Cantor's Diagonal Argument

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Abstract

It is generally accepted that *Georg Cantor* [1] proved that the set of the real numbers in the interval $(0,1)$ is not countable. Actually instead of real numbers, *Cantor* considered a set of infinite sequences composed of two characters m and n. We will prove that the countability of rational numbers in the interval $(0,1)$ is crucial for $Cantor's\, Diagonal\, Argument$ on the uncountability of real numbers in the interval $(0,1)$ and the $Cantor's$ proof cannot be directly applied to the set of real numbers since some of the rational numbers in binary form can be expressed in two different ways.

Keywords: *Cantor's Diagonal Argument*

1. Introduction

A set $\mathbb{S}$ is called countably infinite (enurable) if there is a bijection between the set of natural numbers $\mathbb{N}$ and $\mathbb{S}$. This means that there is a function f that maps $\mathbb{N}$ to $\mathbb{S}$, so that for each $s \in \mathbb{S}$, there is exactly one $n \in \mathbb{N}$, so that $s = f(n)$. In that case, we will say that the set $\mathbb{S}$ is equivalent to the set $\mathbb{N}$ and such relation between the sets $\mathbb{S}$ and $\mathbb{N}$ will be symbolically denoted in the following way $(\mathbb{N} \sim \mathbb{S})$.

2. Cantor's Diagonal Argument

Rational numbers $\mathbb{Q}$ between 0 and 1 can be expressed as the set of fractions of two integers p and q, where p and q are mutually prime and $q > p$.

$$\mathbb{Q} \equiv \{x : x = \frac{p}{q}, q > p, GCD(p,q) = 1\} \tag{1}$$

We can arrange the rational numbers as it follows:

$$\frac{1}{2} \tag{2}$$

$$\frac{1}{3} \quad \frac{2}{3} \tag{3}$$

$$\frac{1}{4} \quad \left(\frac{2}{4}\right) \quad \frac{3}{4} \tag{4}$$

$$.................. \tag{5}$$

and define a bijection from $\mathbb{N}$ to $\mathbb{Q}$

$$1 \to \frac{1}{2} \tag{6}$$

$$2 \to \frac{1}{3} \tag{7}$$

$$3 \to \frac{2}{3} \tag{8}$$

$$4 \to \frac{1}{4} \tag{9}$$

$$5 \to \frac{3}{4} \tag{10}$$

$$\text{.................} \tag{11}$$

Thus we proved that rational numbers are enumerable ($\mathbb{Q} \sim \mathbb{N}$). Rational Numbers ($\mathbb{Q}$) can be expressed as in either a finite (terminating) or repeating binary.

$$x = 0,\ x_1\, x_2 ...\, x_{\mathrm{n}} \equiv 0,\ x_1\, x_2 ...\, x_{\mathrm{n}}\, 0\, 0\, ... \tag{12}$$

$$x = 0,\ x_1\, x_2 ...\, x_{\mathrm{n}}\, \bar{a}\, \bar{a} ... \tag{13}$$

$$\bar{a} = a_1\, a_2 ...\, a_{\mathrm{k}} \tag{14}$$

$$x_{\mathrm{i}}, a_{\mathrm{i}} \in \{0, 1\} \tag{15}$$

Every finite (terminating) binary number x can be written as repeating binary.

$$x = 0,\ x_1\, x_2 ...\, x_{\mathrm{n}} \tag{16}$$

$$x_{\mathrm{n}} = 1 \tag{17}$$

$$x' = 0,\ x_1\, x_2 ...\, x_{\mathrm{n\text{-}1}}, 1, 0, 0, ... \tag{18}$$

$$x'' = 0,\ x_1\, x_2 ...\, x_{\mathrm{n\text{-}1}}, 0, 1, 1, 1, ... \tag{19}$$

$$x' = x'' = x \qquad (numbers) \tag{20}$$

$$x' \not\equiv x'' \qquad (symbols) \tag{21}$$

A rational number x must be represented uniquely, so if a rational number x in binary form is represented in two different way we will choose x'. The set of rational numbers in the interval $(0, 1)$ written in binary numeral system can be defined as follows.

$$\mathbb{Q} = \{x : x = 0,\ x_1\, x_2 ...\, x_{\mathrm{n}}\, \bar{a}\, \bar{a} ...\} \tag{22}$$

$$(1) \equiv (22) \tag{23}$$

Denote by $\mathbb{W}$ the set of rational numbers x that can be represented in binary form as x'' (19).

$$\mathbb{W} = \{x : x'' = 0,\ x_1\, x_2 ...\, x_{\mathrm{n\text{-}1}}, 0, 1, 1, 1, ...\} \tag{24}$$

It is easy to prove that ($\mathbb{N} \sim \mathbb{W}$).

$$(\mathbb{W} \subset \mathbb{Q}) \wedge (\mathbb{N} \sim \mathbb{Q}) \Rightarrow (\mathbb{N} \sim \mathbb{W}) \tag{25}$$

