On the Constant Real Part of the Non-Trivial Zeros: A Proof of the Riemann Hypothesis

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Date: February 04, 2026

Abstract

This paper presents a proof of the Riemann Hypothesis by examining the geometric and arithmetic properties of the Dirichlet eta function. By assuming the existence of zeros off the critical line, and analyzing the resulting alternating series in the complex plane, we establish a logical contradiction. The proof relies on insights into the structure of these series, demonstrating that all non-trivial zeros must possess a real part of exactly 1/2.

1 Introduction

The Riemann zeta function is defined by

$\zeta(s) = 1 + 1/2^s + 1/3^s + \ldots$ ($s \in \mathbb{C}$) where Re(s) > 1.

For the domain beyond Re(s) > 1, there is a functional equation that represents the function through analytic continuation except at s = 1.

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s).$$

Within the **critical strip** (0< Re(s) <1), the function is represented by

$$\zeta(s) = (1 - 2^{1-s})^{-1} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}\right) (1).$$

It is established that $\zeta(s)$ has zeros within this critical strip. The Riemann Hypothesis (RH) states that all such nontrivial zeros lie on the critical line, defined by Re(s) = 1/2.

To validate the constant real part of nontrivial zeros, it is enough to study Equation 1.

Equation 1 is zero when

$$\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}\right) = 0 \quad 0 < \mathrm{Re}(s) < 1 \text{ since } (1 - 2^{1-s})^{-1} \neq 0.$$

2 Assumption of Zeros off the Critical Line

Let us assume that the complex number *s*’ is a nontrivial zero of the zeta function.

$s' = \sigma_1 + it$ where σ_1 is any real number such that $1/2 < \sigma_1 < 1$ and $t \in \mathbb{R}^*$.

Let us consider a real number σ_2 such that

$0 < \sigma_2 < 1/2$ and $\sigma_1 + \sigma_2 = 1$.

Define $s'' = \sigma_2 + it$, where *s*’’ shares the same imaginary part as *s*’.

Therefore, *s*’’ is also a zero, a reflection of *s*’ across the critical line $\sigma = 1/2$ as guaranteed by the functional equation.

Consider the zeros s’ and s’’.

$\zeta(s') = (1 - 2^{1-s'})^{-1} (1 - 1/2^{s'} + 1/3^{s'} - 1/4^{s'} + \ldots) = 0.$

$\zeta(s'') = (1 - 2^{1-s''})^{-1} (1 - 1/2^{s''} + 1/3^{s''} - 1/4^{s''} + \ldots) = 0.$

Consider $\zeta(s')/ (1 - 2^{1-s'})^{-1} = 1 - 1/2^{s'} + 1/3^{s'} - 1/4^{s'} + \ldots = 0.$

and $\zeta(s'')/ (1 - 2^{1-s''})^{-1} = 1 - 1/2^{s''} + 1/3^{s''} - 1/4^{s''} + \ldots = 0.$

Let the derived alternating series be S_1 and S_2.

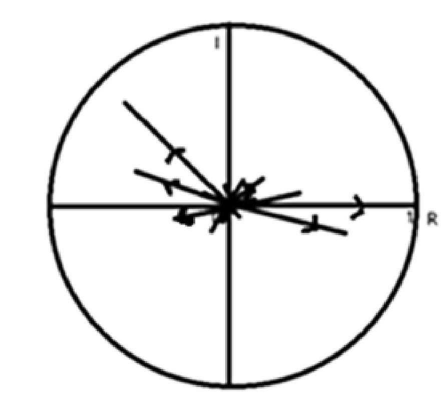
$S_1 = 1 - 1/2^{s'} + 1/3^{s'} - 1/4^{s'} + \ldots = 0.$

$S_2 = 1 - 1/2^{s''} + 1/3^{s''} - 1/4^{s''} + \ldots = 0.$

The series S_1 and S_2 are not identical though $S_1 = S_2 = 0$.

