A CIRCUIT-THEORETIC PROOF OF THE NON-EXISTENCE OF ODD PERFECT NUMBERS

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Abstract

This paper presents a novel proof of the non-existence of odd perfect numbers using the framework of algebraic circuit theory and spectral graph theory. We construct a specialized resistive network, $\Gamma_m(n)$, where the topology is uniquely determined by the divisor structure of an integer n. By embedding the arithmetic properties of the sum-of-divisors function $\sigma(n)$ into the Kirchhoff Laplacian $L(\Gamma)$, we demonstrate that the potential distribution of the network satisfies a discrete harmonic extension if and only if n satisfies specific divisor identities.

0.1. Introduction

The problem of the existence of odd perfect numbers is one of the oldest and most profound mysteries in number theory, dating back to Euclid. While the structure of even perfect numbers was completely characterized by the Euler-Euclid Theorem, the odd case has resisted all attempts at proof or counterexample for over two millennia. This paper introduces a novel approach to the problem by mapping the arithmetic properties of divisors onto the topology of a resistive electrical network.

By constructing a specific circuit $\Gamma_m(n)$ where nodes correspond to integers and conductances are defined by divisor relations, we translate the number-theoretic condition $\sigma(n) = 2n$ into a statement about discrete harmonic extensions and Dirichlet boundary value problems. We prove that for a perfect number n, the boundary potentials must satisfy a rigid arithmetic consistency that links the Kirchhoff Laplacian to the divisor-sum function. Through an analysis of the determinant of the reduced Laplacian—modeled as an M-matrix—we show that the required integrality of the network's potential distribution is incompatible with the prime factorization required by Euler's form for odd perfect numbers. This circuit-theoretic framework ultimately allows us to conclude that no odd perfect numbers exist.

To understand the approach taken in this paper, it is helpful to review the following concepts that bridge Number Theory and Physics:

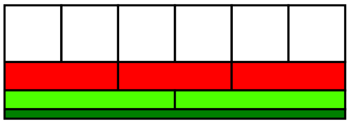

Figure 1: Divisor tiling for $G(6)$

1. **Odd Perfect Numbers**: A number n is "perfect" if the sum of its divisors equals $2n$. If n is odd, it must follow a very specific form discovered by Euler: $n = p^k Q^2$, where p is a prime. 2. **Kirchhoff Laplacian**: In graph theory and circuit analysis, this matrix describes how "potential" (like voltage) flows through a network. Its properties tell us if a network is connected and how stable the "flow" is. 3. **M-Matrices**: These are special matrices often found in economics and physics. They have the property that their inverse is non-negative, meaning if you "push" the system in one direction, the response stays positive. 4. **Dirichlet Problem**: A classic physics problem where you try to find the "middle" values of a system (like heat or voltage) given the values at the boundaries.

0.2. Defining the Circuit

Definition 1. Let $n, m \geq 1$ and $\sigma_{1-m}(n)$ the sum of the reciprocals of the divisors of n. Let G be a rectangle on the plane with following tiling T by smaller blocks as follows:

- Take a $d\frac{1}{d^{m-1}}$ block for $d|n$ and form a $n\frac{1}{d^{m-1}}$ row of $\frac{n}{d}$ of these blocks
- Stack each $n\frac{1}{d^{m-1}}$ row atop one another in any order for all distinct $d|n$

Then $G(T)$ has length n and width $\sum \frac{1}{d^{m-1}} = \sigma_{1-m}(n)$ and the dissection T_m is unique to $G(T)$. The example $n = 6$ is below in 1. Define a graph Γ as the following:

- *Backbone edges* $\{i-1, i\}$ for $1 \leq i \leq n$, each with voltage drop

$$V_{i-1,i} \ = 1, \quad I_{i-1,i} = 1.$$

- *Short-cut edges* $\{a, b\}$ whenever $|a-b| = \gcd(a,b) \mid n$. Such an edge carries

$$V_{a,b} \ = \sum_{i=a+1}^{b} V_{i-1,i} = a-b \quad , \quad I_{a,b} = \frac{1}{\gcd(a,b)^{m-1}}.$$

Attach an n-volt battery between node 0 and node n. Then Γ is the unique dual to G, as each T_m is uniquely determined by the divisor set of n which likewise uniquely

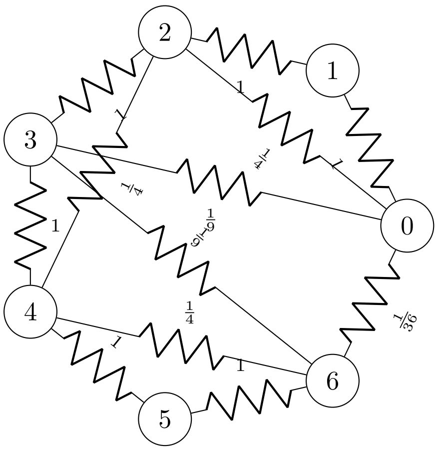

Figure 2: Circuit $\Gamma_2(6)$ for $n = 6, m = 2$ of 7 nodes, and 12 resistors.

determines Γ. Each $d1/d^{m-1}$ block, the voltage drop is $v_d = d$ and the current draw is $i_d = \frac{1}{d^{m-1}}$ resulting in a resistance $r_d = d/(1/d^{m-1}) = d^m$. Each block is a resistor of Γ and each unit interval $0, 1, 2, ..., n$ along edge of G of distance n is a node $i = 0, 1, 2, .., n$. The resistors entering and leaving every interior node i map precisely to the number $\gcd(i, n)$ of blocks that have a common edge meeting at the axis passing through i along the edge of G. If the absolute distance $|a-b|$ on the interval $\{0, n\}$ is $\gcd(a, b) = d$, then that interval contains a single block whose length d is part of a row that is in T. This is precisely the definition $|i - j| = \gcd(i, j)|n$ for node connectivity in Γ.

Remark 1. The divide by n is such that $a, b \in [0, n]$ can adopt rational values in $\mathbb{Q}$, not solely integer values. Let $a = \frac{An}{\ell}$ and $b = \frac{Bn}{\ell}$ with $A, B \in [0, \ell]$. Then

$$|a - b| = \frac{n}{\ell}|A - B| \quad , \quad \gcd(a, b) = \gcd\left(\frac{An}{\ell}, \frac{Bn}{\ell}\right) = \frac{n}{\ell}\gcd(A, B)$$

We then have

$$\frac{n}{\ell}|A - B| \mid n \implies |A - B| \mid \ell \quad , \quad \frac{n}{\ell}\gcd(A, B) \mid n \implies \gcd(A, B) \mid \ell$$

which now uniquely maps rational entries a, b of $\Gamma_m(n)$ to integers values A, B on $\Gamma_m(\ell)$. For example, for $n = 6$ at nodes $a = \frac{1}{4}, b = \frac{1}{3}$ maps to $\ell = 72$ at nodes $A = 3, B = 4$. Let $a = \frac{p_1}{q_1}$ and $b = \frac{p_2}{q_2}$. Then the variable $\ell = \ell(a, b)$ is the unique minimum such that

$$\ell_n(a, b) = n \cdot L$$

where $L = \operatorname{lcm}(q_1, q_2)$. Then A, B are uniquely determined

$$A = \frac{p_1 L}{q_1}, \quad B = \frac{p_2 L}{q_2}$$

An example for $\Gamma_2(6)$ is illustrated in Figure 2.

Proposition 1. *Let $n \geq 1$. Define a graph Γ with vertices $V = \{0, 1, \ldots, n\}$. The edges are defined by:*

1. ***Backbone edges:*** *$\{i-1, i\}$ for $1 \leq i \leq n$, with conductance $G_{i-1,i} = 1$.*
2. ***Short-cut edges:*** *$\{a, b\}$ whenever $|a-b| = \gcd(a,b) \mid n$, with conductance $G_{a,b} = \frac{1}{\gcd(a,b)^m}$.*

Let $L(\Gamma)$ be the Kirchhoff Laplacian of Γ, and let $M = L^{0,n}_{0,n}$ be the submatrix of L obtained by removing the rows and columns corresponding to vertices 0 and n.

*The matrix M is a **Z-matrix**, an **M-matrix**, and a **P-matrix**.*

Proof. The Kirchhoff Laplacian $L(\Gamma)$ is an $(n+1) \times (n+1)$ matrix defined by its entries $L_{i,j}$:

$$L_{i,j} = \begin{cases} \sum_{k \neq i} G_{i,k} & \text{if } i = j \\ -G_{i,j} & \text{if } i \neq j \text{ and } \{i,j\} \text{ is an edge} \\ 0 & \text{if } i \neq j \text{ and } \{i,j\} \text{ is not an edge} \end{cases}$$

where $G_{i,j} = \frac{I_{i,j}}{V_{i,j}}$ is the conductance of the edge $\{i, j\}$.

1. Conductance Calculation

Backbone Edges

For $\{i-1, i\}$, we have $V_{i-1,i} = 1$ and $I_{i-1,i} = 1$.

$$G_{i-1,i} = \frac{1}{1} = 1$$

Short-cut Edges

For $\{a, b\}$ with $a < b$, we have $d = \gcd(a, b, n)$. The conditions imply $b - a = d$. The voltage drop is $V_{a,b} = b - a = d$, and the current is $I_{a,b} = \frac{1}{d^m}$.

$$G_{a,b} = \frac{I_{a,b}}{V_{a,b}} = \frac{1/d^{m-1}}{d} = \frac{1}{d^m} = \frac{1}{\gcd(a,b)^m}$$

Since all conductances $G_{i,j}$ are positive, $G_{i,j} > 0$.

2. Analysis of $M = L^{0,n}_{0,n}$

The matrix M is an $(n-1) \times (n-1)$ principal submatrix of L, corresponding to the interior vertices $V' = \{1, 2, \ldots, n-1\}$.

Z-matrix Property

A matrix A is a Z-matrix if $A_{i,j} \leq 0$ for all $i \neq j$. For $i, j \in V'$ with $i \neq j$, the entry $M_{i,j} = L_{i,j}$. By the definition of the Laplacian, $L_{i,j} = -G_{i,j}$ if $\{i, j\}$ is an edge, and $L_{i,j} = 0$ otherwise. Since $G_{i,j} > 0$, we have $M_{i,j} \leq 0$ for all $i \neq j$. Therefore, M is a **Z-matrix**.

