\})\cap f^{-1}_B(\Delta'))=Tr[\psi\chi_{\Delta'}(B)],\\&\mu_\psi(f^{-1}_{\chi_{\Delta'}(B)}(\{1\}))=\mu_\psi(f^{-1}_B(\Delta'))=Tr[\psi\chi_{\Delta'}(B)],\end{aligned}\tag{73}$$

and we have

$$\begin{aligned}&\mu_\psi(f^{-1}_{\chi_{\Delta'}(B)}(\{1\})\cap\overline{f^{-1}_B(\Delta')})\\&=\mu_\psi(\overline{f^{-1}_{\chi_{\Delta'}(B)}(\{1\})}\cap f^{-1}_B(\Delta'))=0.\end{aligned}\tag{74}$$

Hence, from the lemma (57), we have

$$\begin{aligned}&\mu_\psi(f^{-1}_{\chi_\Delta(A)}(\{1\})\cap f^{-1}_{\chi_{\Delta'}(B)}(\{1\}))\\&=\mu_\psi(f^{-1}_A(\Delta)\cap f^{-1}_B(\Delta')).\end{aligned}\tag{75}$$

Therefore, from (68), we conclude

$$\begin{aligned}&Prob(\Delta,\Delta')^{\psi}_{\theta(A),\theta(B)}\\&=\mu_\psi(f^{-1}_A(\Delta)\cap f^{-1}_B(\Delta')),\end{aligned}\tag{76}$$

which is JD (40). QED.

Now we summarize the inclusion relation provided in this section as follows:

$$\begin{aligned}&\mathrm{HV}\wedge\mathrm{JD}\ (40)\\&\Leftrightarrow\mathrm{HV}\wedge\mathrm{D}\ (34)\wedge\mathrm{FUNC\ A.E.}\ (43)\\&\Leftrightarrow\mathrm{HV}\wedge\mathrm{D}\ (34)\wedge\mathrm{PROD\ A.E.}\ (53).\end{aligned}\tag{77}$$

In what follows, we present two corollaries:

Corollary.

$$\begin{aligned}&\mathrm{HV}\wedge\mathrm{FUNC\ A.E.}\ (43)\Rightarrow\\&\int_{\omega\in\Omega}\mu_\psi(d\omega)f_{g(A)}(\omega)\cdot K(\omega)\\&=\int_{\omega\in\Omega}\mu_\psi(d\omega)g(f_A(\omega))\cdot K(\omega),\end{aligned}\tag{78}$$

for an arbitrary measurable function $K:\omega\mapsto\mathbf{R}$ and an arbitrary state ψ.

Corollary.

$$\begin{aligned}&\mathrm{HV}\wedge\mathrm{PROD\ A.E.}\ (53)\wedge[A,B]=\mathbf{0}\Rightarrow\\&\int_{\omega\in\Omega}\mu_\psi(d\omega)f_{AB}(\omega)\cdot K(\omega)\\&=\int_{\omega\in\Omega}\mu_\psi(d\omega)f_A(\omega)\cdot f_B(\omega)\cdot K(\omega),\end{aligned}\tag{79}$$

for an arbitrary measurable function $K:\omega\mapsto\mathbf{R}$ and an arbitrary state ψ.

In the following, we review the derivation of JD (40) from FUNC A.E. (43).

Theorem.

$$\begin{aligned}&\mathrm{HV}\wedge\mathrm{D}\ (34)\wedge\mathrm{FUNC\ A.E.}\ (43)\\&\Rightarrow\mathrm{HV}\wedge\mathrm{JD}\ (40)\end{aligned}\tag{80}$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each Hermitian operator. Suppose $[A, B] = \mathbf{0}$ holds. It follows from BSF (18), QJD (39), D (34), FUNC A.E. (43), and PROD A.E. (53) that

