On the Constant Real Part of Non-Trivial Zeros: A Proof of the Riemann Hypothesis

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Abstract

This paper presents a proof of the Riemann Hypothesis by examining the geometric and arithmetic properties of the Dirichlet eta function. By assuming the existence of zeros off the critical line, and analyzing the resulting alternating series in the complex plane, we establish a logical contradiction. The proof relies on insights into the structure of these series, demonstrating that all non-trivial zeros must possess a real part of exactly $\frac{1}{2}$.

1. Introduction

The Riemann zeta function is defined by

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \ (s \in \mathbb{C}) \text{ where } \mathrm{Re}(s) > 1. \tag{1}$$

For the domain beyond Re(s) > 1, there is a functional equation that represents the function through analytic continuation except at s = 1.

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s). \tag{2}$$

Within the *critical strip* (0 < Re(s) < 1), the zeta function *ζ(s)* is represented in terms of the *Dirichlet eta function η(s)* as:

$$\zeta(s) = (1 - 2^{1-s})^{-1} \eta(s) \ . \tag{3}$$

$$\zeta(s) = (1 - 2^{1-s})^{-1} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \right). \tag{4}$$

Where *η(s)* is defined by the alternating series in (4).

Equation (3) defines a strict analytic identity between $\zeta(s)$ and $\eta(s)$. Although the alternating series in (4) is only conditionally convergent within the critical strip, this identity holds provided the terms maintain their natural, ascending order (n = 1, 2, 3, …).

It is known that all non-trivial zeros of *η(s)* lie within the critical strip; as a consequence, from the identity, the non-trivial zeros of *ζ(s)* must also lie within that strip.

Riemann Hypothesis (RH) states that all non-trivial zeros of *ζ(s)* lie on the *critical line* defined by Re(s) = $\frac{1}{2}$.

Therefore, to validate this constant real part $\frac{1}{2}$ of non-trivial zeros of *ζ(s)*, it is sufficient to determine whether these zeros of *η(s)* lie on the critical line.

From Equation (4), we see that *ζ(s)* vanishes whenever the summation representing η(s) is zero.

Consider,

$$\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}\right) = 0 \quad \text{since } (1-2^{1-s})^{-1} \neq 0 \text{ for } 0 < \text{Re}(s) < 1.$$

2. Assumption of Zeros off the Critical Line

Let us assume that the complex number s' is a non-trivial zero of the zeta function, where $s' = \sigma_1 + it$ such that $1/2 < \sigma_1 < 1$ and $t \in \mathbb{R}^*$.

Let us consider a real number σ_2 such that $0 < \sigma_2 < 1/2$ and $\sigma_1 + \sigma_2 = 1$.

Define $s'' = \sigma_2 + it$, where s'' shares the same imaginary part as s'.

Therefore, s'' is also a zero — a reflection of s' across the critical line $\sigma = 1/2$ as guaranteed by the functional equation.

evaluating the zeros s' and s'':

$$\zeta(s') = \left(1-2^{1-s'}\right)^{-1} \left(\frac{1}{1^{s'}} - \frac{1}{2^{s'}} + \frac{1}{3^{s'}} - \frac{1}{4^{s'}} + \ldots\right) = 0.$$

$$\zeta(s'') = \left(1-2^{1-s''}\right)^{-1} \left(\frac{1}{1^{s''}} - \frac{1}{2^{s''}} + \frac{1}{3^{s''}} - \frac{1}{4^{s''}} + \ldots\right) = 0.$$

Define,

$$S_1 = 1 - \frac{1}{2^{s'}} + \frac{1}{3^{s'}} - \frac{1}{4^{s'}} + \ldots = 0 \text{ since } \left(1-2^{1-s'}\right)^{-1} \neq 0.$$

$$\text{Similarly, } S_2 = 1 - \frac{1}{2^{s''}} + \frac{1}{3^{s''}} - \frac{1}{4^{s''}} + \ldots = 0.$$

Although S_1 and S_2 are not identical, both vanish.