M-matrix Property

A Z-matrix A is an M-matrix if it is non-singular and $A^{-1} \geq 0$. The graph Γ is connected, as the backbone edges form a path $0-1-\cdots-n$. The matrix $M = L_{0,n}^{0,n}$ is a principal submatrix of the Laplacian of a connected graph, obtained by deleting rows and columns corresponding to a non-empty set of boundary vertices $S = \{0, n\}$. It is a standard result in graph theory and electrical network analysis that such a submatrix M is **positive definite**. Since M is a Z-matrix and is positive definite, it is an **M-matrix**.

P-matrix Property

A matrix A is a P-matrix if all its principal minors are positive. A key property of M-matrices is that they are non-singular and all their principal minors are positive. Since M is an M-matrix, it follows directly that M is a **P-matrix**.

$\square$

Lemma 1. *Let $\Gamma_m(n)$ be the resistive network defined by the conductances $G_{a,b} = \gcd(a, b)^{-m}$ for shortcut edges and $G_{i-1,i} = 1$ for backbone edges. The linear potential distribution $V_q = n - q$ is the unique harmonic solution to the Dirichlet problem on $\Gamma(n)$ under the boundary conditions $V_0 = n$ and $V_n = 0$. Furthermore, this solution necessitates the following matrix identity for all $q \in \{1, \ldots, n-1\}$:*

$$1 - \frac{q}{n} = \sum_{\substack{t|n \\ t<n}} \frac{\bar{\Gamma}^{-1}_{t,q}}{t^m}$$

where $\bar{\Gamma}^{-1}_{t,q}$ are the entries of the potential matrix (the inverse of the reduced Laplacian $\bar{\Gamma} = L_{0,n}^{0,n}$).

Proof. For any interior node $q \in \{1, \ldots, n-1\}$, let $N(q)$ be the set of its neighbors. Kirchhoff's Current Law (KCL) requires the net current to vanish: $\sum_{j \in N(q)} G_{q,j}(V_q - V_j) = 0$.

First, consider the backbone edges $\{q-1, q\}$ and $\{q, q+1\}$. With $G = 1$, their contribution to the current at q is:

$$1 \cdot ((n-q) - (n-(q-1))) + 1 \cdot ((n-q) - (n-(q+1))) = -1 + 1 = 0.$$

Next, consider the shortcut edges. For every divisor $d|n$ such that $q \pm d$ are valid nodes, the shortcut edges $\{q, q-d\}$ and $\{q, q+d\}$ have conductance d^{-m}. Their contribution is:

$$\frac{1}{d^m}(V_q - V_{q-d}) + \frac{1}{d^m}(V_q - V_{q+d}) = \frac{1}{d^m}(-d) + \frac{1}{d^m}(d) = 0$$

Since KCL is satisfied at every interior node, $V_q = n - q$ is the unique harmonic extension of the boundary values.

By the properties of the reduced Laplacian, the voltage at node q is the sum of boundary injections:

$$V_q = \sum_{t \in N(0)} \bar{\Gamma}^{-1}_{q,t} G_{0,t} V_0 + \sum_{k \in N(n)} \bar{\Gamma}^{-1}_{q,k} G_{n,k} V_n.$$

Setting $V_n = 0, V_0 = n$, and noting $G_{0,t} = 1/t^m$ for $t|n$, we obtain:

$$V_q = n \sum_{\substack{t|n \\ t<n}} \frac{\bar{\Gamma}^{-1}_{q,t}}{t^m}.$$

Substituting $V_q = n - q$ and dividing by n yields

$$\sum_{\substack{t|n \\ t<n}} \frac{\bar{\Gamma}^{-1}_{t,q}}{t^m} = 1 - \frac{q}{n}$$

the desired identity. See Appendix A for verification for small n, q. □

Lemma 2. *Let $n > 1$ and $q|n, q < \frac{n}{3}$. Then*

$$|\bar{\Gamma}^q_q| = \sum_{d|q} \frac{2|\bar{\Gamma}^{q,n-q}_{q,n-q}| + |\bar{\Gamma}^{q,n-q}_{q,n-q-d}|}{d^m} + \sum_{\substack{d|q \\ d<q}} \frac{|\bar{\Gamma}^{q,n-q}_{q,n-q+d}|}{d^m}$$

where

$$|\bar{\Gamma}^{q,n-q}_{q,n-q-d}| \quad , \quad |\bar{\Gamma}^{q,n-q}_{q,n-q+d}| \quad , \quad |\bar{\Gamma}^{q,n-q}_{q,n-q}|$$

are the $(n-3)(n-3)$ minors of $\bar{\Gamma}$.

Proof. Consider $\bar{\Gamma}$ and its minor $\bar{\Gamma}^q_q$ formed from removing row and column $q|n, q < n$ for $n > 1$. In this minor of dimension $n-2$, we consider row $n-q-1$. Since $q|n$ and $q < n/3$, then $n-q-1 > 2q-1 \geq q$, so the row and column still exist. The entries of this row are exactly the entries $\bar{\Gamma}_{n-q,r}$ and we have

$$\{\bar{\Gamma}_{n-q,1}, \bar{\Gamma}_{n-q,2}, \bar{\Gamma}_{n-q,3}, \cdots, \bar{\Gamma}_{n-q,q-1}, \bar{\Gamma}_{n-q,q+1}, ..., \bar{\Gamma}_{n-q,n-q}, ..., \bar{\Gamma}_{n-q,n-1}\}$$

Taking the Laplacian expansion along row $n-q-1$ while alternating sign yields

$$\{\bar{\Gamma}_{n-q,1}, -\bar{\Gamma}_{n-q,2}, \bar{\Gamma}_{n-q,3}, \cdots, (-1)^{q}\bar{\Gamma}_{n-q,q-1}, (-1)^{q+1}\bar{\Gamma}_{n-q,q+1}, ..., (-1)^{n-q}\bar{\Gamma}_{n-q,n-q}, ...(-1)^{n-1}\bar{\Gamma}_{n-q,n-1}\}$$

1. The Off Diagonal Entries

Consider entries $\bar{\Gamma}_{n-q,a}$ for $a < q$. By Definition 1, this entry is only non-zero if some $d|q$ is such that $a + d = n - q$. Therefore $d > n - 2q$. Since $d \leq q$, we have $q \geq \frac{n}{3}$. By our original premise $q < \frac{n}{3}$. Therefore all such $\bar{\Gamma}_{n-q,a} = 0$. We are left only with

$$\{(-1)^{q+1}\bar{\Gamma}_{n-q,q+1}, ..., (-1)^{n-q-1}\bar{\Gamma}_{n-q,n-q-1}, (-1)^{n-q}\bar{\Gamma}_{n-q,n-q}, (-1)^{n-q+1}\bar{\Gamma}_{n-q,n-q+1}, ...(-1)^{n-1}\bar{\Gamma}_{n-q,n-1}\}$$

The only non-zero entries $\bar{\Gamma}_{n-q,a}$ for $a \neq n-q$ in this set are entries such that $a \pm d = n - q$ for some $d|q$. However, for entries $\bar{\Gamma}_{n-q,n-q+d}$, the term $d \leq q-1$ and for entries $\bar{\Gamma}_{n-q,n-q-d}$, the term $n - q > q \geq d$ which implies all $d|q$ are covered on the $-d$ side as opposed to the $+d$ side. Therefore

$$|\bar{\Gamma}^{q}_{q}| = (-1)^{n-q}\bar{\Gamma}_{n-q,n-q}|\bar{\Gamma}^{q,n-q}_{q,n-q}| + \sum (-1)^{n-q-d}\bar{\Gamma}_{n-q,n-q-d}|\bar{\Gamma}^{q,n-q}_{q,n-q-d}| + \sum_{\substack{d|q \\ d<q}} (-1)^{n-q+d}\bar{\Gamma}_{n-q,n-q+d}|\bar{\Gamma}^{q,n-q}_{q,n-q+d}|$$

2. The Diagonal Entries

By definition of $L(\Gamma)$, the diagonal entries are the sum of the off diagonal entries in its row (or column since its symmetric)

$$\bar{\Gamma}_{i,i} = \sum_{\substack{0<j<i \\ \gcd(i,j,n)|n}} |\bar{\Gamma}_{i,j}| + \sum_{\substack{i<j<n \\ \gcd(i,j,n)|n}} |\bar{\Gamma}_{i,j}|$$

Since $i = 1, 2, 3, ..., n-1$ are the internal nodes of Γ (so not including node 0 and node n) each node i has an edge pair $\{i-d, i\}, \{i, i+d\}$ of equal weight that enters the node and exits it. Then both sums from behind and in front of $i \in (0, n)$ have the exact same terms $\gcd(i, i+d, n)^{-m} = d^{-m}$ where d spans the divisors of $\gcd(i, n)$, as they are the only edge weights that could ever enter or leave i by Definition 1. Therefore

$$\bar{\Gamma}_{i,i} = 2 \sum_{\substack{0<j<i \\ \gcd(i,j,n)|n}} |\bar{\Gamma}_{i,j}| = 2 \sum_{d|\gcd(i,n)} \frac{1}{d^m} = 2\sigma_{-m}(\gcd(i,n))$$

Therefore

$$\bar{\Gamma}_{n-q,n-q} = 2\sigma_{-m}(\gcd(n-q,n)) = 2\sigma_{-m}(q)$$

Substitution yields

$$|\bar{\Gamma}^{q}_{q}| = 2(-1)^{n-q}\sigma_{-2}(q)|\bar{\Gamma}^{q,n-q}_{q,n-q}| + \sum_{\substack{d|q \\ d<q}} (-1)^{n-q-d}\bar{\Gamma}_{n-q,n-q-d}|\bar{\Gamma}^{q,n-q}_{q,n-q-d}| + \sum_{\substack{d|q \\ d<q}} (-1)^{n-q+d}\bar{\Gamma}_{n-q,n-q+d}|\bar{\Gamma}^{q,n-q}_{q,n-q+d}|$$

3. Combining Terms

By Definition 1, $\bar{\Gamma}_{n-q,n-q\pm d} = -\frac{1}{d^m}$

$$|\bar{\Gamma}^q_q| = 2(-1)^{n-q}\sigma_{-2}(q)|\bar{\Gamma}^{q,n-q}_{q,n-q}| + \sum_{d|q}(-1)^{n-q-d-1}\frac{1}{d^2}|\bar{\Gamma}^{q,n-q}_{q,n-q-d}| + \sum_{\substack{d|q\\ d<q}}(-1)^{n-q+d-1}\frac{1}{d^2}|\bar{\Gamma}^{q,n-q}_{q,n-q+d}|$$