$$
\begin{aligned}
& Prob(\Delta, \Delta')^{\psi}_{\theta(A),\theta(B)} \\
&= Tr[\psi \chi_{\Delta}(A)\chi_{\Delta'}(B)](see(39)) \\
&= Tr[\psi \chi_{\{1\}}(\chi_{\Delta}(A)\chi_{\Delta'}(B))] \\
&= Prob(\{1\})^{\psi}_{\theta(\chi_{\Delta}(A)\chi_{\Delta'}(B))}(see(18)) \\
&= \mu_{\psi}(f^{-1}_{\chi_{\Delta}(A)\chi_{\Delta'}(B)}(\{1\}))(see(34)) \\
&= \mu_{\psi}(\{\omega|\omega \in f^{-1}_{\chi_{\Delta}(A)\chi_{\Delta'}(B)}(\{1\})\}) \\
&= \mu_{\psi}(\{\omega|f_{\chi_{\Delta}(A)\chi_{\Delta'}(B)}(\omega) = 1\}) \\
&= \mu_{\psi}(\{\omega|f_{\chi_{\Delta}(A)}(\omega) \cdot f_{\chi_{\Delta'}(B)}(\omega) = 1\})(see(53)) \\
&= \mu_{\psi}(\{\omega|\chi_{\Delta}(f_A(\omega)) \cdot \chi_{\Delta'}(f_B(\omega)) = 1\})(see(43)) \\
&= \mu_{\psi}(\{\omega|\chi_{\Delta}(f_A(\omega)) = \chi_{\Delta'}(f_B(\omega)) = 1\}) \\
&= \mu_{\psi}(\{\omega|f_A(\omega) \in \Delta \wedge f_B(\omega) \in \Delta'\}) \\
&= \mu_{\psi}(f^{-1}_A(\Delta) \cap f^{-1}_B(\Delta')).
\end{aligned} \tag{81}
$$

QED.

V. THE STATISTICAL KOCHEN-SPECKER THEOREM

We want to modify the KS argument provided in Sec. II so as to relax the assumptions. In fact, VR and FUNC (7) are relaxed into D (34) and FUNC A.E. (43), respectively. In what follows, we assume HV and JD (40) hold. This implies that we can use D (34), FUNC A.E. (43), and PROD A.E. (53).

We shall modify the KS theorem presented in Sec. II for two spin-1/2 systems. Let us try an attempt to assign functions $f : \Omega \mapsto \mathbf{R}$ to the nine operators $\sigma^1_x\sigma^2_y, \sigma^1_y\sigma^2_x, \sigma^1_x\sigma^2_x, \sigma^1_y\sigma^2_y, \sigma^1_z\sigma^2_z, \sigma^1_x, \sigma^1_y, \sigma^2_x, \sigma^2_y$ for an arbitrary state ψ. Then, one can see that

$$
\begin{aligned}
X(\omega) &:= f_{\sigma^1_x\sigma^2_x}(\omega) f_{\sigma^1_y\sigma^2_y}(\omega) f_{\sigma^1_z\sigma^2_z}(\omega) \\
&= f_{\sigma^1_x\sigma^2_x\sigma^1_y\sigma^2_y\sigma^1_z\sigma^2_z}(\omega) = f_{-I}(\omega), (\mu_{\psi} - a.e.) \\
&\Rightarrow \int_{\omega\in\Omega} \mu_{\psi}(d\omega) X(\omega) = E_{\psi}(-I) = -1,
\end{aligned} \tag{82}
$$

where I represents the identity operator for the four-dimensional space. By the way we can factorize two of the terms as $f_{\sigma^1_x\sigma^2_x} = f_{\sigma^1_x} f_{\sigma^2_x}$ and $f_{\sigma^1_y\sigma^2_y} = f_{\sigma^1_y} f_{\sigma^2_y}$ almost everywhere with respect to μ_{ψ}. Further, we have $f_{\sigma^1_x\sigma^2_y} = f_{\sigma^1_x} f_{\sigma^2_y}$ and $f_{\sigma^1_y\sigma^2_x} = f_{\sigma^1_y} f_{\sigma^2_x}$ almost everywhere with respect to μ_{ψ}. Hence we get $f_{\sigma^1_x\sigma^2_x} f_{\sigma^1_y\sigma^2_y} = f_{\sigma^1_x\sigma^2_y} f_{\sigma^1_y\sigma^2_x}$ almost everywhere with respect to μ_{ψ} and

$$
\begin{aligned}
X(\omega) &= f_{\sigma^1_x\sigma^2_x}(\omega) f_{\sigma^1_y\sigma^2_y}(\omega) f_{\sigma^1_z\sigma^2_z}(\omega) \\
&= f_{\sigma^1_x\sigma^2_y}(\omega) f_{\sigma^1_y\sigma^2_x}(\omega) f_{\sigma^1_z\sigma^2_z}(\omega) \\
&= f_{\sigma^1_x\sigma^2_y\sigma^1_y\sigma^2_x\sigma^1_z\sigma^2_z}(\omega) = f_{I}(\omega), (\mu_{\psi} - a.e.) \\
&\Rightarrow \int_{\omega\in\Omega} \mu_{\psi}(d\omega) X(\omega) = E_{\psi}(I) = 1.
\end{aligned} \tag{83}
$$

Noting the relation (82), we see that an attempt to assign functions to the above nine operators for an arbitrary state ψ should fail.

