FAPP treatment of the Hodge conjecture

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Abstract

We treat the Hodge conjecture for all practical purposes (FAPP).

1 Glossary

$\angle A$: angle A .

$\overset{\frown}{AB}$: arc AB .

$\overline{AB}$: line segment AB .

$\triangle ABC$: triangle ABC .

HC: Hodge conjecture .

HT: Hodge theory .

O: the origin (0, 0) .

RHS: right-hand side .

$|x|$: absolute value of x .

2 Introduction

HC and HT having been of some interest [1,2], we treat HC in a FAPP manner, by which we mean that we tackle something practically rather than theoretically.

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3 Tinkering with arcs, circles, etc

We start from the following semicircle .

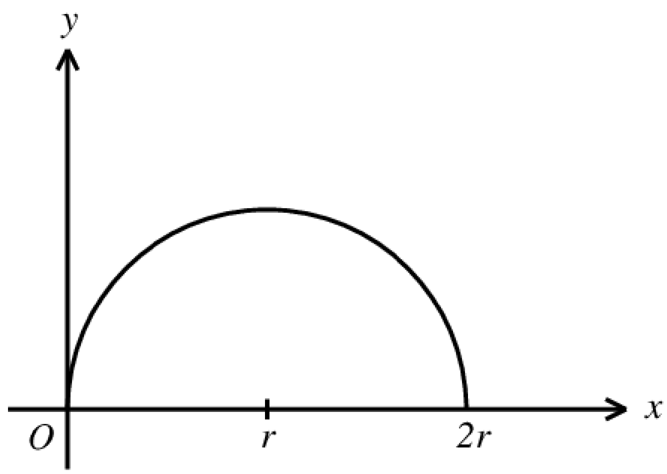

Fig. 1. Semicircle whose radius is r

Next, we consider $\overset{\frown}{BC}$, whose central angle $\angle BAC$ is θ.

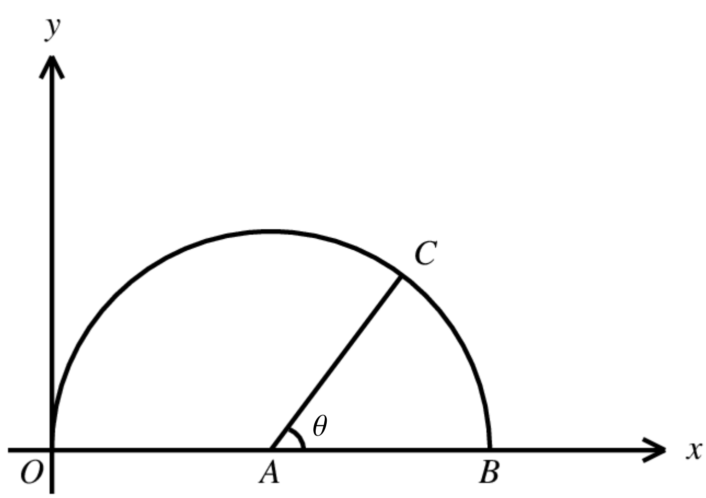

Fig. 2. $\overset{\frown}{BC}$ formed by the central angle θ

Then, we join C and O:

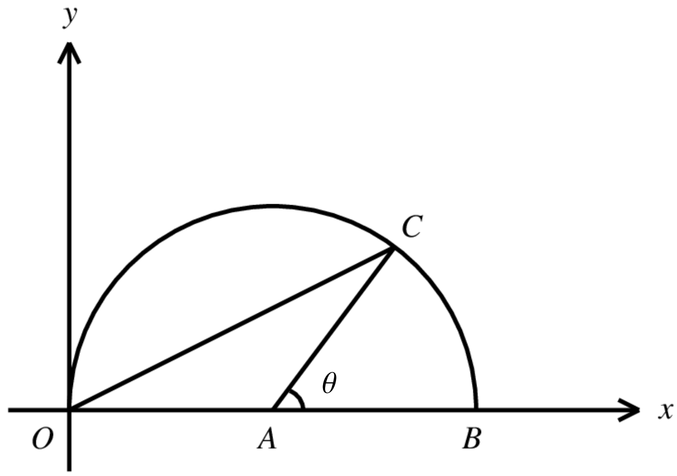

Fig. 3. Addition of $\overline{OC}$ to Fig. 2

Since A is the centre of the semicircle , $\overline{AO} = \overline{AC}$. So by the equality of the base angles of an isosceles triangle , we have $\angle AOC = \angle ACO = \alpha$ as follows.