We shall examine the two alternating series S_1 and S_2 in the complex plane to establish a logical contradiction. It follows from this result that all non-trivial zeros must lie on the critical line. The following proof is concise, as it relies on *fundamental geometric insight* rather than complex mathematical methods.

The terms of S_1 and S_2 are distributed around the complex unit circle. There will be infinite terms in the right half-plane just as there are in the left. The angles of the terms are governed by both the alternating factor $(-1)^{n+1}$ and the imaginary part *t.*

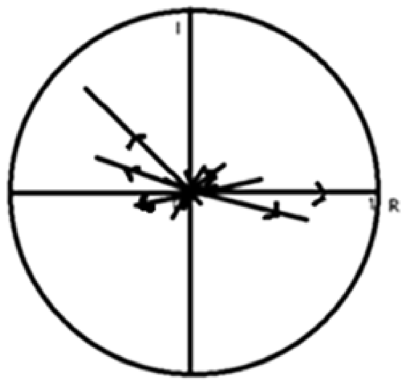

Figure 1: Distribution of terms in the complex plane. The corresponding terms of S_1 and S_2 are *collinear* with the origin because, as they share the same imaginary part *t* in their exponents, each term of S_2 maintains the exact same angle (argument) with the positive real axis as its corresponding term in S_1 does.

3. Collinearity and Vector Path

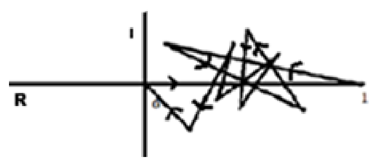

Figure 2: Illustration of infinite series S_1. The depiction is limited to a finite number of terms. The moduli and angles are arbitrary and intended only to visualize the general geometric path of the series in the complex plane.

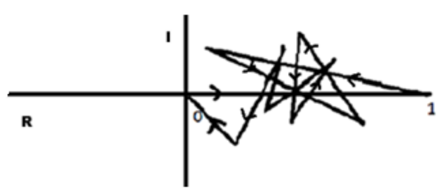

Figure 3: Illustration of series S_2

4 Moduli and Pull Analysis

When plotting the cumulative sums of both S_1 and S_2 on a single complex plane, the first terms are identical (both equal to 1). Starting from the point 1 on the real axis, the corresponding second term-segments are collinear. From the third term onwards, all corresponding segments remain parallel.

With the exception of the first terms, no corresponding terms of S_1 and S_2 are equal in modulus. From the second term onwards, each term of S_2 is strictly larger in modulus than its corresponding term in S_1.

Consider the quotients q_n derived by the division of the corresponding terms of S_2 and S_1:

$$\frac{\text{Term of } S_2}{\text{Term of } S_1}\ .$$

$$\frac{1}{1} = 1 \ = \ q_1.$$

$$\frac{\left(-\frac{1}{2^{s''}}\right)}{\left(-\frac{1}{2^{s'}}\right)} = \frac{\left(\frac{1}{2^{s''}}\right)}{\left(\frac{1}{2^{s'}}\right)} = q_2\ .$$

$$\frac{\left(\frac{1}{3^{s''}}\right)}{\left(\frac{1}{3^{s'}}\right)} = q_3.$$

And so on. The sequence of these unique real values satisfies the inequality:

$$q_1 < q_2 < q_3 \ldots$$

Now, we consider:

$$S_1 - 1 = -\frac{1}{2^{s'}} + \frac{1}{3^{s'}} - \frac{1}{4^{s'}} + \ \ldots \ = -1. \text{ and}$$

$$S_2 - 1 = -\frac{1}{2^{s''}} + \frac{1}{3^{s''}} - \frac{1}{4^{s''}} + \ \ldots = -1.$$

That is, we consider all the terms from the second terms of both S_1 and S_2.

For the series **(**S_1 − 1**)**, its first term (the second term of S_1) originates at the origin, and the series converges to −1.