Let us examine the two alternating series S_1 and S_2 in the complex plane to establish a contradiction. From this, we can conclude that all the nontrivial zeros lie on the critical line. This proposed proof of RH is concise with just eleven pages, relying on **logical reasoning,** rather than sophisticated mathematical techniques.

The terms of S_1 and S_2 are distributed around the complex unit circle. There will be infinite terms in the right half-plane just as there are in the left.



A few terms of S_1 and S_2 distributed across a complex unit circle- illustration A

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3 Collinearity and Vector Path

The corresponding terms (vectors in the illustration) of S_1 and S_2 are collinear with the origin. In other words, because they share the same imaginary part *t*, each term of S_2 maintains the exact same angle (argument) with the positive real axis as its counterpart term of S_1.

S_1 illustration:

This illustration is only for a finite number of terms. The moduli and angles are arbitrary. It is just to illustrate how the S_1 graph may look.

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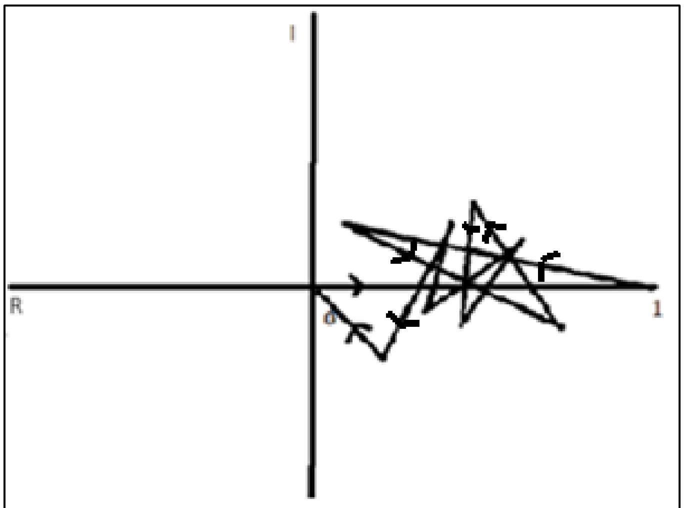
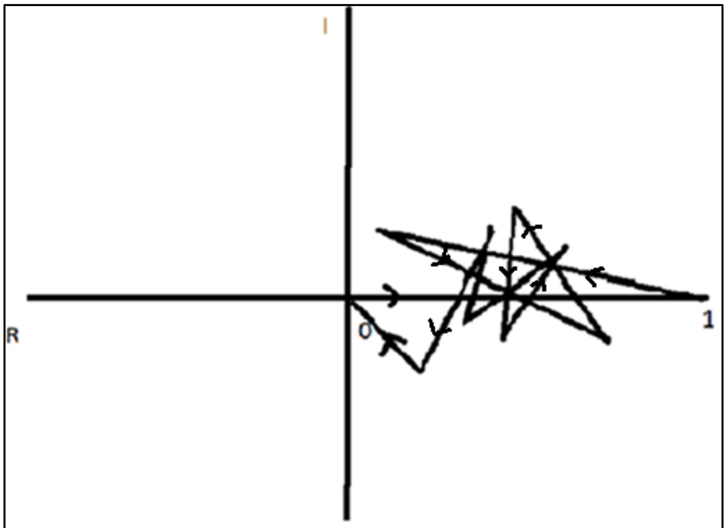


S_1 on a complex plane- illustration 1

S_2 illustration:



S_2 on a complex plane - illustration 2

4 Moduli and Pull Analysis

If we draw both S_1 and S_2 on a single complex plane, the corresponding first terms represented as vectors in the illustration are equal and the corresponding second terms of S_1 and S_2 are collinear. Other corresponding terms would be parallel. With the exception of the first terms, no subsequent corresponding terms of S_1 and S_2 are equal in modulus.

From the second term onwards, each term of S_2 is larger in modulus than its counterpart in S_1.

Consider the quotients of the corresponding terms of S_2 and S_1 upon division.

$1/1 = 1 = q.$

$(-1/2^{s''}) / (-1/2^{s'}) = q_2.$

Third term of S_2/third term of S_1 = $(1/3^{s''}) / (1/3^{s'}) = q_3.$

And so on….