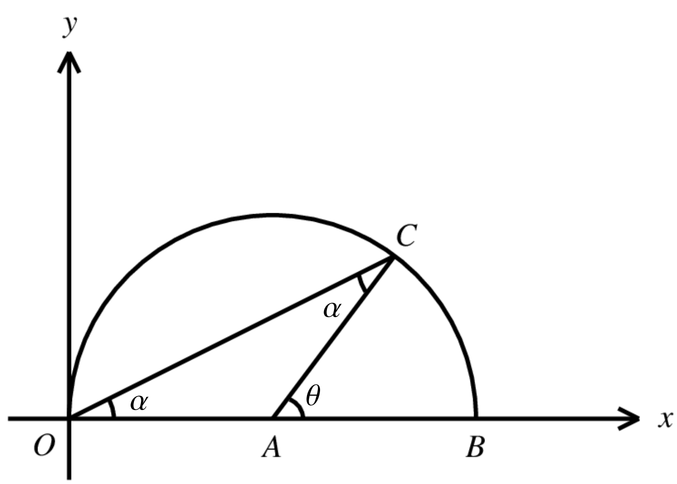

Fig. 4. Showing isosceles triangle $\triangle OAC$ in Fig. 3

Applying the high school exterior angle theorem to $\triangle OAC$, one gets $\alpha + \alpha = \theta$, that is, $\theta = 2\alpha$:

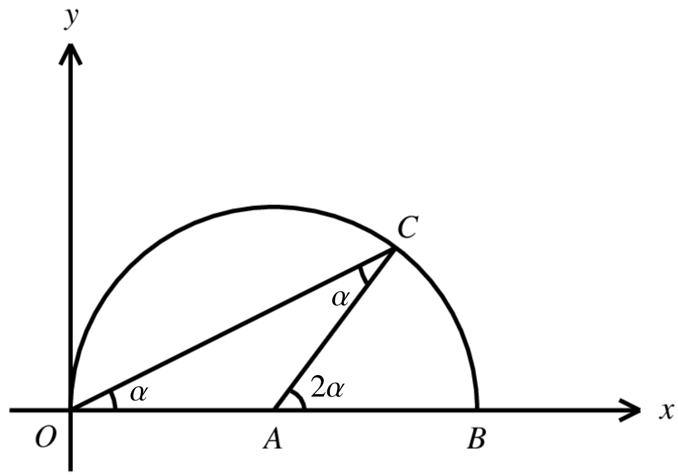

Fig. 5. Rewriting θ in Fig. 4 as 2α

What if we move *A* to *O* as follows?

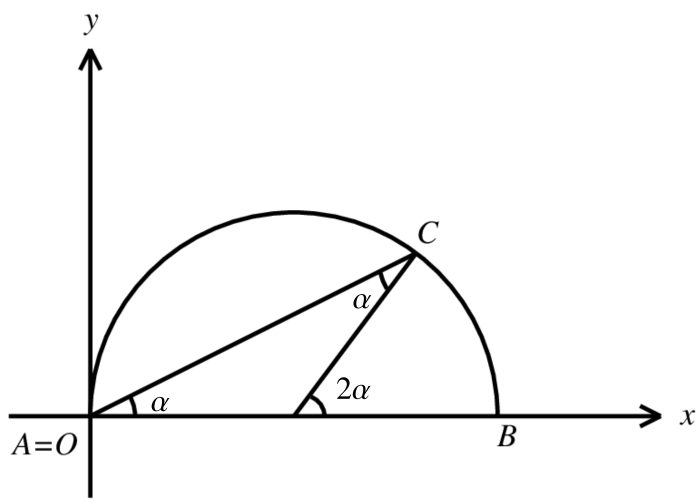

Fig. 6. ‘Identification’ of *A* with *O*

In this case, $\overline{AC}$ has become $\overline{OC}$, and we make the semicircle into a circle, whose centre is *D*:

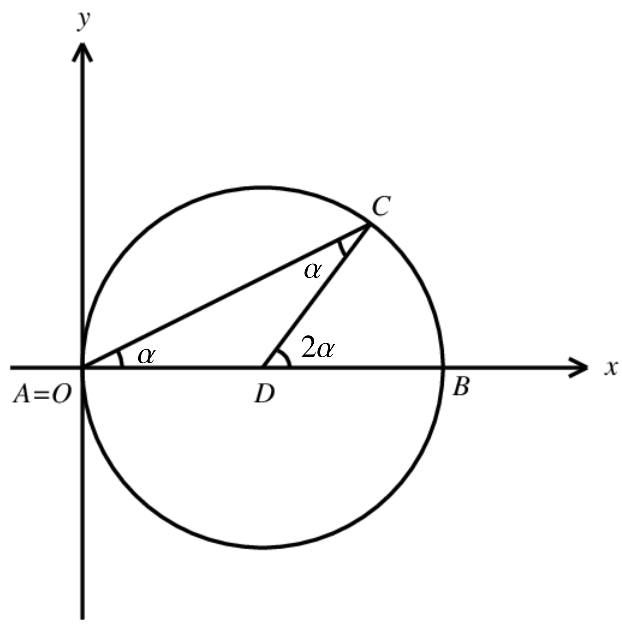

Fig. 7. Drawing a circle using Fig. 6

We can think of another arc, $\overset{\frown}{CE}$ (drawn in red) which is a part of the sector ACE (or OCE):

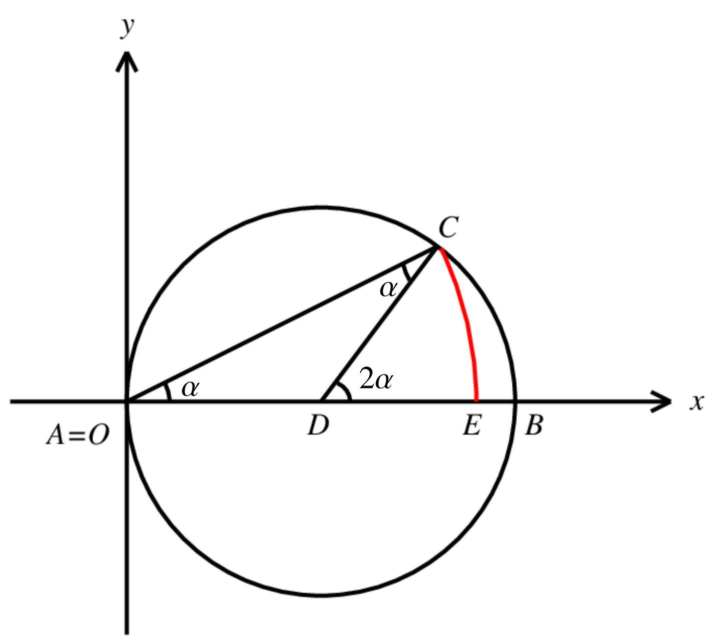

Fig. 8. Addition of $\overset{\frown}{CE}$ to Fig. 7

We try regarding the diameter AB (or OB) as 'radius' of another circle as follows [1] .