We modify another version of the KS theorem presented in Sec. II for three spin-1/2 systems. Let us try an attempt to assign functions to the ten operators $\sigma^1_x\sigma^2_y\sigma^3_y, \sigma^1_y\sigma^2_x\sigma^3_y, \sigma^1_y\sigma^2_y\sigma^3_x, \sigma^1_x\sigma^2_x\sigma^3_x, \sigma^1_x, \sigma^1_y, \sigma^2_x, \sigma^2_y, \sigma^3_x, \sigma^3_y$ for an arbitrary state ψ. Then, one can see that

$$
\begin{aligned}
Y(\omega) &:= f_{\sigma^1_x\sigma^2_y\sigma^3_y}(\omega) f_{\sigma^1_y\sigma^2_x\sigma^3_y}(\omega) f_{\sigma^1_y\sigma^2_y\sigma^3_x}(\omega) f_{\sigma^1_x\sigma^2_x\sigma^3_x}(\omega) \\
&= f_{\sigma^1_x\sigma^2_y\sigma^3_y\sigma^1_y\sigma^2_x\sigma^3_y\sigma^1_y\sigma^2_y\sigma^3_x\sigma^1_x\sigma^2_x\sigma^3_x}(\omega) \\
&= f_{-I}(\omega), (\mu_{\psi} - a.e.) \\
&\Rightarrow \int_{\omega\in\Omega} \mu_{\psi}(d\omega) Y(\omega) = E_{\psi}(-I) = -1,
\end{aligned} \tag{84}
$$

where I represents the identity operator for the eight-dimensional space. By the way, we can factorize each of the four terms almost everywhere with respect to μ_ψ as

$$\begin{aligned}
&f_{\sigma_x^1\sigma_y^2\sigma_y^3}(\omega) = f_{\sigma_x^1}(\omega)f_{\sigma_y^2}(\omega)f_{\sigma_y^3}(\omega),\\
&f_{\sigma_y^1\sigma_x^2\sigma_y^3}(\omega) = f_{\sigma_y^1}(\omega)f_{\sigma_x^2}(\omega)f_{\sigma_y^3}(\omega),\\
&f_{\sigma_y^1\sigma_y^2\sigma_x^3}(\omega) = f_{\sigma_y^1}(\omega)f_{\sigma_y^2}(\omega)f_{\sigma_x^3}(\omega),\\
&f_{\sigma_x^1\sigma_x^2\sigma_x^3}(\omega) = f_{\sigma_x^1}(\omega)f_{\sigma_x^2}(\omega)f_{\sigma_x^3}(\omega), (\mu_\psi - a.e.).
\end{aligned} \tag{85}$$

From PROD A.E. (53), we have $(f_{\sigma_k^j}(\omega))^2 = f_I(\omega), (\mu_\psi - a.e.), (j = 1, 2, 3, k = x, y)$. Here, σ_k^1 means $\sigma_k^1 \otimes I^2 \otimes I^3$ and so on. Omitting two identity operators on the two-dimensional space, we abbreviate those as above. Thus, we get

$$\begin{aligned}
Y(\omega) &= f_{\sigma_x^1\sigma_y^2\sigma_y^3}(\omega)f_{\sigma_y^1\sigma_x^2\sigma_y^3}(\omega)f_{\sigma_y^1\sigma_y^2\sigma_x^3}(\omega)f_{\sigma_x^1\sigma_x^2\sigma_x^3}(\omega)\\
&= (f_{\sigma_x^1}(\omega))^2(f_{\sigma_y^1}(\omega))^2(f_{\sigma_x^2}(\omega))^2(f_{\sigma_y^2}(\omega))^2\\
&\quad\times(f_{\sigma_x^3}(\omega))^2(f_{\sigma_y^3}(\omega))^2\\
&= f_I(\omega)f_I(\omega)f_I(\omega)f_I(\omega)\\
&\quad\times f_I(\omega)f_I(\omega) = f_I(\omega), (\mu_\psi - a.e.)\\
&\Rightarrow \int_{\omega\in\Omega} \mu_\psi(d\omega)Y(\omega) = E_\psi(I) = 1.
\end{aligned} \tag{86}$$

Noting (84), we see that an attempt to assign functions to the above ten operators for an arbitrary state ψ should fail.

These two examples provide a statistical form of the KS theorem which says demolition of either the existence of hidden states or the validity of JD (40). That is, there does not exist any hidden variable theory satisfying JD (40). Noting,

$$\begin{aligned}
&\text{HV} \wedge \text{JD (40)}\\
&\Leftrightarrow \text{HV} \wedge \text{D (34)} \wedge \text{FUNC A.E. (43)},
\end{aligned} \tag{87}$$

VR and FUNC (7) have been relaxed into D (34) and FUNC A.E. (43), respectively. Further, they are of the state-independent form. The quantum state in question can be the thermal state I/d, where the number of the dimension is d, in order to refute any hidden variable theory satisfying JD (40). We have the following result:

Theorem: (*The statistical Kochen-Specker theorem*).