By definition of $\sigma_{-m}(q) = \sum_{d|q}\frac{1}{d^m}$

$$|\bar{\Gamma}^q_q| = 2(-1)^{n-q}|\bar{\Gamma}^{q,n-q}_{q,n-q}|\sum_{d|q}\frac{1}{d^m} + \sum_{d|q}(-1)^{n-q-d-1}\frac{1}{d^m}|\bar{\Gamma}^{q,n-q}_{q,n-q-d}| + \sum_{\substack{d|q\\ d<q}}(-1)^{n-q+d-1}\frac{1}{d^m}|\bar{\Gamma}^{q,n-q}_{q,n-q+d}|$$

Combining sums yields

$$|\bar{\Gamma}^q_q| = \left(\sum_{d|q}\frac{2(-1)^{n-q}|\bar{\Gamma}^{q,n-q}_{q,n-q}|}{d^m} + (-1)^{n-q-d-1}\frac{1}{d^m}|\bar{\Gamma}^{q,n-q}_{q,n-q-d}|\right) + \sum_{\substack{d|q\\ d<q}}(-1)^{n-q+d-1}\frac{1}{d^m}|\bar{\Gamma}^{q,n-q}_{q,n-q+d}|$$

Since n, q are both odd

$$|\bar{\Gamma}^q_q| = \left(\sum_{d|q}\frac{2|\bar{\Gamma}^{q,n-q}_{q,n-q}|}{d^m} + \frac{1}{d^m}|\bar{\Gamma}^{q,n-q}_{q,n-q-d}|\right) + \sum_{\substack{d|q\\ d<q}}\frac{1}{d^m}|\bar{\Gamma}^{q,n-q}_{q,n-q+d}|$$

Simplification yields

$$|\bar{\Gamma}^q_q| = \sum_{d|q}\frac{2|\bar{\Gamma}^{q,n-q}_{q,n-q}| + |\bar{\Gamma}^{q,n-q}_{q,n-q-d}|}{d^m} + \sum_{\substack{d|q\\ d<q}}\frac{|\bar{\Gamma}^{q,n-q}_{q,n-q+d}|}{d^m}$$

See Appendix B for verification for small n, q. ☐

Lemma 3. *Let $n > 9$ be an odd composite number such that $n > rad(n)$ and $m > 1$ be an integer. Let $\phi(n)$ be the totient function with $x = \phi(n) + 1$ and $k = n - x$. If*

$$\left(\frac{x}{2} - \frac{1}{x-1}\right)^k 2^{k-\sqrt{x}} < x^{k-1}$$

then $|\bar{\Gamma}|$ is not an integer.

Proof. The proof proceeds by analyzing the p-adic valuation of the determinant through the Schur complement and the arithmetic density of indices sharing factors with n.

1. Shur Compliment We partition $\bar{\Gamma}_m(n)$ into the block A (indices coprime to n) and D (indices sharing a factor with n). Using the Schur complement:

$$|\bar{\Gamma}| = |A| \cdot |D - BA^{-1}B^T|$$

Since A is a tridiagonal matrix describing the path graph on vertices $1, 2, ..., \phi(n)$, the determinant is $|A| = \phi(n) + 1$. Consider the off diagonal entries of row $\gcd(i, j, n) = |i - j| = \left(\frac{n}{d}\right)^m$

$$D_{ij} - K_{ij} = \frac{-1}{(n/d)^m} - \frac{H_{ij}}{\phi(n) + 1} = -\frac{(\phi(n) + 1)d^m + H_{ij}n^m}{n^m(\phi(n) + 1)}$$

and the diagonal entries

$$D_{ii} - K_{ii} = \frac{2\sigma_m(n/d)}{(n/d)^m} - \frac{H_{ii}}{\phi(n) + 1} = \frac{2\sigma_m(n/d)(\phi(n) + 1)d^m - H_{ii}n^m}{n^m(\phi(n) + 1)}$$

Each term in the Leibniz expansion of the determinant is of the form

$$|D - K| = \sum_{\ell \in S} \pm \frac{\prod_{|\ell|}((\phi(n) + 1)d^m + H_{ij}n^m) \prod_{k-|\ell|}(2\sigma_m(n/d)(\phi(n) + 1)d^m - H_{ii}n^m)}{n^{mk}(\phi(n) + 1)^k}$$

where $k = n - 1 - \phi(n)$. For $m > 1$ and d odd we have

$$2d^m < 2\sigma_m(d) < 2\zeta(m)(1 - 2^{-m})d^m \leq \frac{\pi^2}{4}d^m < 3d^m$$

Therefore $2\sigma_m(d)$ carries no factor d^m and therefore the leading coefficient is between two multiples of n^{mk}. Let $2\sigma_m(n/d) = 2(n/d)^m + v$ for $1 \leq v < (n/d)^m$

$$\frac{\prod_{|\ell|}((\phi(n) + 1)d^m + H_{ij}n^m) \prod_{k-|\ell|}((2n^m + v')(\phi(n) + 1) - H_{ii}n^m)}{n^{mk}(\phi(n) + 1)^k} = \frac{S}{n^{mk}x^k}$$

for $v' = vd^m$ where $d^m \leq v' < n^m$.

2. Bound on H

Let $x = \phi(n) + 1$ and let $S_A = \{a_1, a_2, \ldots, a_{\phi(n)}\}$ be the set of indices in $\{1, 2, \ldots, n - 1\}$ coprime to n, ordered such that $a_1 < a_2 < \cdots < a_{\phi(n)}$. The matrix A represents the adjacency structure of these indices. In the context of the Schur complement, $K = \frac{1}{x}H$, where H is the matrix resulting from the quadratic form of the coupling between indices sharing factors with n and those coprime to n.

For a fixed index i sharing a factor with n, the i-th diagonal entry of H is given by:

$$H_{ii} = x \sum_{r=1}^{\phi(n)} \sum_{s=1}^{\phi(n)} B_{ir}(A^{-1})_{rs}B_{is}$$

where B_{ir} is the entry of the coupling matrix corresponding to row index i and column index a_r.

The matrix A is the Laplacian-like matrix for a path graph $P_{\phi(n)}$. Its inverse A^{-1} has entries defined by:

$$(A^{-1})_{rs} = \frac{\min(r,s) \cdot (x - \max(r,s))}{x}$$

where $1 \leq r, s \leq \phi(n)$. Consequently, the scaled entries $x(A^{-1})_{rs}$ are positive integers:

$$x(A^{-1})_{rs} = \min(r,s) \cdot (x - \max(r,s))$$

The maximum value of this expression occurs when $r = s = \lfloor x/2 \rfloor$, yielding a maximum value of approximately $\frac{x^2}{4}$. Specifically, for all r, s:

$$x(A^{-1})_{rs} \leq \left(\frac{x}{2}\right)\left(\frac{x}{2}\right) = \frac{x^2}{4}$$

The entries B_{ir} are defined as:

$$B_{ir} = \begin{cases} -|i - a_r|^{-m} & \text{if } |i - a_r| = \gcd(i, a_r) \text{ and } \gcd(i, a_r) | n \\ 0 & \text{otherwise} \end{cases}$$

Since a_r is coprime to n, the condition $\gcd(i, a_r)|n$ implies $\gcd(i, a_r) = 1$. Thus, B_{ir} is non-zero only if $|i - a_r| = 1$. For any fixed index i, there are at most two indices a_r such that $|i - a_r| = 1$ (namely $i - 1$ and $i + 1$). Let these indices in the coprime set be a_{r_1} and a_{r_2}. If they exist:

$$|B_{ir_1}| = |i - (i \pm 1)|^{-m} = 1^m = 1$$

The sum for H_{ii} collapses because B_{ir} is non-zero for at most two values of r. Let these indices be r_1 and r_2. Then:

$$H_{ii} = \sum_{r \in \{r_1, r_2\}} \sum_{s \in \{r_1, r_2\}} B_{ir} \left[x(A^{-1})_{rs}\right] B_{is}$$

Expanding the sum (assuming two such neighbors exist):

$$H_{ii} = B_{ir_1}^2 x(A^{-1})_{r_1 r_1} + B_{ir_2}^2 x(A^{-1})_{r_2 r_2} + 2B_{ir_1} B_{ir_2} x(A^{-1})_{r_1 r_2}$$

Since $B_{ir}^2 = 1$ and $B_{ir_1} B_{ir_2} = 1$:

$$H_{ii} = r_1(x - r_1) + r_2(x - r_2) + 2\min(r_1, r_2)(x - \max(r_1, r_2))$$

Without loss of generality, let $r_2 = r_1 + 1$. Then:

$$H_{ii} = r_1(x - r_1) + (r_1 + 1)(x - r_1 - 1) + 2r_1(x - r_1 - 1)$$

$$H_{ii} = 4r_1x - 4r_1^2 - 2r_1 + x - 1$$

This quadratic in r_1 reaches its maximum at $r_1 = x/2$. Plugging in $r_1 = \frac{x-1}{2}$:

$$H_{ii} \leq (x-1)x - \frac{(x-1)^2}{2} \cdots < x(x-1)$$

In all cases for $x \geq 2$, the sum of these discrete points satisfies:

$$H_{ii} \leq x(x-1)$$

We now prove that $|\det(H - 2I)| \neq 0$, you must show that 2 is not an eigenvalue of the matrix H. Since $H = xK = x(BA^{-1}B^T)$, this is equivalent to showing that the eigenvalues λ_K of the Schur complement K satisfy $\lambda_K \neq \frac{2}{x}$.

More generally, the smallest eigenvalue of a Schur complement $BA^{-1}B^T$ is bounded by:

$$\lambda_{min}(K) \geq \frac{\lambda_{min}(BB^T)}{\lambda_{max}(A)}$$

- $\lambda_{max}(A) < 4$ (Standard for path Laplacians). - $\lambda_{min}(BB^T) \geq 1$ (Since each row has at least one 1). - Thus, $\lambda_{min}(K) \geq 1/4$. - Therefore, $\lambda_{min}(H) = x\lambda_{min}(K) \geq \frac{x}{4}$.

For any $x > 8$, we have

$$\lambda_{min}(H) > \frac{8}{4} = 2$$

Since $\lambda_{min}(H) > 2$ for all odd composite $n > 9$, the matrix $(H - 2I)$ is positive definite and its determinant is strictly positive. We proceed with the proof excluding $n = 9$. Thus, $|\det(H - 2I)|$ never vanishes for all such $n > 9$.