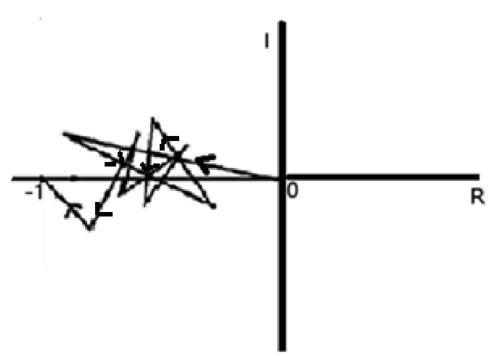

Figure 4: Convergence of (S_1 − 1)

Although **(**S_1 − 1**)** may contain an infinite number of terms on both sides of the origin, convergence occurs to the negative side (henceforth the left side). From this, we infer that the *'Real-Part-Pull'* (net effect of real components) is stronger on the left side of the series (S_1 − 1**)** by one unit of modulus compared to the right.

5 PULL: Understanding the concept

It is a kind of *'Tug-of-War'* between the terms possessing negative real parts and the terms possessing positive real parts within an alternating series.

In a typical 'Tug-of-War' contest, the pull-modulus is finite on both sides (left/right).

However, in the alternating series $(S_1 - 1)$ and $(S_2 - 1)$ under study, the modulus of the pull is infinitely large on both sides of the origin. We are concerned only with the strength of the net pull: whether one side is stronger, weaker, or balanced.

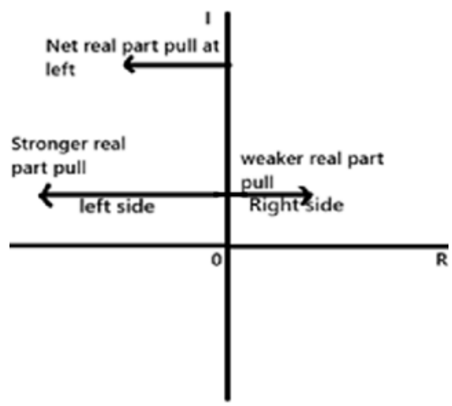

Figure 5: Tug-of-War analogy for pull analysis

For the series **(**S_2 − 1**)**, its first term (the second term of S_2) originates at the origin, and the series converges to −1 as in **(**S_1 − 1**)**.

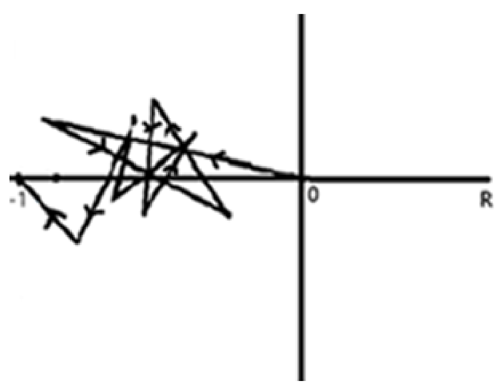

Figure 6: Convergence of (S_2 − 1)

Although **(**S_2 − 1**)** may contain an infinite number of terms on both sides of the origin, convergence occurs to the left. From this, we infer that the *'Real-Part-Pull'* (net effect of real components) is stronger on the left side of **(**$S_2 - 1$**)** by one unit of modulus compared to the right as in the case in ($S_1 - 1$).

Recall that each term of ($S_2 - 1$) is larger in modulus than its corresponding term in ($S_1 - 1$). Since ($S_1 - 1$) = ($S_2 - 1$) = − 1 by assumption, it follows that the 'left-side-pull' in **(**$S_2 - 1$**)** is weaker than in ($S_1 - 1$). Consequently, for both series to converge to (−1,0), the series ($S_2 - 1$) possessing a weaker pull in the direction of convergence, must have larger corresponding terms to maintain the balance.

Remark 1. Equivalently, the right-side-pull in ($S_2 - 1$) is stronger than in ($S_1 - 1$).