For every quantum state described in a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ or $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, $(\text{Dim}(\mathcal{H}_j) = 2, (j = 1, 2, 3))$,

$$\text{HV} \wedge \text{JD (40)} \Rightarrow \perp, \tag{88}$$

where $\perp$ means Contradiction. That is, these two assumptions do not hold at the same time.

These examples are sufficient to show that, if we accept JD (40), HV cannot be possible for any state. They are not of suitable form to test experimentally the KS theorem. Because, in a real experiment, we cannot claim a sharp value as an expected value with arbitrary precision. Therefore, the inequality-form to test the KS theorem is useful [12].

VI. THE STATISTICAL KOCHEN–SPECKER ASSUMPTION AND THE COMMUTING OBSERVABLE ASSIGNMENT RULE

We suppose that both of the identity operator and the projection operator can have a preexistence value as $+1$. We measure them twice (A and B, where $A = B$). Thus the statistical Kochen–Specker theorem is considered.

First, we newly define the functional rule almost everywhere with respect to μ_ψ this:

$$f(g(O)) = g(f(O)), (\mu_\psi - a.e.) \tag{89}$$

where O is a Hermitian operator and f, g are appropriate functions to be used later. Kochen and Specker introduce the above assumption for the hidden variable theory. They want to investigate if the algebraic structure of quantum mechanical operators should be mirrored in the algebraic structure of the preexistence value of the operators. Second, the sum rule almost everywhere with respect to μ_ψ is newly defined this:

$$f(A + B) = f(A) + f(B), (\mu_\psi - a.e.) \tag{90}$$

where A, B are two commuting Hermitian operators. Thirdly, the product rule almost everywhere with respect to μ_ψ is newly defined this:

$$f(A \cdot B) = f(A) \cdot f(B), (\mu_\psi - a.e.). \tag{91}$$

And the commuting observable assignment rule almost everywhere with respect to μ_ψ is defined this:

$$f(A) \cdot f(B) = f(A) + f(B), (\mu_\psi - a.e.). \tag{92}$$

We have [4, 12, 14] the following relation between the rules for commuting observables:

$$\begin{aligned} &\text{The functional rule} \Rightarrow \text{The sum rule} \\ &\text{The functional rule} \Rightarrow \text{The product rule} \end{aligned} \tag{93}$$

and

$$\begin{aligned} &\text{The functional rule} \wedge \text{The sum rule} \wedge \text{The requirement conditions} \\ &\Rightarrow \text{The commuting observable assignment rule} \end{aligned} \tag{94}$$

For example, let us derive the sum rule and the product rule from the functional rule. Suppose now that A and B are two commuting Hermitian operators. In [4, 12, 14], we discuss that we can define functions j and k and there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. Thus we have

$$\begin{aligned} &f(A \cdot B) = f(h(O)) = h(f(O)) = j(f(O)) \cdot k(f(O)) \\ &f(j(O)) \cdot f(k(O)) = f(A) \cdot f(B), (\mu_\psi - a.e.) \end{aligned} \tag{95}$$

where we use the functional rule. We can introduce also a function l such that $A + B = l(O)$ where $l := j + k$. Thus we have

$$\begin{aligned} &f(A + B) = f(l(O)) = l(f(O)) = j(f(O)) + k(f(O)) \\ &f(j(O)) + f(k(O)) = f(A) + f(B), (\mu_\psi - a.e.) \end{aligned} \tag{96}$$

where we use the functional rule. And then, we may introduce a function m, if the following three conditions are satisfied:

1. $m(l(O)) := h(O)$, that is, $m(A + B) = A \cdot B$.
2. $m(j(O)) := j(O)$, that is, $m(A) = A$.
3. $m(k(O)) := k(O)$, that is, $m(B) = B$.

As the first example, we may have

$$A = B = \begin{pmatrix} +1 - \epsilon & 0 \\ 0 & +1 - \epsilon \end{pmatrix}. \tag{97}$$

As the second example, we may have

$$A' = B' = \begin{pmatrix} +1 - \epsilon & 0 \\ 0 & 0 \end{pmatrix}. \tag{98}$$

The value of $\epsilon, (0 \leq \epsilon < 1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Hence we have

$$A + B = \begin{pmatrix} 2(+1 - \epsilon) & 0 \\ 0 & 2(+1 - \epsilon) \end{pmatrix} \text{ and } A \cdot B = \begin{pmatrix} (+1 - \epsilon)^2 & 0 \\ 0 & (+1 - \epsilon)^2 \end{pmatrix}. \tag{99}$$

And we have

$$A' + B' = \begin{pmatrix} 2(+1 - \epsilon) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A' \cdot B' = \begin{pmatrix} (+1 - \epsilon)^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{100}$$