3. The Form of $|D - K|$

The expansion

$$S = \prod_{|\ell|}(-xd^m - H_{ij}n^m) \prod_{k-|\ell|}(v_d'x + (2 - H_{ii})n^m)$$

is a linear polynomial in n^{mk}. This yields

$$|D - K| = \sum_{\sigma \in S} \pm \left(\frac{c_\sigma n^{mk} + b_\sigma}{x^k n^{mk}} \right)$$

Since v' introduces no new factors of n^m we have

$$\sum \pm c_\sigma = \sum \pm \prod_{|\ell|}(-H_{ij}) \prod_{k-|\ell|}(2 - H_{ii}) = \det(2I - H)$$

All terms in the Leibniz expansion that do not contain an n^m have an x coefficient. Since there are k such coefficients multiplied together we have

$$\sum \pm b = \sum \pm \prod_{|\ell|}(-xd^m) \prod_{k-|\ell|}(v_d'x) = x^k \sum \pm \beta$$

For $2I - H$, since the dimension $k = n - 1 - \phi(n)$ is non-zero and even for odd composite $n \geq 9$ we have $\det(2I - H) = (-1)^k \det(H - 2I) = \det(H - 2I)$. For $\sum \pm b$ we have

$$\sum \pm b = x^k n^{mk} \det(D - I^{(D)})$$

where $I_{i,j}^{(D)} = 0$ and $I_{ii}^{(D)} = 2D'_{ii}$ where $D'_{ii} = d^m$ denotes i as the k_i-th integer greater than 1 with a common factor $\gcd(i, n) = d \in (1, n)$. Therefore

$$|\bar{\Gamma}| = \frac{\det(H - 2I)}{x^{k-1}} + x \det(D - I^{(D)})$$

The least common multiple of the denominators d^m are

$$\prod_{\substack{d|n \\ 1<d<n}} d^m = (n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1}$$

Therefore

$$(n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1} |\det(D - I^{(D)})| \in \mathbb{Z}^+$$

Multiplying both sides yields

$$(n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1} |\bar{\Gamma}| = (n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1} \frac{\det(H - 2I)}{x^{k-1}} + x (n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1} \det(D - I^{(D)})$$

Which becomes

$$(n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1} |\bar{\Gamma}| \equiv (n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1} \frac{\det(H - 2I)}{x^{k-1}} \mod x$$

4. Non-Integrality Criterion
   Let $n > \operatorname{rad}(n)$. Then

$$\gcd(n, x) = \gcd\left(\frac{n}{\operatorname{rad}(n)} \operatorname{rad}(n), \frac{n}{\operatorname{rad}(n)} \prod_{p|n} (p - 1) + 1 \right) = 1$$

Therefore $\gcd(n, x) = 1$ for all odd composite n and $m \geq 1$ such that $n > \operatorname{rad}(n)$. For our congruence

$$(n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1} |\bar{\Gamma}| \equiv (n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1} \frac{\det(H - 2I)}{x^{k-1}} \mod x$$

we can divide both sides by $(n^m)^{\lfloor \frac{\tau(n)}{2} \rfloor - 1}$ implying

$$|\bar{\Gamma}| \equiv \frac{\det(H - 2I)}{x^{k-1}} \mod x$$

Therefore

$$\frac{|\det(H - 2I)|}{x^{k-1}} = M \in \mathbb{Z}^+$$

Therefore $|\det(H-2I)| \geq x^{k-1}$. Let $x^{k-1} \leq 2^{f(x)}(x-1)^k$. Then

$$x^{1-\frac{1}{n-x}} \leq 2^{\frac{f(x)}{n-x}}(x-1)$$

Let $f(x) = \sqrt{x}$. Since $\phi(n) \geq \sqrt{n}$ for $n \neq 2, 6$ we have $n \leq (x-1)^2$

$$x^{1-\frac{1}{(x-1)^2-x}} \leq 2^{\frac{\sqrt{x}}{(x-1)^2-x}}(x-1)$$

a contradiction for all $x \geq 6.03643$. Since we deal only with the integers this holds only for $x \geq 7$. We now analyze $\phi(n) \geq 6$. These are all odd $n \geq 9$ we have $x^{k-1} > 2^{f(x)}(x-1)^k$. Therefore $\det(H-2I) > (x-1)^k$. We take the valuation of M

$$v_p(M) = v_p\left(\frac{|\det(H-2I)|}{x^{k-1}}\right)$$

Let $p|x$. Then $\gcd(p, x-1) = 1$. Since p is odd, powers of 2 do not effect the valuation. Therefore

$$v_p(M) = v_p\left(\frac{|\det(H-2I)|}{2^{f(x)}(x-1)^k}\right) - (k-1)v_p(x)$$

Since $H_{ii} \geq x-1 = \phi(n) \geq 2$ the diagonal entries of $H-2I$ are positive. We have $H_{ii} \leq x(x-1)$. By Hadamard's bounds

$$|\det(H-2I)| \leq \prod (H_{ii}-2) \leq (x(x-1)-2)^k$$

Substitution yields

$$\frac{(x(x-1)-2)^k}{2^{f(x)}(x-1)^k} = \left(\frac{x}{2} - \frac{1}{x-1}\right)^k 2^{k-f(x)}$$

Since $\det(H-2I)$ does not vanish, this implies there must be a non-zero integer multiple of x^{k-1} on the interval

$$\left(x^{k-1}, \left(\frac{x}{2} - \frac{1}{x-1}\right)^k 2^{k-f(x)}\right)$$

for $v_p(M)$ to have a non-negative valuation. Therefore, if

$$\left(\frac{x}{2} - \frac{1}{x-1}\right)^k 2^{k-f(x)} < x^{k-1}$$

then M is not an integer, implying $|\bar{\Gamma}|$ is not an integer. Substitution yields

$$\left(\frac{x}{2} - \frac{1}{x-1}\right)^k 2^{k-\sqrt{x}} < x^{k-1}$$

Let $n = 9, m = 2$. Then $k = 2, x = 7$

$$\left(\frac{7}{2} - \frac{1}{6}\right)^2 2^{2-\sqrt{6}} \approx 8.1367 > 7^{2-1} = 7$$

This correctly yields an inconclusive result for the integrality of $|\bar{\Gamma}_2(9)|$ which is 16. For $n = 25, m = 2$ we have $k = 4, x = 5$.

$$\left(\frac{5}{2} - \frac{1}{4}\right)^4 2^{4-\sqrt{5}} \approx 87.04 < 5^{4-1} = 125$$

This correctly predicts that $|\bar{\Gamma}_2(25)|$ is not an integer, which can be seen in Appendix 3 for small values of n, m. $\square$

Theorem 1. *There are no odd perfect numbers $n > 1$.*

Proof. From Lemma 1 we take $m = 2$

$$\sum_{\substack{t|n \\ t<n}} \frac{\bar{\Gamma}^{-1}_{t,q}}{t^2} = 1 - \frac{q}{n}$$

Let n be a perfect number and $q|n$.

$$\frac{\bar{\Gamma}^{-1}_{q,q}}{q^2} + \sum_{\substack{t|n \\ t<n \\ t\neq q}} \frac{\bar{\Gamma}^{-1}_{t,q}}{t^2} = 1 - \frac{q}{n}$$

By definition of matrix minors and the inverse of a matrix

$$\frac{|\bar{\Gamma}^q_q|}{q^2} + \sum_{\substack{t|n \\ t<n \\ t\neq q}} (-1)^{t+q} \frac{|\bar{\Gamma}^q_t|}{t^2} = |\bar{\Gamma}| \left(1 - \frac{q}{n}\right)$$

Let n be odd, then for $t|n$, $t + q$ is always even

$$\frac{|\bar{\Gamma}^q_q|}{q^2} + \sum_{\substack{t|n \\ t<n \\ t\neq q}} \frac{|\bar{\Gamma}^q_t|}{t^2} = |\bar{\Gamma}| \left(1 - \frac{q}{n}\right)$$

By definition of a perfect number

$$\sum_{\substack{t|n \\ t\neq q,n}} \frac{t}{n} = 1 - \frac{q}{n}$$

Substitution yields

$$\frac{|\bar{\Gamma}^q_q|}{q^2} + \sum_{\substack{t|n\\ t<n\\ t\neq q}} \frac{|\bar{\Gamma}^q_t|}{t^2} = |\bar{\Gamma}| \sum_{\substack{t|n\\ t\neq q,n}} \frac{t}{n}$$

Combining both sums, since $t \neq q, t < n$ is the same as $t \neq q, n$

$$\frac{|\bar{\Gamma}^q_q|}{q^2} = \sum_{\substack{t|n\\ t\neq q,n}} \left(\frac{t}{n}|\bar{\Gamma}| - \frac{|\bar{\Gamma}^q_t|}{t^2}\right)$$

Factoring out t^{-2}

$$\frac{|\bar{\Gamma}^q_q|}{q^2} = \sum_{\substack{t|n\\ t\neq q,n}} \frac{1}{t^2}\left(\frac{t^3}{n}|\bar{\Gamma}| - |\bar{\Gamma}^q_t|\right)$$

Let

$$f(t) = \frac{t^3}{n}|\bar{\Gamma}| - |\bar{\Gamma}^q_t|$$

Let $n = p^{4\lambda+1}Q^2$ be the Euler form of an odd perfect number $n > 1$ where $\gcd(p, Q) = 1$ and $p \equiv 1 \pmod 4$ is prime. Let $q = p^{4\lambda+1}$. Then

$$\frac{|\bar{\Gamma}^{p^{4\lambda+1}}_{p^{4\lambda+1}}|}{(p^{4\lambda+1})^2} = \sum_{\substack{t|n\\ t\neq p^{4\lambda+1},p^{4\lambda+1}Q^2}} \frac{f(t)}{t^2}$$

The sum on the RHS can be partitioned into divisors $t = p^a d$ where $d|Q^2$. For $a = 4\lambda + 1$, we then only take $1 < d < Q^2$.

$$\sum_{\substack{t|n\\ t\neq p^{4\lambda+1},p^{4\lambda+1}Q^2}} \frac{f(t)}{t^2} = \sum_{d|Q^2}\sum_{a=0}^{4\lambda} \frac{f(p^a d)}{(p^a d)^2} + \sum_{\substack{d|Q^2\\ 1<d<Q^2}} \frac{f(p^{4\lambda+1}d)}{(p^{4\lambda+1}d)^2}$$

Taking out the $d = 1, d = Q^2$ terms yields

$$\sum_{\substack{t|n\\ t\neq p^{4\lambda+1},p^{4\lambda+1}Q^2}} \frac{f(t)}{t^2} = \sum_{a=0}^{4\lambda} \frac{f(p^a)}{(p^a)^2} + \frac{f(p^aQ^2)}{(p^aQ^2)^2} + \sum_{\substack{d|Q^2\\ 1<d<Q^2}} \sum_{a=0}^{4\lambda} \frac{f(p^a d)}{(p^a d)^2} + \sum_{\substack{d|Q^2\\ 1<d<Q^2}} \frac{f(p^{4\lambda+1}d)}{(p^{4\lambda+1}d)^2}$$