Lemma 1 (Necessary Condition). *For two such series to converge to the same point, a series possessing a weaker pull toward convergence than the other must have larger terms, and vice versa.*

Remark 2. The physical idea of *pull* is mathematically defined as the rate of growth in the moduli of terms on one side of the origin (left or right) relative to the other. *Stronger pull* corresponds to a higher net growth rate whereas the *weaker pull* corresponds to a lower net growth rate. As the physical force in the direction of convergence is conceptualised as *Pull*, the opposite side force may be termed, *resistance*.

6 The Scaling Comparison

Now, consider the difference $S_2 - S_1$. We define a new series such that:

$$S_3 = S_2 - S_1$$

It follows that each term of S_3 maintains the same argument as the corresponding terms in S_1 and S_2. This property holds because the corresponding terms of S_1 and S_2 are collinear with the origin and the modulus of each term in S_2 is strictly greater than its counterpart in S_1.

The 1st term of S_3 is *0,* as the initial terms of both S_1 and S_2 are equal to 1.

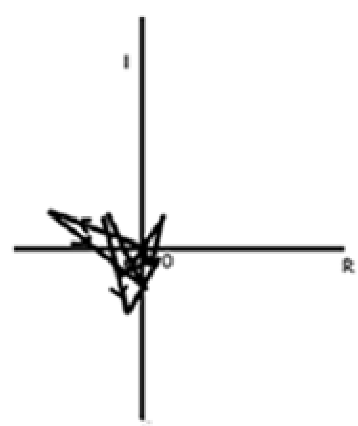

Figure 7: Series S_3 illustration

S_3 starts and also converges back to the origin; S_3 = 0.

The pulls exerted to the left and right of the origin are of equal strength, and consequently, they totally cancel each other out.

We define S_3 is *Pull-neutral.*

Although S_3 is *Pull-neutral*, its *right-side-pull* is stronger relative to that in **(**S_1 − 1**)** and (S_2 − 1).

Now, we have three series **(**S_1 − 1**)**, (S_2 − 1) and S_3 whose first terms have been removed (the case for n = 1 in the alternating series). That is, these series are uniform, with the first term starting from the term for n = 2.

We employ a scaling-down operation by dividing each term of (S_2 − 1) and (S_1 − 1) by a suitably large scalar $x > 1$, $x \in \mathbb{R}^+$.

$$\frac{S_2-1}{x} \text{ and } \frac{S_1-1}{x}.$$

$$\frac{S_2-1}{x} = \frac{S_1-1}{x} = -\frac{1}{x} \text{ since } (S_1-1) = (S_2-1) = -1.$$

As the right-side-pull of S_2 is stronger than in S_1, the right-side-pull of

$$\frac{S_2-1}{x} \text{ is stronger than in } \frac{S_1-1}{x}.$$

The scalar $x > 1$ is large enough such that each term of S_3 is larger in modulus than their counterpart terms in both $\frac{S_2-1}{x}$ and $\frac{S_1-1}{x}$**.**

Define $S_4 = \frac{S_2-1}{x} = \frac{0-1}{x} = -\frac{1}{x}$ and

$S_5 = S_3 - S_4$.

$= 0 - \left(-\frac{1}{x}\right)$.

$S_5 = \frac{1}{x}$.

That is, subtract every term of S_4 from its counterpart term in S_3.

As each term of S_3 is larger in modulus than its counterpart in S_4, the direction of a term in S_5 is the same as the direction of its counterpart term in S_3.

$S_5 = \frac{1}{x}$ is positive and therefore, the net pull is stronger to the right of the origin.