Let us take the function m as $m(2(+1 - \epsilon)) = (+1 - \epsilon)^2$ and $m(+1 - \epsilon) = +1 - \epsilon$. Therefore we can introduce the function m that satisfies the requirement conditions. And let us take the function m as $m(2(+1 - \epsilon)) = (+1 - \epsilon)^2$,

$m(+1-\epsilon) = +1-\epsilon$, and $m(0) = 0$. Therefore we can introduce the function m that satisfies the requirement conditions for both the cases. Then we have, using the function m,

$$\begin{aligned}
& f(A)\cdot f(B) = f(A\cdot B) = f(h(O)) = f(m(l(O))) \\
& = l(f(m(O))) \\
& = j(f(m(O))) + k(f(m(O))) \\
& = f(m(j(O))) + f(m(k(O))) \\
& = f(A) + f(B), (\mu_\psi - a.e.)
\end{aligned} \tag{101}$$

where we use the functional rule and the product rule. We have the commuting observable assignment rule

$$f(A)\cdot f(B) = f(A) + f(B), (\mu_\psi - a.e.). \tag{102}$$

Theorem.

The operation Addition $(f(A) + f(B))$ is equal to the operation Multiplication $(f(A)\cdot f(B))$ when $A = B = \begin{pmatrix} +1-\epsilon & 0 \\ 0 & +1-\epsilon \end{pmatrix}$ or $\begin{pmatrix} +1-\epsilon & 0 \\ 0 & 0 \end{pmatrix}$ and the quantum state is $|\uparrow\rangle$ that is a simultaneous eigenstate of A, B.

And then, in these cases, we can describe an assignment rule that might be called commuting observable assignment rule. The set for commuting observable assignment rule is different from our common set in which Newton's mechanics is created. In the set holding commuting observable assignment rule, the operation Addition and the operation Multiplication are the same as each other. In the normal set where Newton's mechanics is held, the operation Addition and the operation Multiplication are different from each other.

VII. THE INCONSISTENCY WITH HIDDEN VARIABLE THEORIES

Let $H(x) \equiv \begin{pmatrix} +1 & 0 \\ 0 & +x \end{pmatrix}$ be a quantum observable described by a parameter x. We consider the two patterns of the observables this:

1. $H(0) = \begin{pmatrix} +1 & 0 \\ 0 & 0 \end{pmatrix}$ (Projection operator)

2. $H(+1) = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}$ (Identity operator)

Let $|\uparrow\rangle$ be an eigenstate of $H(x)$ such that $H(x)|\uparrow\rangle = +1|\uparrow\rangle$. The preexistence value of quantum measurement outcome is $+1$ in the ideal case.

We are in the inconsistency with hidden variable theories introduced by Kochen and Specker when the following physical situation happens twice: The preexistence value is $+1$ by measuring quantum observable $H(x)$ in the quantum state $|\uparrow\rangle$. It turns out that we cannot assign the preexistence value for quantum measurement outcome as $+1$ when measuring quantum observable $H(x)$ in the quantum state $|\uparrow\rangle$.

We consider a value V which is the sum of the two preexistence values in a thought experiment. The preexistence value of outcome is $+1$. If the number of outcomes is two, then we have

$$V = (+1) + (+1) = +2. \tag{103}$$

We derive a necessary condition of the value V.

On the other hand, we can depict the preexistence values using v_1, v_2 this: $v_1 = +1$ and $v_2 = +1$. Let us write V this:

$$V = v_1 + v_2. \tag{104}$$

In the following, we evaluate another value of V and derive a necessary condition. Then, we have, using the commuting observable assignment rule

$$V = v_1 + v_2 = v_1 \times v_2 = +1, \tag{105}$$

where $f(A) + f(B) = f(A) \times f(B)$ is used. Here $f(\cdot) = \exp(\mathrm{tr}[\rho(\cdot)])$ and the quantum state ρ is $|\uparrow\rangle\langle\uparrow|$.

We cannot assign simultaneously the same two truth values ("1" and "1") or ("0" and "0") for the two propositions (103) and (105). We derive the inconsistency with hidden variable theories for quantum measurement outcome on $H(x)$. In the case that $x = 0$, Projection operator does not have a counterpart in physical reality. In the case that

$x = +1$, Identity operator does not have a counterpart in physical reality. Our simplified version says that we cannot assign a preexistence value as $+1$ to both quantum operators (the projection operator and the identity operator) even though the assignment is independent from each other. They does not have a counterpart in physical reality in the sense that the inconsistency with hidden variable theories is proposed by Kochen and Specker.

Theorem: (*The statistical Kochen–Specker theorem based on the identity operator or the projection operator*).