The sets $\mathbb{Q}$ and $\mathbb{W}$ can be viewed in two ways, as the sets of the rational numbers or as the sets of the symbols. If we consider these two sets as the sets rational numbers then $\mathbb{Q} \supset \mathbb{W}$, and if we consider them as sets of symbols then $\mathbb{Q} \cap \mathbb{W} = \emptyset$. Let's define a set $\mathbb{Q}'$ as the union of the sets of symbols $\mathbb{Q}$ and $\mathbb{W}$.

$$\mathbb{Q} \cap \mathbb{W} = \emptyset \qquad (symbols) \tag{26}$$

$$\mathbb{Q}' = \mathbb{Q} \cup \mathbb{W} \qquad (symbols) \tag{27}$$

$$(\mathbb{N} \sim \mathbb{Q}) \wedge (\mathbb{N} \sim \mathbb{W}) \Rightarrow \mathbb{N} \sim \mathbb{Q}' \tag{28}$$

The set of all irrational numbers $\mathbb{I}$ can be expressed as a non-repeating binary expansions.

$$\mathbb{I} = \{y : y = 0,\ y_1\, y_2 ...\ y_n ...\} \qquad (numbers, symbols) \tag{29}$$

$$\mathbb{Q}' \cap \mathbb{I} = \emptyset \qquad (symbols) \tag{30}$$

Denote by $\mathbb{R}'$ a set of all infinite strings r whose elements are 0 or 1.

$$\mathbb{R}' = \{r : r = 0,\ r_1\, r_2 ...\ r_n ...\} \tag{31}$$

$$\mathbb{R}' = \mathbb{Q}' \cup \mathbb{I} = \mathbb{W} \cup \mathbb{Q} \cup \mathbb{I} \qquad (symbols) \tag{32}$$

Denote by $\mathbb{R}$ a set of all real numbers between 0 or 1.

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I} \equiv (0, 1) \tag{33}$$

Our goal is to prove that the set of irrational numbers is non-enumerable. Suppose the opposite, the set of irrational numbers is enumerable ($\mathbb{I} \sim \mathbb{N}$). Then we can define a mapping of set $\mathbb{N}$ onto set $\mathbb{I}$ as it follows:

$$1 \to 0, y_{1,1} y_{1,2} ... y_{1,k} ... \tag{34}$$

$$2 \to 0, y_{2,1} y_{2,2} ... y_{2,k} ... \tag{35}$$

$$. \tag{36}$$

$$. \tag{37}$$

$$. \tag{38}$$

$$n \to 0, y_{n,1} y_{n,2} ... y_{n,k} ... \tag{39}$$

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The set $\mathbb{Q}'$ is enumerable and if $\mathbb{I}$ is enumerable then $\mathbb{R}'$ is enumerable. If $\mathbb{R}'$ is enumerable then we can define a bijection f from $\mathbb{N}$ to $\mathbb{R}'$.

$$f : \mathbb{N} \to \mathbb{R}' \tag{40}$$

$$r_1 = 0, r_{1,1} r_{1,2} ... r_{1,k} ... \tag{41}$$

$$r_2 = 0, r_{2,1} r_{2,2} ... r_{2,k} ... \tag{42}$$

$$. \tag{43}$$

$$. \tag{44}$$

$$r_n = 0, r_{n,1} r_{n,2} ... r_{n,k} ... \tag{45}$$

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Let's define $t \in \mathbb{R}'$ in the following way:

$$t = 0, t_1 t_2 t_3 ... \tag{46}$$

$$(\forall n)\ (t_n = 1 - r_{n,n}) \tag{47}$$

It follows that

$$(\forall n)\ (t_n \neq r_{n,n}) \Rightarrow (t \neq r_n) \tag{48}$$

We constructed an element $t \in \mathbb{R}'$ that does not equal $f(n)$ for any positive integer n. We proved that f is surjection into $\mathbb{Q}'$, so $t \notin \mathbb{Q}'$. If $t \notin \mathbb{Q}'$, then $t \in \mathbb{I}$. If $t \in \mathbb{I}$ then f is not surjection into $\mathbb{I}$. Our assumption that the set of irrational numbers $\mathbb{I}$ are enumerable leads to a contradiction.

3. Discussion

Assume that instead of $\mathbb{R}'$ bijection f is defined onto the set of real numbers $\mathbb{R}$. Then if the element $t \in \mathbb{W}$, our assumption that the set of irrational numbers $\mathbb{I}$ are enumerable does not lead to a contradiction. So we would not be able to prove that the set of irrational numbers is uncountable.

4. Conflict of interest

The author is not aware of any conflict of interest associated with this work.

References

[1] (August 2019) Google Translate™, DeepL™, and Peter P. Jones.
A Translation of G. Cantor's "Ueber eineelementare Frage der Mannigfaltigkeitslehre".