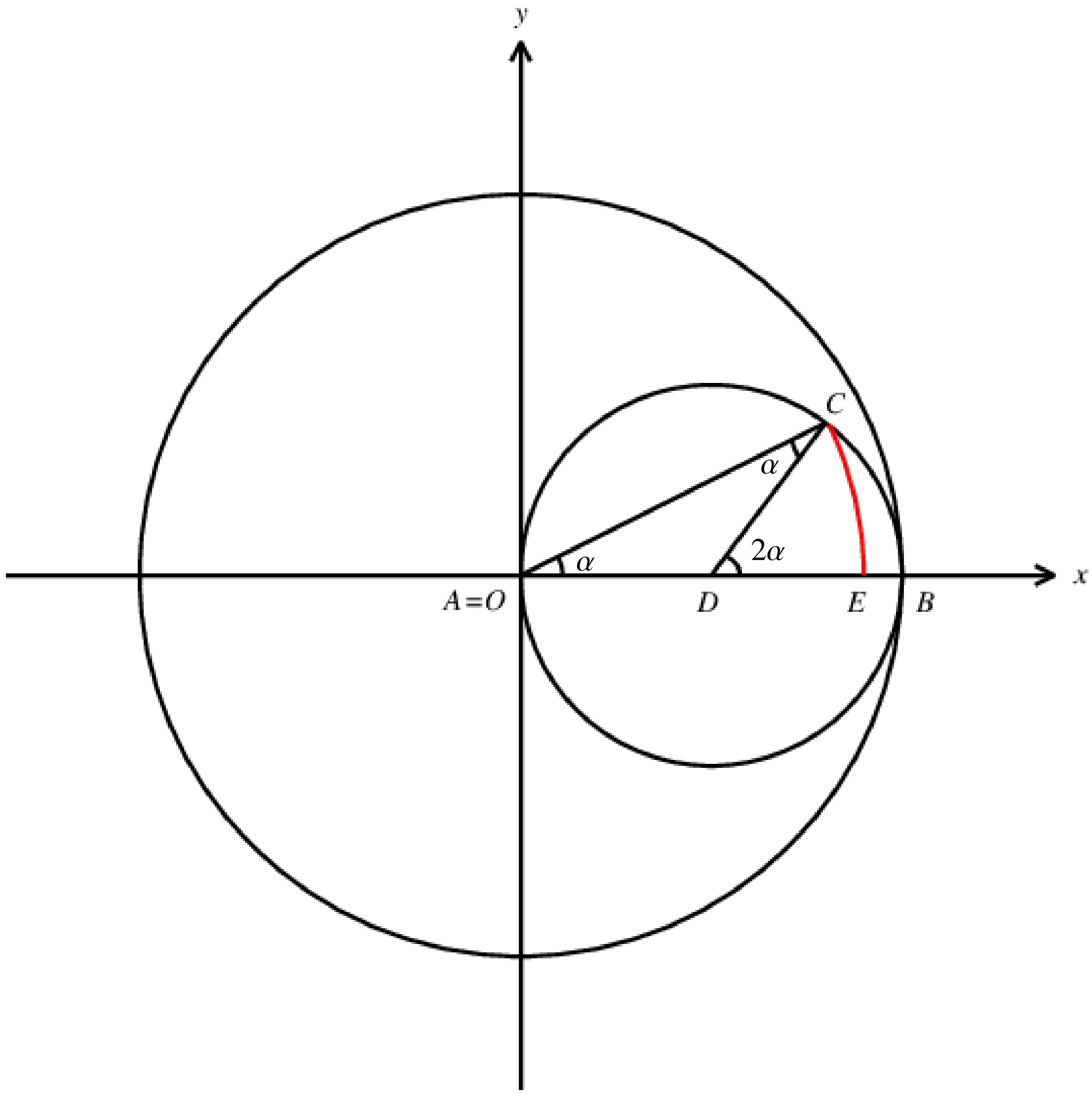

Fig. 9. Drawing another circle using Fig. 8

Extending $\overline{AC}$ (or $\overline{OC}$) in the upper right direction results in the formation of the intersection F, and we can think of yet another arc, $\overset{\frown}{BF}$ (drawn in blue), as shown below.

[1] At this stage, some might recall the Hawaiian earring [3, ***Fig. 2.4***].