We have been in the inconsistency with hidden variable theories introduced by Kochen and Specker when the following physical situation happens twice: The preexistence value is $+1$ by measuring quantum observable $H(x)$ in the quantum state $|\uparrow\rangle$. It has turned out that we cannot assign the preexistence value for quantum measurement outcome as $+1$ when measuring quantum observable $H(x)$ in the quantum state $|\uparrow\rangle$.

VIII. THE STATISTICAL KOCHEN–SPECKER THEOREM BASED ON THE IDENTITY OPERATOR

We restrict ourselves to the identity operator. We want to modify the KS argument provided in Sec. VII so as to relax the assumptions. In fact, VR and FUNC (7) are relaxed into D (34) and FUNC A.E. (43), respectively. In what follows, we assume HV and JD (40) hold. This implies that we can use D (34), FUNC A.E. (43), and PROD A.E. (53).

We shall modify the KS theorem presented in Sec. VII for an identity operator on the two-dimensional space. Let us try an attempt to assign function $f : \Omega \mapsto \mathbf{R}$ to the identity operator on the two-dimensional space I for an arbitrary state ψ. D (34) says $f_I(\omega) = +1, (\mu_\psi - a.e.)$. Then, one can see that

$$\sum_{\omega\in\{\omega_1,\omega_2\}} f_I(\omega) = f_I(\omega_1) + f_I(\omega_2) = +2, (\mu_\psi - a.e.). \tag{106}$$

By the way ($\epsilon = 0$) the operation Addition $(f(A)+f(B))$ is equal to the operation Multiplication $(f(A)\cdot f(B))$ almost everywhere with respect to μ_ψ when $A = B = I$ and the quantum state is $|\uparrow\rangle$ that is a simultaneous eigenstate of A, B. Hence we have $f(A) \cdot f(B) = f(A) + f(B), (\epsilon = 0)$ almost everywhere with respect to μ_ψ. Hence we get $f_I(\omega_1) + f_I(\omega_2) = f_I(\omega_1) \times f_I(\omega_2)$ almost everywhere with respect to μ_ψ and

$$\sum_{\omega\in\{\omega_1,\omega_2\}} f_I(\omega) = f_I(\omega_1) + f_I(\omega_2) = f_I(\omega_1) \times f_I(\omega_2) = +1, (\mu_\psi - a.e.). \tag{107}$$

Noting the relation (106), we see that an attempt to assign function to the identity operator on the two-dimensional space for an arbitrary state ψ should fail.

IX. THE INCOMPLETENESS IN A REAL EXPERIMENT

In a real experiment, there are no perfect detectors, but the good ones with some errors. There is an unforeseen effect that an imperfect detector does not count even though the particle indeed passes through the detector (the quantum efficiency). There is also an unforeseen effect that an imperfect detector counts even though the particle does not pass through the detector (the dark count). In this case, we increase measurement outcomes to even number $2N(\gg 1)$ and then we change such errors into trivial things. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

We suppose that an identity operator on the two-dimensional space can have a preexistence value for quantum measurement outcome as $+1$. Suppose now that $I \equiv |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$ is the identity operator on the two-dimensional space. We define I this:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{108}$$

Let $|\uparrow\rangle$ be an eigenstate of I such that $I|\uparrow\rangle = +1|\uparrow\rangle$. The preexistence value for quantum measurement outcome is $+1$ in the ideal case. Let $|\downarrow\rangle$ be the other eigenstate of I such that $I|\downarrow\rangle = +1|\downarrow\rangle$. When we consider a quantum optical experiment, we have the following relations with the photon polarization states:

$$\begin{aligned} &|\uparrow\rangle \leftrightarrow |H\rangle, \\ &|\downarrow\rangle \leftrightarrow |V\rangle, \end{aligned} \tag{109}$$

where $|H\rangle$ is a quantum state interpreted by a horizontally polarized photon and $|V\rangle$ is a quantum state interpreted by a vertically polarized photon.

Let us introduce the random noise admixture $\rho_{\rm noise}(=\frac{1}{2}I)$ into the quantum state $|\uparrow\rangle$, where the operator I is the two-dimensional identity operator. We consider the noisy quantum state emerged from an imperfect source this:

$$\rho=(1-\epsilon)|\uparrow\rangle\langle\uparrow|+\epsilon\times\rho_{\rm noise}. \tag{110}$$

The value of $\epsilon(<1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Then we have ${\rm tr}[\rho I]=+1-\epsilon$. The preexistence value for quantum measurement outcome is $+1-\epsilon$.

