

# Analytical Study of the Rhombicuboctahedron Using the Theory of Polygon

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## 1. Introduction

A rhombicuboctahedron (also called small rhombicuboctahedron) is an Archimedean solid which has 8 congruent equilateral triangular and 18 congruent square faces each having equal edge length, 48 edges & 24 vertices [1]. It is created/generated either by shifting/translating all 8 equilateral triangular faces of a regular octahedron [2,3] radially outwards by the same distance without any other transformation (i.e. rotation, distortion etc.) or by shifting/translating all 6 square faces of a cube (regular hexahedron) radially outwards by the same distance without any other transformation (i.e. rotation, distortion etc.) till either the vertices, initially coincident, of each four equilateral triangular faces of the octahedron form a square of the same edge length or the vertices, initially coincident, of each three square faces of the cube form an equilateral triangle of the same edge length. Both the methods create the same solid having 8 congruent equilateral triangular & 18 congruent square faces each having equal edge length. Thus, the solid generated is called rhombicuboctahedron or small rhombicuboctahedron [4].

In this paper, the circumscribed radius of a rhombicuboctahedron is derived using an alternative geometrical approach based on HCR's Theory of Polygon [5,6]. An explicit analytical expression is obtained for the radius of the circumscribed sphere passing through all 24 congruent vertices of a rhombicuboctahedron with a given edge length. Using the same theoretical framework, closed-form formulae are subsequently derived for the normal distances of the equilateral triangular and square faces from the centre, the total surface area, and the enclosed volume. In addition, analytical expressions are presented for the solid angles subtended at the centre by each equilateral triangular face and each square face, the dihedral angles between any two faces meeting at a vertex, and the solid angle subtended by the rhombicuboctahedron at any of its 24 identical vertices.

## 2. Derivation of radius $R$ of circumscribed sphere of a rhombicuboctahedron

Consider a rhombicuboctahedron with edge length  $a$  & centre  $O$  such that all its 24 identical vertices lie on a spherical surface of radius  $R$ . Drop the perpendiculars  $OM$  &  $ON$  from the centre  $O$  to the centres  $M$  &  $N$  of an equilateral triangular face  $ABC$  & adjacent square face  $ABDE$  respectively. Let  $H_T$  &  $H_S$  be the normal distances of equilateral triangular face  $ABC$  & square face  $ABDE$  respectively (as shown in fig-1).

In right  $\Delta AMO$  (see fig-1), using Pythagorean theorem, we get

$$OM = \sqrt{(OA)^2 - (AM)^2} \quad \left( AM = \text{circumscribed radius} = \frac{a}{\sqrt{3}} \right)$$

$$H_T = \sqrt{R^2 - \left(\frac{a}{\sqrt{3}}\right)^2} = \sqrt{R^2 - \frac{a^2}{3}} = \sqrt{\frac{3R^2 - a^2}{3}} \quad \dots \dots \dots (I)$$

Similarly, right  $\Delta ANO$  (see fig-1), using Pythagorean theorem, we get

$$ON = \sqrt{(OA)^2 - (AN)^2} \quad \left( AN = \text{circumscribed radius} = \frac{a}{\sqrt{2}} \right)$$

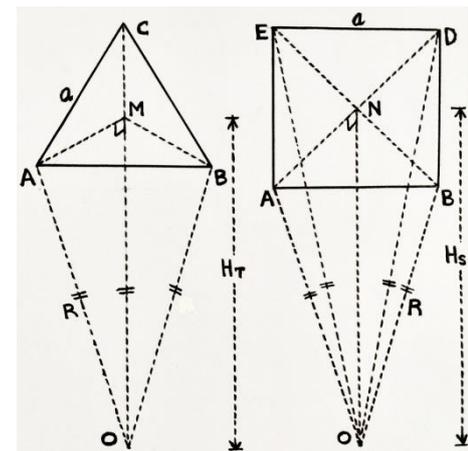


Figure 1: Equilateral triangular face  $ABC$  & square face  $ABDE$  are at normal distances  $H_T$  &  $H_S$  respectively from the centre  $O$  and  $OA = OB = OC = OD = OE = R$ .

$$H_S = \sqrt{R^2 - \left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{R^2 - \frac{a^2}{2}} = \sqrt{\frac{2R^2 - a^2}{2}} \dots \dots \dots (II)$$

We know that the solid angle ( $\omega$ ) subtended by any regular polygonal plane with  $n$  no. of sides each of length  $a$  at any point lying at a distance  $H$  on the vertical axis passing through the centre, is given by HCR's Theory of Polygon [5,6] as follows

$$\omega = 2\pi - 2n \sin^{-1} \left( \frac{2H \sin \frac{\pi}{n}}{\sqrt{4H^2 + a^2 \cot^2 \frac{\pi}{n}}} \right)$$