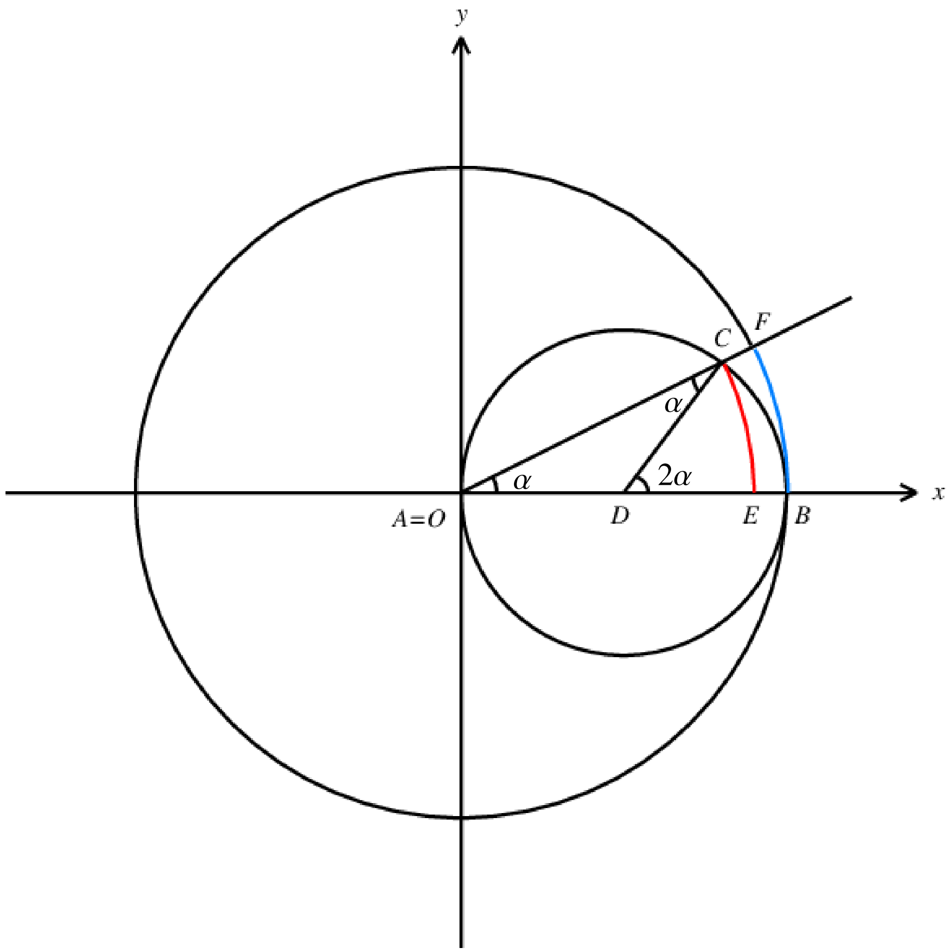

Fig. 10. Showing minor sector *ABF* (or *OBF*)

We put together the results we have so far obtained as below.

Table

Sector	Circular arc	Central angle and its degree
ACE (or *OCE*)	$\overset{\frown}{CE}$	$\angle CAE$ (or $\angle COE$), α
BCD	$\overset{\frown}{BC}$	$\angle BDC$, 2α
BAF (or *BOF*)	$\overset{\frown}{BF}$	*BAF* (or $\angle BOF$), α

4 Discussion

Since we have stressed the role of FAPP, we are justified in saying that our reasoning has been somewhat sloppy. However, putting such (inevitable) sloppiness aside for a while, we should like to start our discussion.

At the outset, the length L of an arc of a circle with radius r and central angle θ (measured in radians) is $L = \theta r$. We repeatedly use this relation in what follows [2] . We compute the lengths of some arcs in Fig. 10. For example, $\overset{\frown}{CE} = \alpha \cdot \overline{AC}$. Using the law of cosines , we have

$$\overline{AC} = \sqrt{\overline{AD}^2 + \overline{CD}^2 - 2\overline{AD} \cdot \overline{CD} \cdot \cos(\pi - 2\alpha)}\,. \tag{1}$$

Using the angle addition formula , we compute $\cos(\pi - 2\alpha) = \cos\pi\cos 2\alpha + \sin\pi\sin 2\alpha = -\cos 2\alpha$. So $\overline{AC} = \sqrt{r^2 + r^2 - 2 \cdot r \cdot r \cdot (-\cos 2\alpha)} = \sqrt{2r^2(1 + \cos 2\alpha)}$. To this, we apply the formula $\cos 2\theta = 2\cos^2\theta - 1$, one of the double-angle formulae , to compute

$$\overline{AC} = \sqrt{2r^2(1 + \cos 2\alpha)} = \sqrt{2r^2(1 + 2\cos^2\alpha - 1)} = \sqrt{4r^2\cos^2\alpha} = 2r\cos\alpha. \tag{2}$$

Since $\overline{AC} = 2r\cos\alpha$, $\overset{\frown}{CE} = 2r\alpha\cos\alpha$. Besides, we get $\overset{\frown}{BC} = 2\alpha \cdot r = 2r\alpha$ and $\overset{\frown}{BF} = \alpha \cdot 2r = 2r\alpha$.