However the quantum state ρ is not an eigenstate of I. Thus we change the definition of the measured observable and of the quantum state this:

$$I'=\begin{pmatrix}+1-\epsilon & 0\\ 0 & +1-\epsilon\end{pmatrix} \tag{111}$$

and

$$\rho'=|\uparrow\rangle\langle\uparrow|. \tag{112}$$

Then the quantum state ρ' is an eigenstate of I'. Further we have ${\rm tr}[\rho' I']=+1-\epsilon$.

The odd number preexistence values are $+1-\epsilon$ by measuring the identity operator I' in the quantum state ρ', the even number preexistence values are the same $+1-\epsilon$ as by measuring the same identity operator I' in the same quantum state ρ'. That is, the number of outcomes is even number $2N$.

We consider a following value $V_i(i=1,2,...,N)$ which is the sum of the two preexistence values in the thought optical experiment:

$$V_i=(+1-\epsilon)+(+1-\epsilon)=2(+1-\epsilon). \tag{113}$$

We introduce the following function $S(N)$ which is the sum over i of V_i in a thought optical experiment:

$$S(N)=V_1+V_2+...+V_N. \tag{114}$$

If the number of outcomes is even number $2N$, then we have

$$\begin{aligned}S(N)&=V_1+V_2+...+V_N=N\Big((+1-\epsilon)+(+1-\epsilon)\Big)\\&=N\big(2(+1-\epsilon)\big)=2N(+1-\epsilon),\end{aligned} \tag{115}$$

where we use $V_i=(+1-\epsilon)+(+1-\epsilon)=2(+1-\epsilon)$. We derive here a necessary condition for the function value $S(N)$.

On the other hand, we can depict the preexistence values using $v_1,v_2,v_3,...,v_{2N}$ this: $v_1=+1-\epsilon$, $v_2=+1-\epsilon$, $v_3=+1-\epsilon$,..., $v_{2N}=+1-\epsilon$. We can write V_i this:

$$V_i=v_{2i-1}+v_{2i}, \tag{116}$$

where

$$v_{2i-1}=+1-\epsilon,\ v_{2i}=+1-\epsilon. \tag{117}$$

Thus, we have

$$V_i=(+1-\epsilon)+(+1-\epsilon). \tag{118}$$

In the following we evaluate another value of V_i and derive a necessary condition for the function value $S(N)$. Then, we have, using the commuting observable assignment rule,

$$V_i=v_{2i-1}+v_{2i}=v_{2i-1}\times v_{2i}=(+1-\epsilon)^2, \tag{119}$$

where $f(A)\cdot f(B)=f(A)+f(B)$ is used. Here $f(\cdot)={\rm tr}[\rho'(\cdot)]$ and the quantum state ρ' is $|\uparrow\rangle\langle\uparrow|$. Thus, we have

$$\begin{aligned}S(N)&=V_1+V_2+...+V_N\\&=N(+1-\epsilon)^2.\end{aligned} \tag{120}$$

We cannot assign simultaneously the same two truth values ("1" and "1") or ("0" and "0") for the two propositions (115) and (120). We derive the inconsistency within the hidden variable theory.

Theorem: (*The statistical and experimental accessible Kochen–Specker theorem based on an identity operator on the two-dimensional space*).

We have been in the inconsistency within the hidden variable theory when quantum measurement outcomes are even number $2N(\gg 1)$, the odd number preexistence values are $+1-\epsilon$ by measuring the identity operator I' in the quantum state ρ', the even number preexistence values are the same $+1-\epsilon$ as by measuring the same identity operator I' in the same quantum state ρ'.

X. THE STATISTICAL AND EXPERIMENTAL ACCESSIBLE KOCHEN–SPECKER THEOREM BASED ON AN IDENTITY OPERATOR ON THE TWO-DIMENSIONAL SPACE

We want to modify the KS argument provided in Sec. IX so as to relax the assumptions. In fact, VR and FUNC (7) are relaxed into D (34) and FUNC A.E. (43), respectively. In what follows, we assume HV and JD (40) hold. This implies that we can use D (34), FUNC A.E. (43), and PROD A.E. (53).

We shall modify the KS theorem presented in Sec. IX for an identity operator. Let us try an attempt to assign function $f:\Omega\mapsto\mathbf{R}$ to the identity operator I' for an arbitrary state ψ. D (34) says $f_{I'}(\omega)=+1-\epsilon,(\mu_\psi-a.e.)$. Then, one can see that

$$N\times\sum_{\omega\in\{\omega_1,\omega_2\}}f_{I'}(\omega)=N(f_{I'}(\omega_1)+f_{I'}(\omega_2))=2N(+1-\epsilon),(\mu_\psi-a.e.). \tag{121}$$

