Guessing that the Riemann Hypothesis is unprovable using Non-Standard Analysis

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Abstract

Riemann Hypothesis has been the unsolved conjecture for 164 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"(B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is 1/2. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I guess the independence (unprovability) of the Riemann Hypothesis. I consider the axiomatic system in which the Riemann Hypothesis holds and the axiomatic system in which the Riemann Hypothesis does not hold. Finally, I guess that the Riemann Hypothesis is unprovable.

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I think in Non-standard analysis and it is delicate at infinity, I think only finite area.

The Riemann Hypothesis is that the real part of the non-trivial zero of the ζ function is 1/2. (Ivić[2]p4)

Proposition 1. *The only non-trivial zero point of the Riemann Zeta function is on $Re(s) = 1/2$.*

If $\zeta(s) = 0, Re(s) = 1/2$, then for infinitesimal ϵ_0, we can set $\zeta(s + \epsilon_0) = 0, Re(s + \epsilon_0) \neq 1/2$. This thing means Riemann Zeta function is continue and holomorphic function at $s + \epsilon_0$ in the Non-Standard Analysis. This condition alone is sufficient to show that the Riemann hypothesis is unprovable. This impossibility of proving includes a scenario that I do not want to think about. It is impossible to prove, but it is possible to disprove it. This means that I can find a counterexample to the zero point from a small number, but the Riemann Hypothesis cannot be proven in this direction.

Note that the Riemann Hypothesis is true or false. However, in some axiomatic system, the Riemann Hypothesis do not hold. So, the positive proposition of the Rieman Hypothesis cannot prove, and the negative proposition of the Riemann Hypothesis is always provable.

If the model in nonstandard analysis includes a positive proof of the Riemann Hypothesis, which has common ground with normal mathematics, there is no way that a negative model can be created in nonstandard analysis. The negative proposition of the Riemann Hypothesis, is provable, but the positive proposition is unprovable since the negation can be created by one of the extensions.

2 On the Equivalence Proposition

Möbius Function Version

Theorem 1. *(Ivić[2]p48, Titchmarsh[5]p370, Theorem 14.25)*

$$the\ Riemann\ Hypothesis \Leftrightarrow \sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

The following is a known short proof:
Define $M(x)$ as follows:

$$M(x) := \sum_{n=1}^{[x]} \mu(n)$$

Then

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \int_{x=0.1}^{\infty} x^{-s} d(M(x))$$

$d(M(x))$ is the Stieltjes integral of $M(x)$.

$$= [M(x)x^{-s}]_{0.1}^{\infty} + s \int_{x=0.1}^{\infty} M(x)x^{-s-1}dx = s \int_{x=0.1}^{\infty} M(x)x^{-s-1}dx$$

Here, we analytically continue $\frac{1}{\zeta(s)}$ until this value is finite. When $\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$ holds, this integral is finite for $Re(s) > 1/2$. We obtain that there are no zeros of the zeta function for $Re(s) > 1/2$. Furthermore, by combining this with a discussion of functional inequalities, we obtain the Riemann hypothesis.

Conversely, if we assume the Riemann hypothesis,

$$s \int_{x=0.1}^{\infty} M(x)x^{-s-1}dx$$

$Re(s) > 1/2 + \epsilon > 1/2$, and is not infinite.

$$|M(x)| < Km^{\frac{1}{2}+\epsilon}$$

We obtain. ϵ can be arbitrarily small.

Let s be a nontrivial zero with real part $1/2$, and let the point $s+\epsilon_0$ to the right of the infinitesimal ϵ_0 be a zero of the Riemann zeta function. Even if there is no clear precedent within a finite range, considering the above proof, $\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$ and $\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon_0+\epsilon})$ are simultaneously and consistently true. These two equations are identical to the previous equations, which, as we will see later, is a manifestation of the unprovability of the Riemann hypothesis.

Another one

$$If\ the\ Riemann\ Hypothesis\ is\ true \Leftrightarrow \sum_{\rho} \frac{Re(\rho)}{|\rho|^2} = \sum_{\rho} \frac{\frac{1}{2}}{|\rho|^2}$$

Assuming the Riemann Hypothesis is true, Since ρ and $1-\rho$ correspond one-to-one,

$$\frac{x}{x^2+y^2} + \frac{(1-x)}{(1-x)^2+y^2} = \frac{\frac{1}{2}}{x^2+y^2} + \frac{\frac{1}{2}}{(1-x)^2+y^2}$$

This is obtained. If the Riemann hypothesis is false, then $x < 1-x$, and

$$(1/2 - x)(\frac{1}{x^2+y^2} - \frac{1}{(1-x)^2+y^2}) > 0$$

From this,

$$\frac{x}{x^2+y^2} + \frac{(1-x)}{(1-x)^2+y^2} < \frac{\frac{1}{2}}{x^2+y^2} + \frac{\frac{1}{2}}{(1-x)^2+y^2}$$

This leads to the question.

Let $\rho = 1/2 + it$ be a zero of the Riemann zeta function, and take the infinitesimal right-hand side, $\rho + \epsilon_0$. Formally, the counterexample in nonstandard analysis is that

$$\sum_{\rho} \frac{Re(\rho + \epsilon_0)}{|\rho + \epsilon_0|^2} = \sum_{\rho} \frac{\frac{1}{2}}{|\rho + \epsilon_0|^2}$$

holds true, and in this sense, the impossibility of proof still appears. Furthermore, counterexamples that approach the real part $1/2$ line infinitely often

pose a challenge to proof, but as this example shows, the difficulty is hidden by the equations.

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