In **Table**, we notice that the central angle α corresponds to $\overset{\frown}{CE}$ and $\overset{\frown}{BF}$, whereas the central angle 2α corresponds to $\overset{\frown}{BC}$ only. Based on this distinction, we tentatively call views pertaining to the former and the latter 'α-view' and '2α-view', respectively. 2α being $2 \cdot \alpha$, such 'twofoldness' reminds us of this: "$SU(2)$ is a double cover of $SO(3)$", which is the standard mantra meaning that there is twice as much rotation in the quaternions as there is in our space-time [4]. Since a point in four-dimensional space with Cartesian coordinates (u, x, y, z) may be represented by a quaternion $P = u + xi + yj + zk$, we roughly regard '2α-view' as four-dimensional.

Talking of quaternions, quaternionic Kähler manifold has been known [5]. And a Kähler manifold is not unrelated to HT . So our '2α-view' doesn't seem irrelevant to HC.

As we have already seen, '2α-view' is unique in the sense that it pertains to only one arc, unlike 'α-view' that is related to two arcs. Thinking of such 'uniqueness' as one of its assets, we hope it will shed some light on HC. On the other hand, we guess that 'non-uniqueness' seen in 'α-view' is causative of some elusiveness of HC [3] .

Incidentally, what if we consider ellipse instead of circle? Frankly speaking, considering arc length of an ellipse might be more fruitful than just tinkering with circles. That said, arc length of an ellipse is given by incomplete elliptic integral , which seems less tractable than computing arc lengths of circles. Another worrisome thing is that we have presented neither proof of HC nor its counterexample. However and finally, despite these apprehensions, we wonder if such 'elliptisation' will lead to a deeper understanding of HC.

Acknowledgment. We should like to thank the developers of Okular and PostScript for their indirect help, which enabled us to prepare figures for submission.

[2] We can also apply this relation to Fig.'s. See *e.g.*, Fig.'s 1 and 2, in which we can get $\overset{\frown}{BC} = \theta r$ using the relation.

[3] By 'elusiveness', we mean that HC *itself* seems largely unsettled for some reason.

References

[1] Voisin, C., “The Hodge Conjecture,” p. 521 in Nash, J. F. Jr. and Rassias, M. T. eds. “Open Problems in Mathematics,” Springer 2016 .

[2] Cattani, E., Zein, F. E., Griffiths, P. A., and Tráng, L. D. eds., “Hodge Theory,” Princeton University Press 2014 .

[3] Basener, W. F., “Topology and Its Applications,” Wiley-Interscience 2006 p. 60.

[4] Morris, D., “Quaternions,” Abane & Right 2015 p. 76.

[5] Harvey, R. F., “Spinors and Calibrations,” Academic Press 1989 pp. 92-93.

5 Appendix

5.1 Some computational subtleties

We elaborate on some subtle matters. With regard to (1), due to the law of cosines, we have

$$\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 - 2\overline{AD}\cdot\overline{CD}\cdot\cos(\pi - 2\alpha).$$

Therefore, more precisely,

$$\overline{AC} = \pm\sqrt{\overline{AD}^2 + \overline{CD}^2 - 2\overline{AD}\cdot\overline{CD}\cdot\cos(\pi - 2\alpha)}\,. \tag{3}$$

However, it is hard to imagine $\overline{AC} < 0$, since line segments have a defined length. So we drop the negative root of (3) to get (1). Regarding (2), more precisely, recalling $\sqrt{x^2} = |x|$, we should have written

$$\sqrt{4r^2\cos^2\alpha} = |2r\cos\alpha|\ . \tag{4}$$

That said, since $|ab| = |a||b|$, the RHS of (4) equals

$$|2r||\cos\alpha|.$$

Since we suppose that $r > 0$, $2r > 0$, and thus $|2r| = 2r$. Moreover, supposing $0 < \alpha < \frac{\pi}{2}$, we have $0 < \cos\alpha < 1$, and thus $|\cos\alpha| = \cos\alpha$. Eventually, the RHS of (4) coincides with the rightmost-hand side of (2). Taken together, it has turned out that our computations were slightly hasty, which doesn’t seem to pass as an excuse for our sloppiness mentioned in **Discussion**, though.

5.2 Is $\geq 5-$ dimensional FAPP approach applicable to HC?

We have failed to treat or rule out the cases where dimensions are $5, 6, 7, \ldots$. So our response to this question is

Maybe.