By the way the operation Addition $(f(A)+f(B))$ is equal to the operation Multiplication $(f(A)\cdot f(B))$ almost everywhere with respect to μ_ψ when $A=B=\begin{pmatrix}+1-\epsilon & 0\\ 0 & +1-\epsilon\end{pmatrix}$ and the quantum state is $|\uparrow\rangle$ that is a simultaneous eigenstate of A,B. Hence we have $f(A)\cdot f(B)=f(A)+f(B)$ almost everywhere with respect to μ_ψ. Hence we get $f_{I'}(\omega)+f_{I'}(\omega)=f_{I'}(\omega)\times f_{I'}(\omega)$ almost everywhere with respect to μ_ψ and

$$N\times\sum_{\omega\in\{\omega_1,\omega_2\}}f_{I'}(\omega)=N(f_{I'}(\omega_1)+f_{I'}(\omega_2))=N(f_{I'}(\omega_1)\times f_{I'}(\omega_2))=N(+1-\epsilon)^2,(\mu_\psi-a.e.). \tag{122}$$

Noting the relation (121), we see that an attempt to assign function to the identity operator for an arbitrary state ψ should fail.

XI. THE MAIN THEOREM

Theorem: (*The statistical and experimental accessible Kochen–Specker theorem based on an identity operator on the two-dimensional space*).

$$\mathrm{HV}\wedge\mathrm{JD}\ (40)\Rightarrow\mathrm{HV}\wedge f_{I'}(\omega)=+1-\epsilon,(\mu_\psi-a.e.)\wedge\mathrm{FUNC\ A.E.}\ (43)\Rightarrow\perp, \tag{123}$$

where $\perp$ means Contradiction. That is, these assumptions do not hold at the same time. We say "from this definition D (34), it is required that the value of $\omega(I)$ is $+1$ almost everywhere, i.e., $f_I(\omega)=+1,(\mu_\psi-a.e.)$. Here, I represents the identity operator on the two-dimensional space." Thus, we can use the weak assumption $f_{I'}(\omega)=+1-\epsilon,(\mu_\psi-a.e.)$ instead of the strong assumption D (34) in order to construct our theorem. We do not already need the weak hidden variable theory satisfying D (34).

XII. CONCLUSIONS AND DISCUSSIONS

In conclusions, we have investigated the significant striking aspect of the most basic two quantum observables: (1) an observable that is obtained though a single projection operator and (2) an observable that is obtained though a single identity operator. Here, both of quantum observables have been well studied by Kochen and Specker in terms of the inconsistency with the hidden variable theory. They have shown that some value assignment to elements of a set of projection operators and the identity operator should fail.

Now, we have greatly simplified the Kochen–Specker theorem. We here have formulated the basic two quantum observables this: $H(x)\equiv\begin{pmatrix}+1 & 0\\ 0 & +x\end{pmatrix},(x=0,+1)$. We have been in the inconsistency with hidden variable theories introduced by Kochen and Specker when the following physical situation happens twice: The preexistence value has been $+1$ by measuring quantum observable $H(x)$ in the quantum state $|\uparrow\rangle$. It has turned out that we cannot assign the preexistence value for quantum measurement outcome as $+1$ when measuring quantum observable $H(x)$ in the quantum state $|\uparrow\rangle$.

In the case that $x=0$, Projection operator has not had a counterpart in physical reality. In the case that $x=+1$, Identity operator has not had a counterpart in physical reality. Our simplified version has said that we cannot assign a preexistence value as $+1$ to both quantum operators (the projection operator and the identity operator) even though the assignment is independent from each other. They have not had a counterpart in physical reality in the sense that the inconsistency with hidden variable theories is proposed by Kochen and Specker.

And finally, we have been able to use the weak assumption (the identity operator possesses a preexistence value as $+1$) instead of the strong assumption (the existence of the weak hidden variable theory) in order to verify the Kochen–Specker theorem.

We can extend our result into the N-dimensional space. Let a quantum operator $H_N(x)$ be this:

$$H_N(x) \equiv |1\rangle\langle 1| + x|2\rangle\langle 2| + ... + x|N\rangle\langle N|. \tag{124}$$

Then we have

1. $H_N(0) = |1\rangle\langle 1|$ (Projection operator)

2. $H_N(+1) = |1\rangle\langle 1| + |2\rangle\langle 2| + ... + |N\rangle\langle N|$ (the N-dimensional identity operator)

We are in the inconsistency with hidden variable theories introduced by Kochen and Specker when the following physical situation happens twice: The preexistence value is $+1$ by measuring quantum observable $H_N(x)$ in the quantum state $|1\rangle$. It turns out that we cannot assign the preexistence value for quantum measurement outcome as $+1$ when measuring quantum observable $H_N(x)$ in the quantum state $|1\rangle$. In the case that $x = 0$, Projection operator does not have a counterpart in physical reality. In the case that $x = +1$, the N-dimensional identity operator does not have a counterpart in physical reality.

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