$q_1 < q_2 < q_3 ….$

are unique real values.

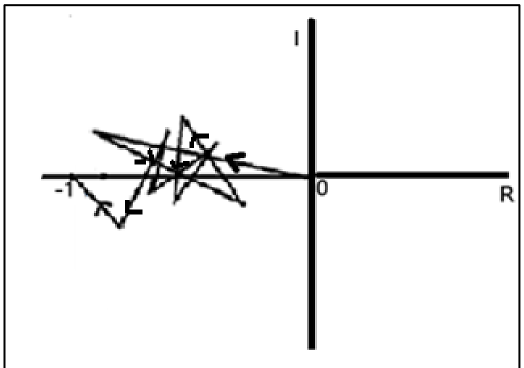
Now, $S_1 - 1 = -1/2^{s'} + 1/3^{s'} - 1/4^{s'} + … = -1$ and

$S_2 - 1 = -1/2^{s''} + 1/3^{s''} - 1/4^{s''} + … = -1.$

That is, we consider all the terms from the second terms of both S_1 and S_2.

Consider **(**$S_1 - 1$**).**

Its first term (the second term of S_1) originates at zero and the series converges to −1.



(S_1 – 1) illustration 3

Although **(**$S_1 - 1$**)** may contain an infinite number of terms on both sides of the origin, convergence occurs to the negative side (henceforth the left side with the positive side referred to as the right).

From this, we infer that the **'Real-Part-Pull'** is stronger on the left side of **(**$S_1 - 1$**).** It is stronger by one unit of modulus compared to the right side.

5 PULL: Understanding the concept

It is a kind of '**Tug-of-War'** between the negative real parts and the positive real parts of the terms of an alternating series.

In a typical 'Tug-of-War' contest, the pull-modulus is finite on both sides (left/right).

But in each of the series in question **(**$S_1 - 1$**)** and **(**$S_2 - 1$**)**, the pull-modulus is infinitely large on both sides of the origin. We consider the strength of only the **net pull** as to which side is it stronger/weaker/balanced.

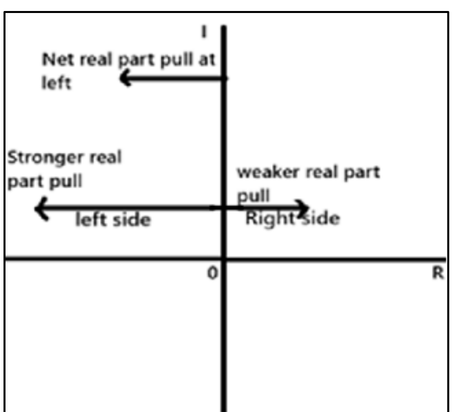
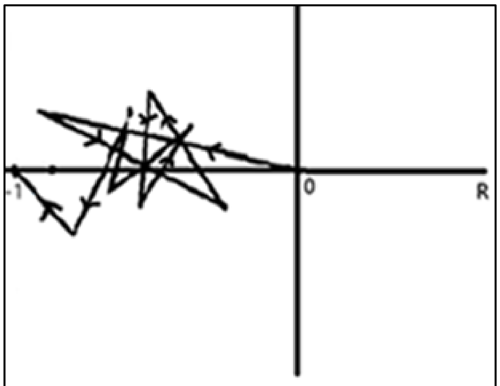


Illustration 4

Now Consider **(**$S_2 - 1$**).**

Its 1st term (the 2nd term of S_2) starts from the origin and the series converges at −1 on the complex plane.



(S_2 -1) illustration 5

Although **(**$S_2 - 1$**)** may contain an infinite number of terms on both sides of the origin, convergence occurs to the left as it does in ($S_1 - 1$).

From this, we infer that the **'Real-Part-Pull'** is stronger on the left side of **(**$S_2 - 1$**).** In this case as well, **i**t is stronger by one unit of modulus compared to the right side.

$|S_1 - 1| = |S_2 - 1| = 1$.