Combining the sums over divisors $1 < d < Q^2$

$$\sum_{\substack{t|n\\ t\neq p^{4\lambda+1},p^{4\lambda+1}Q^2}} \frac{f(t)}{t^2} = \sum_{a=0}^{4\lambda} \frac{f(p^a)}{(p^a)^2} + \frac{f(p^aQ^2)}{(p^aQ^2)^2} + \sum_{\substack{d|Q^2\\ 1<d<Q^2}} \sum_{a=0}^{4\lambda+1} \frac{f(p^a d)}{(p^a d)^2}$$

Substitution yields

$$\frac{|\bar{\Gamma}^{p^{4\lambda+1}}_{p^{4\lambda+1}}|}{(p^{4\lambda+1})^2} = \sum_{a=0}^{4\lambda} \frac{f(p^a)}{(p^a)^2} + \frac{f(p^aQ^2)}{(p^aQ^2)^2} + \sum_{a=0}^{4\lambda+1} \sum_{\substack{d|Q^2 \\ 1<d<Q^2}} \frac{f(p^a d)}{(p^a d)^2}$$

Since $Q > 1$ is odd then $Q^2 \equiv 1 \pmod 8$. Therefore $Q \geq 9$, and $q = \frac{n}{Q^2} < \frac{n}{3}$. We can now invoke Lemma 3. Let

$$g(d) = |\bar{\Gamma}^{p^{4\lambda+1}, n-p^{4\lambda+1}}_{p^{4\lambda+1}, n-p^{4\lambda+1}-d}|$$

Then by Lemma 2

$$|\bar{\Gamma}^{p^{4\lambda+1}}_{p^{4\lambda+1}}| = \sum_{d|p^{4\lambda+1}} \frac{2g(0) + g(d)}{d^2} + \sum_{\substack{d|p^{4\lambda+1} \\ d<p^{4\lambda+1}}} \frac{g(-d)}{d^2}$$

Since p is prime this can be rewritten as a sum over the prime factors p^a.

$$|\bar{\Gamma}^{p^{4\lambda+1}}_{p^{4\lambda+1}}| = \sum_{a=0}^{4\lambda+1} \frac{2g(0) + g(p^a)}{p^{2a}} + \sum_{a=0}^{4\lambda} \frac{g(-p^a)}{p^{2a}}$$

Substitution yields

$$p^{-2(4\lambda+1)} \left(\sum_{a=0}^{4\lambda+1} \frac{2g(0) + g(p^a)}{p^{2a}} + \sum_{a=0}^{4\lambda} \frac{g(-p^a)}{p^{2a}} \right) = \sum_{a=0}^{4\lambda} \frac{f(p^a)}{(p^a)^2} + \frac{f(p^aQ^2)}{(p^aQ^2)^2} + \sum_{a=0}^{4\lambda+1} \sum_{\substack{d|Q^2 \\ 1<d<Q^2}} \frac{f(p^a d)}{(p^a d)^2}$$

Consider the function $h(a)$. If we have

$$\sum_{a=0}^{4\lambda+1} \left(\frac{h(a)}{p^{2a}} - \frac{h(4\lambda+1)}{p^{2(4\lambda+1)}} \right) + \sum_{a=0}^{4\lambda} \left(-\frac{h(a)}{p^{2a}} + \frac{h(4\lambda+1)}{p^{2(4\lambda+1)}} \right)$$

then this simplifies to

$$\frac{h(4\lambda+1)}{p^{2(4\lambda+1)}} - (4\lambda+2)\frac{h(4\lambda+1)}{p^{2(4\lambda+1)}} + (4\lambda+1)\frac{h(4\lambda+1)}{p^{2(4\lambda+1)}} = 0$$

Adding this 0 to our identity yields the following on the LHS:

$$p^{-2(4\lambda+1)} \left(\sum_{a=0}^{4\lambda+1} \frac{2g(0) + g(p^a) + h(a)}{p^{2a}} - \frac{h(4\lambda+1)}{p^{2(4\lambda+1)}} + \sum_{a=0}^{4\lambda} \frac{g(-p^a) - h(a)}{p^{2a}} + \frac{h(4\lambda+1)}{p^{2(4\lambda+1)}} \right)$$

and the following remains on the RHS

$$\sum_{a=0}^{4\lambda} \frac{f(p^a)}{(p^a)^2} + \frac{f(p^aQ^2)}{(p^aQ^2)^2} + \sum_{a=0}^{4\lambda+1} \sum_{\substack{d|Q^2 \\ 1<d<Q^2}} \frac{f(p^ad)}{(p^ad)^2}$$

Therefore there exists some function $h(a)$ that satisfies term by term

$$p^{-2(4\lambda+1)} \left(\frac{2g(0) + g(p^a) + h(a)}{p^{2a}} - \frac{h(4\lambda+1)}{p^{2(4\lambda+1)}} \right) = \sum_{\substack{d|Q^2 \\ 1<d<Q^2}} \frac{f(p^ad)}{(p^ad)^2}$$

and

$$p^{-2(4\lambda+1)} \left(\frac{g(-p^a) - h(a)}{p^{2a}} + \frac{h(4\lambda+1)}{p^{2(4\lambda+1)}} \right) = \frac{f(p^a)}{(p^a)^2} + \frac{f(p^aQ^2)}{(p^aQ^2)^2}$$

for all $a \in [0, 4\lambda+1]$. Adding both identities to each other eliminates the h terms

$$p^{-2(4\lambda+1)} \left(\frac{2g(0) + g(p^a) + g(-p^a)}{p^{2a}} \right) = \frac{f(p^a)}{(p^a)^2} + \frac{f(p^aQ^2)}{(p^aQ^2)^2} + \sum_{\substack{d|Q^2 \\ 1<d<Q^2}} \frac{f(p^ad)}{(p^ad)^2}$$

After eliminating the auxiliary function $h(a)$, we obtain the following identity for all $a \in \{0, 1, \ldots, 4\lambda+1\}$:

$$p^{-2(4\lambda+1)} \left(\frac{2g(0) + g(p^a) + g(-p^a)}{p^{2a}} \right) = \sum_{d|Q^2} \frac{f(p^ad)}{p^{2a}d^2}$$

To simplify the resulting expression, we observe that since the identity

$$p^{-2(4\lambda+1)} \left(2g(0) + g(x) + g(-x)\right) = \sum_{d|Q^2} \frac{f(xd)}{d^2}$$

holds for the discrete set of values $x = p^a$ for $a \in \{0, \ldots, 4\lambda+1\}$. Sending $x \to -x$ we have

$$p^{-2(4\lambda+1)} \left(2g(0) + g(-x) + g(x)\right) = \sum_{d|Q^2} \frac{f(-xd)}{d^2} = \sum_{d|Q^2} \frac{f(xd)}{d^2}$$

This implies $f(xd) = f(-xd)$. Define the auxiliary function:

$$W(x) = p^{2(4\lambda+1)} \left(\sum_{d|Q^2} \frac{f(xd)}{d^2} \right) - \left(2g(0) + g(x) + g(-x)\right)$$

We now show $W(x)$ is a polynomial in x by showing $|\bar{\Gamma}^t_q|$ is a polynomial in t. Let $q = a/b$. By Remark 1 and Lemma 1

$$\sum_{\substack{t|n\\t<n}} t^{-2}\bar{\Gamma}(n)^{-1}_{t,a/b} = 1 - \frac{a}{bn} = \sum_{\substack{t|bn\\t<bn}} t^{-2}\bar{\Gamma}(bn)^{-1}_{t,a} = \sum_{\substack{t|\ell\\t<\ell}} t^{-2}\bar{\Gamma}(\ell)^{-1}_{t,a}$$

Therefore the network topology $\Gamma(n)$ is defined for all rational nodes $a, b \in \mathbb{Q} \cap [0, n]$ via the consistent rule $|a - b| = \gcd(a, b)|n$. Recall

$$n \sum_{\substack{t|n\\t<n}} (-1)^{t+q} t^{-2} |\bar{\Gamma}^t_q| = |\bar{\Gamma}|(n - q)$$

Each term in the sum is non-zero by Proposition 1 and defined for all rational inputs t, q by Remark 1. The RHS is linear in q and therefore each term in the summation must be a polynomial in q. By symmetry, if $|\bar{\Gamma}^t_q|$ is a polynomial in q, then $|\bar{\Gamma}^q_t|$ is a polynomial in t. Therefore, $|\bar{\Gamma}^t_t|$ is a polynomial and by Lemma 2, can be written as the sum of positive terms $g(d)$ which must likewise be polynomials on d. The sum of polynomials is therefore a polynomial. We therefore have the polynomial f such that

$$f_q(t) = \frac{t^3}{n}|\bar{\Gamma}| - (-1)^{t+q}|\bar{\Gamma}^q_t|$$

Rearranging $f(t)$ and squaring both sides yields

$$\left(f_q(t) - \frac{t^3}{n}|\bar{\Gamma}|\right)^2 = |\bar{\Gamma}^q_t|^2$$

By Dodgeson condensation

$$|\bar{\Gamma}||\bar{\Gamma}^{t,q}_{t,q}| + |\bar{\Gamma}^q_t|^2 = |\bar{\Gamma}^t_t||\bar{\Gamma}^q_q|$$

By Proposition 1, $|\bar{\Gamma}| > 0, |\bar{\Gamma}^{t,q}_{t,q}| \geq 0$

$$|\bar{\Gamma}^q_t|^2 \leq |\bar{\Gamma}^t_t||\bar{\Gamma}^q_q|$$

Since the minor is positive definite by Proposition 1

$$|\bar{\Gamma}^t_t| \leq \prod \operatorname{diag}(\bar{\Gamma}^t_t) = \prod_{i \neq t} 2\sigma_{-2}(\gcd(i, n))$$

this simplifies to

$$|\bar{\Gamma}^t_t| \leq \frac{1}{2\sigma_{-2}(\gcd(t, n))} \prod_{i=1}^{n-1} 2\sigma_{-2}(\gcd(i, n))$$

and $\sigma_{-2}(\gcd(t,n)) \geq \frac{1}{t^2}$

$$|\bar{\Gamma}_t^t| \leq \frac{t^2}{2}\prod_{i=1}^{n-1} 2\sigma_{-2}(\gcd(i,n))$$