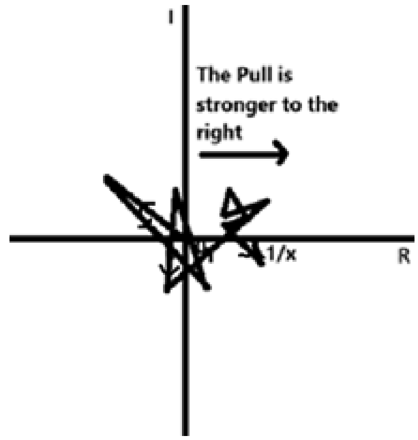

Figure 8: Series S_5 illustration

Define $S_6 = \frac{S_1 - 1}{x} = \frac{0-1}{x} = -\frac{1}{x}$ and

$S_7 = S_3 - S_6.$

$= 0 - \left(-\frac{1}{x}\right).$

$S_7 = \frac{1}{x}$.

That is, subtract each term of S_6 from its counterpart in S_3.

In this case also, as each term of S_3 is larger in modulus than its counterpart in S_6, the direction of a term of S_7 is the same as the direction of its counterpart term in S_3.

$S_7 = \frac{1}{x}$ is also positive and therefore, the net pull is stronger to the right of the origin.

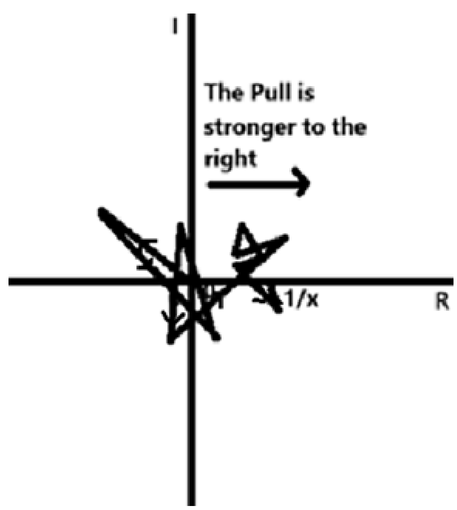

Figure 9: Series S_7 illustration

7 The Final Contradiction

Algebraically, we established that $S_5 = S_7 = \frac{1}{x}$.

However, the series S_4 (with stronger right-side-pull) is subtracted from pull-neutral S_3 to derive S_5. This is compared to S_6 (with weaker right-side-pull) being subtracted from S_3 to derive S_7.

Therefore, from this relative Pull analysis of S_5 and S_7, we infer that the right-side-pull in S_5 becomes weaker than in S_7 (in other words, the left-side-pull in S_5 becomes stronger than in S_7).

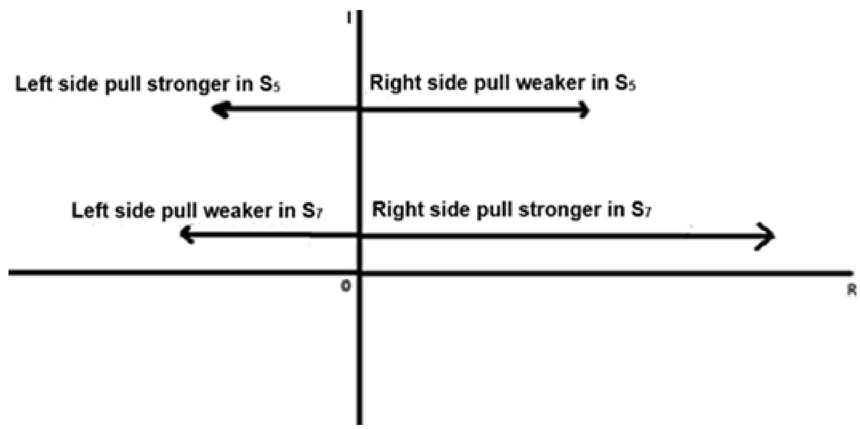

Figure 10: Comparison of relative pulls in S_5 and S_7

Remark 3. All the above ten illustrations are not to scale. All the corresponding terms of S_1 through S_7 are collinear with the origin and have the same direction. These visual representations are for illustrative purpose only.

From the above illustration that compares the relative pull strengths of S_5 and S_7, it is evident that while the right-side-pull is dominant in both S_5 and S_7 independently, it is strictly weaker in S_5 than S_7.