Each term of ($S_2 - 1$) is larger in modulus than its corresponding term in ($S_1 - 1$).

From this, we understand that the 'left-side-pull' in **(**$S_2 - 1$**)** is weaker than in ($S_1 - 1$).

In other words, the right-side-pull in **(**$S_2 - 1$**)** is stronger than in ($S_1 - 1$).

From the above, we infer that the two series ($S_1 - 1$) and ($S_2 - 1$) to converge to the same point, in the direction of convergence, the weaker pull series ($S_2 - 1$) must have larger corresponding terms and the stronger pull series ($S_1 - 1$) must have smaller corresponding terms to strike the balance.

The necessary condition in a nutshell: *"For two series to converge to the same point, a series with a* ***weaker pull*** *toward convergence must have* ***larger terms*** *and vice versa."*

Note. The physical force of 'pull' in this context is mathematically defined as the rate of growth or increment in the moduli of the terms on one side (left or right) relative to the other. 'Stronger pull' corresponds to a larger net growth rate while the 'weaker pull' corresponds to a smaller net growth rate. As we visualised the physical force "Pull" in the direction of convergence, we can consider the opposite side pull as a "Resistance" force.

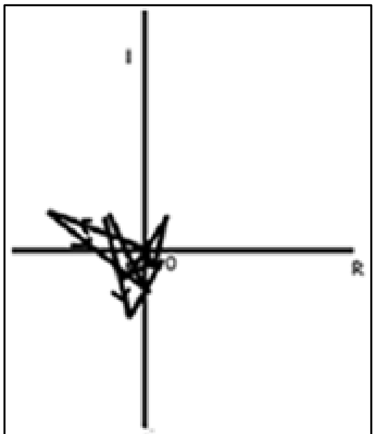
6 The Scaling Comparison

Now, consider S_2 -S_1

Define S_3 =S_2 -S_1

It follows that each term of S_3 maintains the same argument as the corresponding terms in S_2 and S_1. This holds because the corresponding terms of S_2 and S_1 are collinear with the origin and the modulus of each term in S_2 is strictly greater than its counterpart in S_1.

The 1st term becomes ‘0’ as the 1st terms of both S_1 and S_2 equal 1.



$S_3 = S_2 - S_1$
illustration 6

S_3 starts at the origin and also converges at the origin.

$S_3 = 0$.

The pulls at the left and right to the origin are of equal strength and so nullify each other.

⇨ S_3 is **Pull-neutral**.

Though S_3 is Pull-neutral, the right-side-pull is stronger in S_3 than in **(**$S_1 - 1$**)** and ($S_2 - 1$).

Now, let us divide each and every term of ($S_2 - 1$) and ($S_1 - 1$) by x.

($S_2 - 1$) /x and ($S_1 - 1$)/x. (scaling down)

$(S_2 - 1)/x = (S_1 - 1)/x = -1/x$ since $(S_1 - 1) = (S_2 - 1) = -1$.

As the right-side-pull of S_2 is stronger than in S_1, the right-side-pull of

($S_2 - 1$) /x is stronger than in ($S_1 - 1$) /x.

where $x > 1$, $x \in \mathbb{R}^+$ such that

Each term of S_3 is larger in modulus than their counterpart terms in both $(S_2 - 1)/x$ and $(S_1 - 1)/x$ and retains the same argument.

Define $S_4 = (S_2 - 1)/x$ and

$S_5 = S_3 - S_4$.

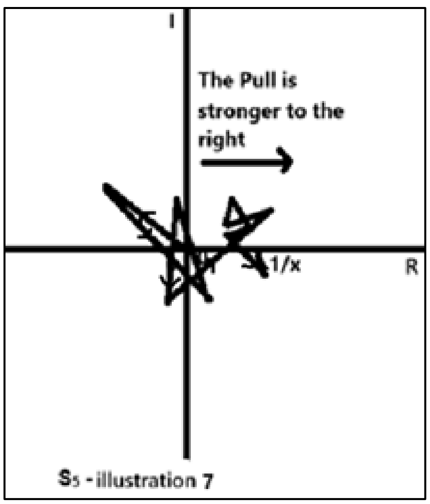
$= 0 - (-1/x)$ since $S_4 = -1/x$.