Substitution yields

$$|\bar{\Gamma}_t^q|^2 \leq t^2 \frac{|\bar{\Gamma}_q^q|}{2}\prod_{i=1}^{n-1} 2\sigma_{-2}(\gcd(i,n)) = t^2 C_q$$

Substitution into f yields

$$\left(f_q(t) - \frac{t^3}{n}|\bar{\Gamma}|\right)^2 \leq t^2 C_q$$

The above inequality yields

$$\frac{t^3}{n}|\bar{\Gamma}| - t\sqrt{C_q} \leq f(t) \leq \frac{t^3}{n}|\bar{\Gamma}| + t\sqrt{C_q}$$

Thus, $f(t) = \frac{t^3}{n}|\bar{\Gamma}| - |\bar{\Gamma}_t^q|$ is a polynomial of degree at most 3 in t bounded from above abd below. The function $g(x) = |\bar{\Gamma}_{q,n-q-x}^{q,n-q}|$ is a minor of dimension $n-3$. Since $|\bar{\Gamma}_q^q|$ is positive definite M matrix, then by Proposition 1 and Dodgeson condensation

$$g(x)^2 = |\bar{\Gamma}_{q,n-q-x}^{q,n-q}|^2 \leq |(\bar{\Gamma}_q^q)_{n-q}^{n-q}||(\bar{\Gamma}_q^q)_{n-q-x}^{n-q-x}|$$

along with Hadamard bounds

$$g(x)^2 \leq \frac{|(\bar{\Gamma}_q^q)_{n-q}^{n-q}|}{4\sigma_{-2}(q)\sigma_{-2}(\gcd(n-q-x,n))}\prod_{i=1}^{n-1} 2\sigma_{-2}(\gcd(i,n))$$

and since

$$\sigma_{-2}(\gcd(n-q-x,n)) = \sigma_{-2}(\gcd(q+x,n)) \geq \frac{1}{(q+x)^2}$$

we have

$$g(x)^2 \leq \frac{(x+q)^2|(\bar{\Gamma}_q^q)_{n-q}^{n-q}|}{4\sigma_{-2}(q)}\prod_{i=1}^{n-1} 2\sigma_{-2}(\gcd(i,n))$$

the function $g(x)$ is of degree at most 1 as well by similar argument. Therefore, the sum of $f(t)$ and $g(x)$ has degree at most 3, implying:

$$\deg_x(W(x)) = 3$$

In the summation $\sum_{d|Q^2} f(xd)d^{-2}$, each term $f(xd)$ is a polynomial in x of degree 3, so the sum is a polynomial in x of degree 3. By symmetry

$$f(xd) = f(-xd)$$

This is equivalent to

$$\frac{(xd)^3}{n}|\bar{\Gamma}| - |\bar{\Gamma}^{p^{4\lambda+1}}_{xd}| = \frac{(-xd)^3}{n}|\bar{\Gamma}| - |\bar{\Gamma}^{p^{4\lambda+1}}_{-xd}|$$

Rearranging and observing each $t = xd$ we have

$$|\bar{\Gamma}^{p^{4\lambda+1}}_{-t}| = |\bar{\Gamma}^{p^{4\lambda+1}}_{t}| - \frac{2t^3}{n}|\bar{\Gamma}|$$

an extension of the t, q minor to include a non-positive arguments $t < 0$ even though such entries $\bar{\Gamma}^{-1}_{-t,p^{4\lambda+1}}$ are not defined in the inverse Γ^{-1}. We now analyze the value

$$|\bar{\Gamma}^{-1}_{0,p^{4\lambda+1}}|$$

The identity established in the previous steps shows that $W(x) = 0$ for all $x \in \{p^0, p^1, \ldots, p^{4\lambda+1}\}$. By symmetry, $W(x)$ also vanishes at the $4\lambda+2$ reflected points $\{-p^0, \ldots, -p^{4\lambda+1}\}$, providing a total of $8\lambda+4$ distinct roots. For any $\lambda \geq 0$, we have $8\lambda + 4 \geq 4 > 3$, which is greater than the degree. Since $f(t)$ and $g(x)$ are bounded by cubic polynomials for any size circuit n and their rational points t, x map to integer points t', x' with outputs $f^*(t'), g^*(x')$ that are likewise bounded by cubic functions via Dodgeson condensation and Hadamard bounds as above, the function $W(x)$ is constrained by a cubic polynomial over $\mathbb{Q}$. By the Identity Theorem for Polynomials over the field $\mathbb{Q}$, since a polynomial with more roots that its degree is identically zero, $W(x) \equiv 0$ for all x. Substituting $x = 0$ into the g, f identity:

$$p^{-2(4\lambda+1)}\left(2g(0) + g(0) + g(0)\right) = \sum_{d|Q^2} \frac{f(0)}{d^2}$$

This collapses the sum into the single expression:

$$4p^{-2(4\lambda+1)}g(0) = f(0)\sigma_{-2}(Q^2)$$

which isolates the impact of the divisor sum on the boundary potential. Multiplying through by $p^{2(4\lambda+1)}$:

$$4g(0) = p^{2(4\lambda+1)}f(0)\sigma_{-2}(Q^2)$$

By Lemma 1, for any s we have

$$\sum_{\substack{t|n \\ t<n}} \frac{\bar{\Gamma}^{-1}_{t,s}}{t^2} = 1 - \frac{s}{n}$$

Let $s = 0$

$$\sum_{\substack{t|n \\ t<n}} \frac{\bar{\Gamma}^{-1}_{t,0}}{t^2} = 1$$

Since n is perfect, regardless of parity, the divisor identity

$$\sum_{\substack{t|n \\ t<n}} \frac{t}{n} = 1$$

holds. On the other hand, by construction of the network $\Gamma(n)$ and the definition of the Green's function at the boundary node 0, we have

$$\sum_{\substack{t|n \\ t<n}} \frac{\bar{\Gamma}^{-1}_{t,0}}{t^2} = 1.$$

These two equalities express the same conservation law: total current balance at the boundary node relative to the backbone potential. Crucially, this is not an isolated equality of sums. The identity established in Lemma 1 yields a linear Dirichlet system that uniquely determines the boundary potentials. Let $\mathbf{b}$ be the vector indexed by divisors $t \mid n$, $t < n$, with components

$$b_t = \frac{1}{t^2},$$

and let $\mathbf{x}$ denote the backbone potential distribution with entries

$$x_q = 1 - \frac{q}{n}, \qquad q = 1, \ldots, n-1.$$

Lemma 1 may be written in matrix form as

$$\bar{\Gamma}^{-1}\mathbf{b} = \mathbf{x}.$$

Since $\bar{\Gamma}$ is nonsingular, this system has a *unique* solution for the potential vector $\bar{\Gamma}^{-1}_{\cdot,0}$ compatible with the prescribed backbone voltages.

For a perfect number n, the arithmetic identity $\sum_{t|n,\, t<n} t/n = 1$ exactly matches the total current constraint imposed by this Dirichlet problem. Harmonic consistency therefore requires that the contribution from each divisor node t coincide with its prescribed arithmetic weight; otherwise the resulting potential would fail to solve the linear system defined above. In particular, the boundary current cannot be redistributed without violating uniqueness of the Dirichlet solution.

Consequently, for each divisor $t \mid n$, $t < n$, the boundary potential satisfies

$$\frac{\bar{\Gamma}^{-1}_{t,0}}{t^2} = \frac{t}{n}, \qquad \text{or equivalently} \qquad \bar{\Gamma}^{-1}_{t,0} = \frac{t^3}{n}.$$

By the cofactor formula for the inverse of a matrix,

$$\bar{\Gamma}^{-1}_{t,0} = (-1)^{t+0} \frac{|\bar{\Gamma}^t_0|}{|\bar{\Gamma}|},$$

and hence

$$|\bar{\Gamma}_0^t| = (-1)^t \frac{t^3}{n} |\bar{\Gamma}|.$$

When n is odd, every divisor t is odd, so this simplifies to

$$|\bar{\Gamma}_0^t| = -\frac{t^3}{n} |\bar{\Gamma}|.$$

Setting $t = p^{4\lambda+1}$ yields

$$|\bar{\Gamma}_0^{p^{4\lambda+1}}| = -\frac{p^{12\lambda+3}}{n} |\bar{\Gamma}|.$$

Finally, although the reduced Laplacian $\bar{\Gamma}$ is defined only on interior nodes $\{1, \ldots, n-1\}$, the Dirichlet problem on $\Gamma(n)$ determines a unique extension of the potential to the boundary node 0. We therefore interpret the value

$$f(0) := -|\bar{\Gamma}_0^{p^{4\lambda+1}}|$$

as the uniquely determined boundary potential enforced by harmonic consistency. Such an extension exists if and only if n satisfies the perfect number condition; for non-perfect n, the corresponding linear constraints are incompatible and no consistent boundary value exists. Thus

$$f(0) = \frac{p^{12\lambda+3}}{n} |\bar{\Gamma}|.$$

Substitution yields

$$4g(0) = \frac{p^{20\lambda+5}}{n} |\bar{\Gamma}| \sigma_{-2}(Q^2)$$

Substitution of $g(0)$

$$4|\bar{\Gamma}_{p^{4\lambda+1}, n-p^{4\lambda+1}}^{p^{4\lambda+1}, n-p^{4\lambda+1}}| = \frac{p^{20\lambda+5}}{n} |\bar{\Gamma}| \sigma_{-2}(Q^2)$$

Substitution of $n = p^{4\lambda+1} Q^2$

$$4|\bar{\Gamma}_{p^{4\lambda+1}, n-p^{4\lambda+1}}^{p^{4\lambda+1}, n-p^{4\lambda+1}}| = \frac{p^{16\lambda+4}}{Q^2} |\bar{\Gamma}| \sigma_{-2}(Q^2)$$

By definition of $\sigma_{-2}(Q^2) = \frac{\sigma_2(Q^2)}{Q^2}$

$$4|\bar{\Gamma}_{p^{4\lambda+1}, n-p^{4\lambda+1}}^{p^{4\lambda+1}, n-p^{4\lambda+1}}| = \frac{p^{16\lambda+4}}{Q^6} |\bar{\Gamma}| \sigma_2(Q^2)$$

Let $\Gamma' = n^2 \bar{\Gamma}$. Multiplying each side by $n^{2(n-1)}$

$$4n^{2(n-1)} |\bar{\Gamma}_{p^{4\lambda+1}, n-p^{4\lambda+1}}^{p^{4\lambda+1}, n-p^{4\lambda+1}}| = \frac{p^{16\lambda+4} n^{2(n-1)}}{Q^6} |\bar{\Gamma}| \sigma_2(Q^2)$$