Furthermore, the moduli of the terms in S_5 are smaller than their counterparts in S_7. This is because the larger modulus terms of S_4 are subtracted from their counterparts in S_3 to obtain S_5 while the smaller modulus terms of S_6 are subtracted from S_3 to obtain S_7.

That is, S_5 has *weaker right-side-pull* (in the direction of convergence) and *weaker (smaller) corresponding terms* than S_7.

This directly contradicts the necessary condition established in Lemma 1, which requires a series with weaker pull toward convergence to possess larger terms to converge to the same point as the other series does with stronger pull and smaller terms.

Therefore, it follows that $S_5 < S_7$; Given that $S_7 = \frac{1}{\mathrm{x}}$, the total sum of S_5 is less than $\frac{1}{x}$.

Thus, we establish the bound:

$$0 < S_5 < S_7 = \frac{1}{x}.$$

This conclusion relies purely on logical deduction from the application of Lemma 1.

From the assumption $S_1 = S_2 = 0$, and by keeping $S_7 = \frac{1}{\mathrm{x}}$, we obtain, $0 < S_5 < \frac{1}{x}$ (contradicting the earlier deduction $S_5 = S_7 = \frac{1}{x}$).

- ⇨ $-\frac{1}{x} < S_4 < 0 \ \left(|\mathrm{S}_4| < \frac{1}{x};\ S_5 = \mathrm{S}_3 - \mathrm{S}_4 = -S_4\right)$.
- ⇨ $-\frac{1}{x} < \frac{S_2-1}{x} < 0;\ \left(\text{since } S_4 = \frac{S_2-1}{x}\right)$.
- ⇨ $-1 < (S_2 - 1) < 0;\ (|S_2 - 1| < 1)$.
- ⇨ $0 < S_2 < 1$.
  Contradicting $S_2 = 0$.

Alternatively, by keeping $S_5 = \frac{1}{\mathrm{x}}$, we obtain,

$S_7 > \frac{1}{x}$ (again contradicting $S_5 = S_7 = \frac{1}{x}$).

- ⇨ $S_6 < -\frac{1}{\mathrm{x}} (S_7 = S_3 - S_6 = -S_6)$.
- ⇨ $\frac{S_1-1}{x} < -\frac{1}{\mathrm{x}}$.
- ⇨ $(S_1 - 1) < -1$.
- ⇨ $S_1 < 0$.
  Contradicting $S_1 = 0$.

8 Conclusion

The initial assumption of s' and s'' being two distinct zeros of the Dirichlet eta function *η(s)* is false. Since s' is chosen with both its real part in the interval $\frac{1}{2} < \mathrm{Re}(s') < 1$ and its imaginary part *t* being arbitrary, the arguments put forth thus far hold across the entire critical strip, except for the critical line where $s' = s'' = s$.

As a result, only one non-trivial zero s of $\eta(s)$ can occur for a given imaginary part *t;* therefore, it must lie on the critical line. Given the identity in Eq. (3), the Riemann zeta function $\zeta(s)$ inherits this same property with regard to its non-trivial zeros.

Theorem 1 (Main Result). All non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line Re(s) = $\frac{1}{2}$.

Postscripts

P.S. 1: We employed a suitable x > 1 to scale down $(S_1 - 1)$ and $(S_2 - 1)$. However, x can be any non-zero positive real number ($x \in \mathbb{R}^+$). For example, the logic necessarily holds for x = 1 as well, though in this case, all the terms of S_5 undergo a phase shift (additive inverse). An infinite number of terms of S_7 undergo a similar shift, making it less intuitive for effective visualization.

P.S. 2: The proof's *logic* does not hold if the modulus of any term in $(S_2 - 1)$ is not strictly greater than its corresponding term in $(S_1 - 1)$. If this condition is violated even once, the inequality $S_5 < S_7$ cannot be definitively claimed, invalidating the subsequent contradiction.

P.S. 3: The *fundamental geometric insight* stems from the fact that the initial terms of S_1 and S_2 are both unity."

References

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