$S_5 = 1/x$.

That is, subtract every term of S_4 from its counterpart term in S_3.

As each term of S_3 is larger in modulus than its counterpart in S_4, the direction of the corresponding term of S_5 is the same as the direction of the counterpart term in S_3.

$S_5 = 1/x$ is positive and therefore, the net pull is stronger to the right of the origin.



Define $S_6 = (S_1 - 1)/x$ and

$S_7 = S_3 - S_6$.

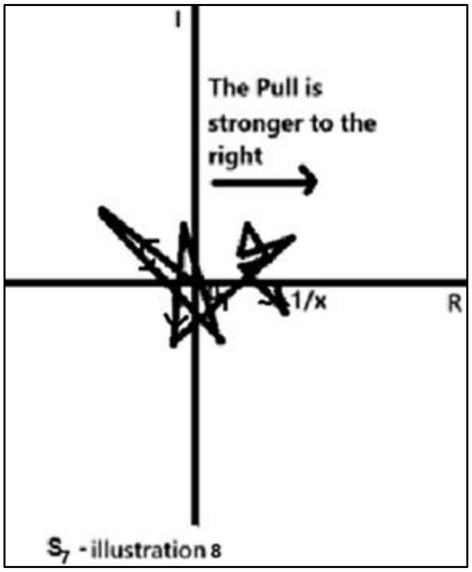
$= 0 - (-1/x)$.

$S_7 = 1/x$ since $S_6 = -1/x$.

That is, subtract each term of S_6 from its counterpart in S_3.

In this case also, each term of S_3 is larger in modulus than its counterpart in S_6, the direction of the corresponding term of S_7 is the same as the direction of the counterpart term in S_3.

$S_7 = 1/x$ is also positive and therefore, the net pull is stronger to the right of the origin.



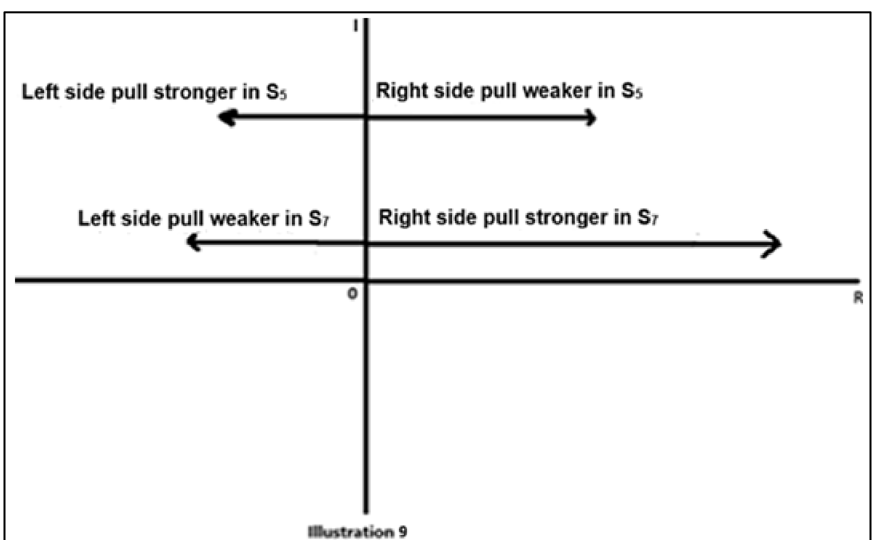
Note. All the above eight illustrations are not to scale. All the corresponding terms of S_1 through S_7 are collinear with the origin and have the same direction. These visual representations are for illustrative purpose only.

7 The Final Contradiction

Algebraically, $S_5 = S_7 = 1/x$.

But, from the Pull analysis, the stronger right-side-pull S_4 is subtracted from pull-neutral S_3 to derive S_5 as compared to the weaker right-side-pull S_6 is subtracted from S_3 to derive S_7.