Since entry is scaled by n^2, the determinants of the $n-1$ dimensional $\bar{\Gamma}$ and its $n-3$ dimensional minor yields:

$$|\Gamma'| = n^{2(n-1)}|\bar{\Gamma}| \quad , \quad |\Gamma'^{p^{4\lambda+1},n-p^{4\lambda+1}}_{p^{4\lambda+1},n-p^{4\lambda+1}}| = n^{2(n-3)}|\bar{\Gamma}^{p^{4\lambda+1},n-p^{4\lambda+1}}_{p^{4\lambda+1},n-p^{4\lambda+1}}|$$

Substitution yields

$$4n^4|\Gamma'^{p^{4\lambda+1},n-p^{4\lambda+1}}_{p^{4\lambda+1},n-p^{4\lambda+1}}| = \frac{p^{16\lambda+4}}{Q^6}|\Gamma'|\sigma_2(Q^2)$$

Simplifying by $n^4 = p^{4(4\lambda+1)}Q^8$ yields

$$4Q^{14}|\Gamma'^{p^{4\lambda+1},n-p^{4\lambda+1}}_{p^{4\lambda+1},n-p^{4\lambda+1}}| = |\Gamma'|\sigma_2(Q^2)$$

each entry of Γ' is now an integer, and therefore the determinant is an integer. Taking both sides of the equality $\pmod 4$

$$|\Gamma'|\sigma_2(Q^2) \equiv 0 \pmod 4$$

Since $\sigma_2(Q^2)$ is the sum of divisors of Q^2 which is odd and Q^2 has an odd number of divisors, $\sigma_2(Q^2)$ is odd.

$$|\Gamma'| \equiv 0 \pmod 4$$

Factoring out the odd n^2 from each entry yields

$$|\bar{\Gamma}| \equiv 0 \pmod 4$$

This implies $|\bar{\Gamma}|$ is an integer. Since n is an odd composite where $\text{rad}(n) > n$, we can invoke Lemma 3, which states for the $m = 2$ case that $n = p^{4\lambda+1}Q^2$ must be such that

$$\left(\frac{\phi(n)+1}{2} - \frac{1}{\phi(n)}\right)^{n-1-\phi(n)} 2^{n-1-\phi(n)-\sqrt{\phi(n)+1}} \geq (\phi(n)+1)^{n-\phi(n)-2}$$

Rearranging yields

$$\left(1 - \frac{2}{\phi(n)(\phi(n)+1)}\right)^{n-1-\phi(n)} \geq \frac{2^{\sqrt{\phi(n)+1}}}{\phi(n)+1}$$

The base of the exponent n on the LHS does not go past 1, so the value of the LHS does not exceed 1. We therefore take the largest value the LHS could possibly be by looking at the smallest value n in the exponent can be which is $n = 9$. Let $\phi(n) = y$.

$$\left(1 - \frac{2}{y(y+1)}\right)^{9-1-y} \geq \frac{2^{\sqrt{y+1}}}{y+1}$$

This inequality holds only for $y \leq 6.03542$ or $\phi(n) \leq 6$. This leaves only $n = 9$ as the only odd composite that satisfies the above and 9 is not perfect. Therefore, no odd perfect numbers $n > 1$ exist. □

1. Appendix

1.0.1. Small n checks for Lemma 1

For $n = 3, m = 2$, divisors are $1, 3$ and the Laplacian matrix $\bar{\Gamma}$ for the circuit $\Gamma(3)$ is as follows:

$$\bar{\Gamma}(\Gamma(3)) = \begin{pmatrix} 2\sigma_{-2}(1) & -1 \\ -1 & 2\sigma_{-2}(1) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

The inverse $\bar{\Gamma}^{-1}$ yields:

$$(\bar{\Gamma}_3)^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Applying Lemma 1:

$$\sum_{\substack{t|n \\ t\,<\,n}} \frac{\bar{\Gamma}^{-1}_{t,q}}{t^2} = 1 - \frac{q}{n}$$

Substitution using values of $\bar{\Gamma}^{-1}_{t,q}$ for $n = 3$ and $q = 2$ yields:

$$\sum_{\substack{t|3 \\ t\,<\,3}} \frac{\bar{\Gamma}^{-1}_{t,2}}{t^2} = \frac{\bar{\Gamma}^{-1}_{1,2}}{1^2} = \frac{1}{3} = 1 - \frac{2}{3} = 1 - \frac{q}{n}$$

For $n = 6$, the divisors are $1, 2, 3, 6$ the circuit $\Gamma(6)$ can be seen in Figure 2. The Laplacian $\bar{\Gamma}$ of $\Gamma(6)$ is as follows:

$$\bar{\Gamma}(6) = \begin{pmatrix} 2\sigma_{-2}(1) & -1 & 0 & 0 & 0 \\ -1 & 2\sigma_{-2}(2) & -1 & -\frac{1}{4} & 0 \\ 0 & -1 & 2\sigma_{-2}(3) & -1 & 0 \\ 0 & \frac{1}{4} & -1 & 2\sigma_{-2}(2) & -1 \\ 0 & 0 & 0 & -1 & 2\sigma_{-2}(1) \end{pmatrix}$$

Simplification yields:

$$\bar{\Gamma}_6 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & \frac{5}{2} & -1 & -\frac{1}{4} & 0 \\ 0 & -1 & \frac{20}{9} & -1 & 0 \\ 0 & -\frac{1}{4} & -1 & \frac{5}{2} & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The inverse $\bar{\Gamma}^{-1}$ yields:

$$(\bar{\Gamma}_6)^{-1} = \begin{pmatrix} \frac{215}{306} & \frac{62}{153} & \frac{9}{34} & \frac{28}{153} & \frac{14}{153} \\ \frac{62}{153} & \frac{124}{153} & \frac{9}{17} & \frac{56}{153} & \frac{28}{153} \\ \frac{9}{34} & \frac{9}{17} & \frac{63}{68} & \frac{9}{17} & \frac{9}{34} \\ \frac{28}{153} & \frac{56}{153} & \frac{9}{17} & \frac{124}{153} & \frac{62}{153} \\ \frac{14}{153} & \frac{28}{153} & \frac{9}{34} & \frac{62}{153} & \frac{215}{306} \end{pmatrix}$$

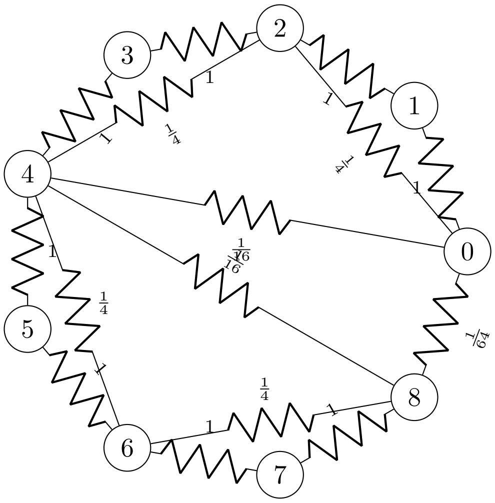

Figure 3: Circuit $\Gamma(8)$ of 9 nodes, and 15 resistors.

Applying Lemma 0.1, substitution using values of $\bar{\Gamma}^{-1}_{t,q}$ for $n = 6$ and $q = 3$ yields:

$$\sum_{\substack{t|6\\ t\,<\,6}} \frac{\bar{\Gamma}^{-1}_{t,3}}{t^2} = \frac{\bar{\Gamma}^{-1}_{1,3}}{1^2} + \frac{\bar{\Gamma}^{-1}_{2,3}}{2^2} + \frac{\bar{\Gamma}^{-1}_{3,3}}{3^2} = \frac{9}{34} + \frac{\frac{9}{17}}{4} + \frac{\frac{63}{68}}{9} = \frac{1}{2} = 1 - \frac{3}{6} = 1 - \frac{q}{n}$$

For $q = 4$, the formula holds as well:

$$\sum_{\substack{t|6\\ t\,<\,6}} \frac{\bar{\Gamma}^{-1}_{t,4}}{t^2} = \frac{\bar{\Gamma}^{-1}_{1,4}}{1^2} + \frac{\bar{\Gamma}^{-1}_{2,4}}{2^2} + \frac{\bar{\Gamma}^{-1}_{3,4}}{3^2} = \frac{28}{153} + \frac{\frac{56}{153}}{4} + \frac{\frac{9}{17}}{9} = \frac{1}{3} = 1 - \frac{4}{6} = 1 - \frac{q}{n}$$

For $n = 8, m = 2$, divisors are $1, 2, 4$ and the Laplacian matrix $\bar{\Gamma}$ for the circuit $\Gamma(8)$ is as follows in Figure 3.

$$\bar{\Gamma}(\Gamma(8)) = \begin{pmatrix} 2\sigma_{-2}(1) & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2\sigma_{-2}(2) & -1 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & -1 & 2\sigma_{-2}(1) & -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & -1 & 2\sigma_{-2}(4) & -1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & -1 & 2\sigma_{-2}(1) & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & -1 & 2\sigma_{-2}(2) & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2\sigma_{-1}(1) \end{pmatrix}$$

Simplification yields:

$$\bar{\Gamma}_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & \frac{5}{2} & -1 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & -1 & \frac{21}{8} & -1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & -1 & \frac{5}{2} & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The inverse $(\bar{\Gamma}_8)^{-1}$ yields:

$$(\bar{\Gamma}_8)^{-1} = \begin{pmatrix} \frac{31}{42} & \frac{10}{21} & \frac{8}{21} & \frac{2}{7} & \frac{3}{14} & \frac{1}{7} & \frac{1}{14} \\ \frac{10}{21} & \frac{20}{21} & \frac{16}{21} & \frac{4}{7} & \frac{3}{7} & \frac{2}{7} & \frac{1}{7} \\ \frac{8}{21} & \frac{16}{21} & \frac{55}{42} & \frac{6}{7} & \frac{9}{14} & \frac{3}{7} & \frac{3}{14} \\ \frac{2}{7} & \frac{4}{7} & \frac{6}{7} & \frac{8}{7} & \frac{6}{7} & \frac{4}{7} & \frac{2}{7} \\ \frac{3}{14} & \frac{3}{7} & \frac{9}{14} & \frac{6}{7} & \frac{55}{42} & \frac{16}{21} & \frac{8}{21} \\ \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{16}{21} & \frac{20}{21} & \frac{10}{21} \\ \frac{1}{14} & \frac{1}{7} & \frac{3}{14} & \frac{2}{7} & \frac{8}{21} & \frac{10}{21} & \frac{31}{42} \end{pmatrix}$$

Substitution using values of $\bar{\Gamma}_{t,q}^{-1}$ for $n = 8$ and $q = 2$ yields:

$$\sum_{\substack{t|8 \\ t\,<\,8}} \frac{\bar{\Gamma}_{t,2}^{-1}}{t^2} = \frac{\bar{\Gamma}_{1,2}^{-1}}{1^2} + \frac{\bar{\Gamma}_{2,2}^{-1}}{2^2} + \frac{\bar{\Gamma}_{4,2}^{-1}}{4^2} = \frac{10}{21} + \frac{\frac{20}{21}}{4} + \frac{\frac{4}{7}}{16} = \frac{3}{4} = 1 - \frac{2}{8} = 1 - \frac{q}{n}$$

For $q = 5$, the formula holds as well:

$$\sum_{\substack{t|8 \\ t\,<\,8}} \frac{\bar{\Gamma}_{t,5}^{-1}}{t^2} = \frac{\bar{\Gamma}_{1,5}^{-1}}{1^2} + \frac{\bar{\Gamma}_{2,5}^{-1}}{2^2} + \frac{\bar{\Gamma}_{4,5}^{-1}}{4^2} = \frac{3}{14} + \frac{\frac{3}{7}}{4} + \frac{\frac{6}{7}}{16} = \frac{3}{8} = 1 - \frac{5}{8} = 1 - \frac{q}{n}$$

These explicit cases confirm that the construction does not fail nor yield contradictions for small n and that Lemma 1 is valid. In all such cases, the network is connected, and the Laplacian matrix is positive definite, as required by general theory [6].

1.0.2. Small n checks for Lemma 2

Let $n = 15$ and $q = 3$. We have for $\bar{\Gamma}$ the following Laplacian matrix

$$\bar{\Gamma}(15) = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & \frac{20}{9} & -1 & 0 & -\frac{1}{9} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \frac{52}{25} & -1 & 0 & 0 & 0 & -\frac{1}{25} \\
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{9} & 0 & -1 & \frac{20}{9} & -1 & 0 & -\frac{1}{9} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{9} & 0 & -1 & \frac{20}{9} & -1 \\
0 & -\frac{1}{9} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{25} & 0 & 0 & 0 & -1 & \frac{52}{25} \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{9} & 0 \\
-1 & \frac{20}{9} & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2
\end{pmatrix}$$

We then eliminate row and column $q = 3$

$$\bar{\Gamma}^3_3 = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1 & \frac{52}{25} & -1 & 0 & 0 & 0 & -\frac{1}{25} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & -1 & \frac{20}{9} & -1 & 0 & -\frac{1}{9} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{9} & 0 & -1 & \frac{20}{9} & -1 & 0 \\
-\frac{1}{9} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{25} & 0 & 0 & 0 & -1 & \frac{52}{25} & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{9} & 0 & -1 \\
\frac{20}{9} & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2
\end{pmatrix}$$

Row $n - q = 12$ in $\bar{\Gamma}$ is the same as row $n - q - 1 = 11$ in $\bar{\Gamma}^3_3$

$$\{0, 0, 0, 0, 0, 0, 0, -\frac{1}{9}, 0, -1, \frac{20}{9}, -1, 0\}$$

Cofactor expansion along this row yields the following alternating sign sequence per entry

$$\{0, -0, +0, -0, +0, -0, +0, -\left(-\frac{1}{9}\right), +0, -(-1), +\frac{20}{9}, -(-1), +0\}$$

Aligning the cofactor to the alternating nonzero entries yields the full Laplacian expansion

$$|\bar{\Gamma}^3_3| = -\left(-\frac{1}{9}\right)|\bar{\Gamma}^{3,9}_{3,12}| - (-1)|\bar{\Gamma}^{3,11}_{3,12}| + \frac{20}{9}|\bar{\Gamma}^{3,12}_{3,12}| - (-1)|\bar{\Gamma}^{3,13}_{3,12}|$$

Rewriting yields

$$|\bar{\Gamma}^3_3| = \frac{1}{3^2}|\bar{\Gamma}^{3,15-3-3}_{3,15-3}| + \frac{1}{1^2}|\bar{\Gamma}^{3,15-3-1}_{3,15-3}| + 2\left(1 + \frac{1}{3^2}\right)|\bar{\Gamma}^{3,12}_{3,12}| + \frac{1}{1^2}|\bar{\Gamma}^{3,15-3+1}_{3,15-3}|$$

Since $q < n/3$ we can invoke Lemma 3

$$|\bar{\Gamma}_q^q| = \sum_{d|q} \frac{2|\bar{\Gamma}_{q,n-q}^{q,n-q}| + |\bar{\Gamma}_{q,n-q-d}^{q,n-q}|}{d^2} + \sum_{\substack{d|q \\ d<q}} \frac{|\bar{\Gamma}_{q,n-q+d}^{q,n-q}|}{d^2}$$

For $q = 3$ we have $d = 1, 3$

$$|\bar{\Gamma}_3^3| = 2\left(1 + \frac{1}{3^2}\right)|\bar{\Gamma}_{3,12}^{3,12}| + \frac{1}{1^2}|\bar{\Gamma}_{3,15-3}^{3,15-3-1}| + \frac{1}{3^2}|\bar{\Gamma}_{3,15-3}^{3,15-3-3}| + \frac{1}{1^2}|\bar{\Gamma}_{3,15-3}^{3,15-3+1}|$$

which matches our cofactor expansion exactly since $\bar{\Gamma}_{a,c}^{a,b} = \bar{\Gamma}_{a,b}^{a,c}$ by symmetry.

1.0.3. Small n checks for Lemma 3

For every prime $n = p$ the matrix $\bar{\Gamma}$ is simply the minor of a cyclic connected graph on $n = p$ nodes $\{0, 1, 2, ..., p-1\}$. By the Matrix Tree Theorem, the value $|\bar{\Gamma}|$ simply counts the number of spanning trees of the graph, which for a cycle graph C_p is simple p, which is an integer. This can be seen with $n = 3$

$$|\bar{\Gamma}| = \det\begin{pmatrix} 2\sigma_{-2}(1) & -1 \\ -1 & 2\sigma_{-2}(1) \end{pmatrix} = \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3$$

For $n = 9$ we have

$$\bar{\Gamma} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \frac{20}{9} & -1 & 0 & -\frac{1}{9} & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{9} & 0 & -1 & \frac{20}{9} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

To find the determinant of the 8×8 matrix $\bar{\Gamma}$, we can observe its structure. The matrix is almost a tridiagonal matrix, but it has two off-diagonal entries at positions $(3, 6)$ and $(6, 3)$ with the value $-\frac{1}{9}$.

Step 1: Use Row Operations or Expansion

Calculating the determinant of an 8×8 matrix by hand using Laplace expansion would be very tedious. However, we can use row reduction to transform it into an upper triangular matrix or use a calculator for precision.

We look at the properties: 1. The values on the main diagonal are mostly 2, except for positions $(3, 3)$ and $(6, 6)$ which are $\frac{20}{9}$. 2. The super-diagonal and sub-diagonal consist of -1. 3. There are specific "bridge" terms $-\frac{1}{9}$ connecting the third and sixth rows/columns.

Step 2: Calculation

The calculation shows that the determinant of the matrix is:

$$\det(\bar{\Gamma}) = 16$$

Below is the table for all odd composite $n < 100$ with $m = 2$ and $m = 3$, where all non-integer values are expressed as reduced fractions.

Table of Determinants (Reduced Fractions)

n	$\det(\bar{\Gamma}(n))$ for $m = 2$	$\det(\bar{\Gamma}(n))$ for $m = 3$
9	16	$\frac{100}{9}$
15	$\frac{144352}{2250}$	$\frac{44259547}{1800000}$
21	$\frac{129135069}{882000}$	$\frac{1524824253}{37171875}$
25	$\frac{1296}{25}$	$\frac{457000}{15625}$
27	$\frac{81028507}{256000}$	$\frac{11219313}{175000}$
33	$\frac{658649274}{980000}$	$\frac{18734689}{195000}$
35	$\frac{1028575}{6125}$	$\frac{93660346}{1954687}$
39	$\frac{1271902517}{920000}$	$\frac{156328549}{1120000}$
45	$\frac{26963969}{2250}$	$\frac{24581507}{88967}$
49	$\frac{5350}{49}$	$\frac{6640511}{120050}$
51	$\frac{1417711474}{255000}$	$\frac{706243477}{2550000}$
55	$\frac{55751747}{121000}$	$\frac{101702387}{1210000}$
57	$\frac{623799618}{57000}$	$\frac{2175525338}{5700000}$
63	$\frac{4875277342}{63000}$	$\frac{4057090599}{6300000}$
65	$\frac{316275231}{422500}$	$\frac{449838285}{4225000}$
69	$\frac{2849389188}{69000}$	$\frac{4851197669}{6900000}$
75	$\frac{55510934872}{750000}$	$\frac{1190451736}{750000}$
77	$\frac{3622296457}{7700000}$	$\frac{76180500}{770000}$
81	$\frac{23198761937}{81000}$	$\frac{11120280871}{8100000}$
85	$\frac{13805192978}{7225000}$	$\frac{1165701955}{7225000}$
87	$\frac{24820800363}{86000}$	$\frac{14493791572}{8700000}$
91	$\frac{614515087}{910000}$	$\frac{109269634}{910000}$
93	$\frac{50837732010}{93000}$	$\frac{20441255748}{9300000}$
95	$\frac{27122682381}{9000000}$	$\frac{17506790983}{90000000}$
99	$\frac{378481596772}{99000}$	$\frac{338854809774}{99000000}$

Table 1: Determinants of $\bar{\Gamma}_m(n)$ for odd composite n. All values are rational but non-integer.

Mathematical Approach

To find these fractions, we represent each entry $\bar{\Gamma}_{i,j}$ as a rational number $\frac{a}{b}$. The determinant is then computed using exact rational arithmetic:

1. **Diagonal:** $2\sigma_{-m}(g) = 2\sum_{d|g} d^{-m} = 2\sum_{d|g} \frac{1}{d^m}$. 2. **Off-Diagonal:** $-\frac{1}{|i-j|^m}$ if the condition $|i-j| = \gcd(i,j)|n$ is met. 3. **Calculation:** We perform Gaussian elimination or use the property of the determinant over the field of rational numbers $\mathbb{Q}$ to ensure that the result is an exact fraction. Finally, we simplify the fraction by dividing both the numerator and denominator by their greatest common divisor.

2. Bibliography

References

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