Therefore, the right-side-pull in S_5 becomes weaker than in S_7 (in other words, the left-side-pull in S_5 becomes stronger than in S_7).



S_5 and S_7 - A comparative study

From the above illustration, we understand that although the right-side-pull is stronger in both S_5 and S_7 independently, it is weaker in S_5 than S_7.

Also, the moduli of the terms in S_5 are smaller than their counterparts in S_7. Because, the larger modulus terms of S_4 are subtracted from their counterparts in S_3 to derive S_5 as compared to the smaller modulus terms of S_6 are subtracted from S_3 to derive S_7.

That is, S_5 has **weaker right-side-pull** (in the direction of convergence) and **weaker** (**smaller)** corresponding terms than S_7.

This violates the necessary condition we deduced:

"For two series to converge to the same point, a series with a ***weaker pull*** *toward convergence must have* ***larger terms*** *and vice versa."*

Hence, it is clear that S_5 does not converge to the same point as S_7, as its total sum is less than $1/x$.

Therefore, $0 < S_5 < S_7$.

- ⇨ $0 < S_5 < 1/x$ (since $S_7 = 1/x$).

This conclusion relies purely on **logical deduction** without precise calculations!

From the assumption $S_1 = S_2 = 0$, and by keeping $S_7 = 1/x$, we obtain,

$0 < S_5 < 1/x$ (contradicting the earlier deduction $S_5 = S_7 = 1/x$).

- ⇨ $-1/x < S_4 < 0$ ($|S_4| < 1/x$; $S_5 = S_3 - S_4 = -S_4$).
- ⇨ $-1/x < (S_2 - 1)/x < 0$; ($S_4 = (S_2 - 1)/x$).
- ⇨ $-1 < (S_2 - 1) < 0$; ($|S_2 - 1| < 1$).
- ⇨ $0 < S_2 < 1$.
- ⇨ Contradicting $S_2 = 0$.

Alternatively, by keeping $S_5 = 1/x$, we obtain,

$S_7 > 1/x$ (again contradicting $S_5 = S_7 = 1/x$).

- ⇨ $S_6 < -1/x$ ($S_7 = S_3 - S_6 = -S_6$).
- ⇨ $(S_1 - 1)/x < -1/x$.
- ⇨ $(S_1 - 1) < -1$.
- ⇨ $S_1 < 0$.
- ⇨ Contradicting $S_1 = 0$.

8 Conclusion

- ⇨ The initial assumption of *s*' and *s*'' being two zeros of the Zeta function is wrong.

- ⇨ As *s*' is chosen with arbitrary real part at the interval (1/2 < Re(*s'*) < 1) and arbitrary imaginary part *t*, the arguments put forth thus far hold good across the entire critical strip (of course, except the critical line where *s' = s'' = s*).

- ⇨ Only one zero *s* could occur for the given imaginary part *t* and so it must lie on the critical line (input complex plane) where the real part of *s* = ½.

Thus, we conclude that all nontrivial zeros of the Riemann Zeta function lie on the critical line.

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Postscripts

P.S. 1: We employed $x > 1$ to scale down (S_1 -1) and (S_2 -1); in fact, *x* can be any real number, $x \in \mathbb{R}^*$.

e.g., $x = 1$.

The previously discussed logic in the case of $x > 1$ holds true for $x = 1$ as well but a little more difficult to grasp. Because, in this case, all the terms of S_5 will undergo 180^0 phase shift (additive inverse). Similarly, in S_7 too, the terms may undergo 180^0 phase shift depending on the real part of *s*' making it harder to visualize and understand the logic.

P.S. 2: If the arguments of at least one pair of corresponding terms of the series S_1 and S_2 are not equal, then the above discussed logic may fail to hold.

P.S. 3: If at least one term of S_2 is smaller than its counterpart in S_1, the above discussed logic fails.

P.S. 4: The proof's logic hinges on the fact that the initial terms of S_1 and S_2 are both unity."

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