Now, substituting the corresponding values in above general formula i.e. number of sides  $n = 3$  (for equilateral  $\Delta$ ), length of each side  $a = AB = a$ , & normal height  $H = OM = H_T$ , the solid angle  $\omega_T$  subtended by the equilateral triangular face ABC at the centre O of rhombicuboctahedron (as shown in above fig-1), is obtained as follows

$$\begin{aligned} \omega_T &= 2\pi - 2 \times 3 \sin^{-1} \left( \frac{2 \left( \sqrt{\frac{3R^2 - a^2}{3}} \right) \sin \frac{\pi}{3}}{\sqrt{4 \left( \sqrt{\frac{3R^2 - a^2}{3}} \right)^2 + a^2 \cot^2 \frac{\pi}{3}}} \right) = 2\pi - 6 \sin^{-1} \left( \frac{2 \left( \sqrt{\frac{3R^2 - a^2}{3}} \right) \frac{\sqrt{3}}{2}}{\sqrt{\frac{12R^2 - 4a^2}{3} + a^2 \left( \frac{1}{\sqrt{3}} \right)^2}} \right) \\ &= 2\pi - 6 \sin^{-1} \left( \frac{\sqrt{3R^2 - a^2}}{\sqrt{\frac{12R^2 - 4a^2 + a^2}{3}}} \right) = 2\pi - 6 \sin^{-1} \left( \frac{\sqrt{3R^2 - a^2}}{\sqrt{4R^2 - a^2}} \right) = 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{3R^2 - a^2}{4R^2 - a^2}} \right) \\ &= 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{a^2 \left( \frac{3R^2}{a^2} - 1 \right)}{a^2 \left( \frac{4R^2}{a^2} - 1 \right)}} \right) = 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{\frac{3R^2}{a^2} - 1}{\frac{4R^2}{a^2} - 1}} \right) = 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{3 \left( \frac{R}{a} \right)^2 - 1}{4 \left( \frac{R}{a} \right)^2 - 1}} \right) \end{aligned}$$

Now, for ease of calculation, let's assume  $\frac{R}{a} = x$  ( $\forall x > 1$ ) then we get

$$\omega_T = 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{3x^2 - 1}{4x^2 - 1}} \right) \dots \dots \dots (III)$$

Similarly, substituting the corresponding values in above general formula i.e. number of sides  $n = 4$  (for square), length of each side  $a = AB = a$ , & normal height  $H = ON = H_S$ , the solid angle  $\omega_S$  subtended by the square face ABDE at the centre O of rhombicuboctahedron (as shown in the above fig-1), is obtained as follows

$$\omega_S = 2\pi - 2 \times 4 \sin^{-1} \left( \frac{2 \left( \sqrt{\frac{2R^2 - a^2}{2}} \right) \sin \frac{\pi}{4}}{\sqrt{4 \left( \sqrt{\frac{2R^2 - a^2}{2}} \right)^2 + a^2 \cot^2 \frac{\pi}{4}}} \right) = 2\pi - 8 \sin^{-1} \left( \frac{2 \left( \sqrt{\frac{2R^2 - a^2}{2}} \right) \frac{1}{\sqrt{2}}}{\sqrt{\frac{8R^2 - 4a^2}{2} + a^2(1)^2}} \right)$$

$$\begin{aligned}
 &= 2\pi - 8 \sin^{-1} \left( \frac{\sqrt{2R^2 - a^2}}{\sqrt{\frac{8R^2 - 4a^2 + 2a^2}{2}}} \right) = 2\pi - 8 \sin^{-1} \left( \frac{\sqrt{2R^2 - a^2}}{\sqrt{4R^2 - a^2}} \right) = 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{2R^2 - a^2}{4R^2 - a^2}} \right) \\
 &= 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{a^2 \left( \frac{2R^2}{a^2} - 1 \right)}{a^2 \left( \frac{4R^2}{a^2} - 1 \right)}} \right) = 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{\frac{2R^2}{a^2} - 1}{\frac{4R^2}{a^2} - 1}} \right) = 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{2 \left( \frac{R}{a} \right)^2 - 1}{4 \left( \frac{R}{a} \right)^2 - 1}} \right)
 \end{aligned}$$

Now, for ease of calculation, let's assume  $\frac{R}{a} = x$  ( $\forall x > 1$ ) then we get

$$\omega_S = 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right) \dots \dots \dots (IV)$$

But we know that solid angle, subtended by any closed surface at any point inside it, is always equal to  $4\pi$  sr [7]. Since a rhombicuboctahedron is a closed surface consisting of 8 congruent equilateral triangular faces and 18 congruent square faces therefore the solid angle subtended by all 26 faces at the centre O of rhombicuboctahedron must be equal to  $4\pi$  sr as follows

$$8(\omega_T) + 18(\omega_S) = 4\pi$$

Substituting the corresponding values of solid angles  $\omega_T$  &  $\omega_S$  from above Eq.(III) & (IV) as follows

$$\begin{aligned}
 &8 \left( 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{3x^2 - 1}{4x^2 - 1}} \right) \right) + 18 \left( 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right) \right) = 4\pi \\
 &16\pi - 48 \sin^{-1} \left( \sqrt{\frac{3x^2 - 1}{4x^2 - 1}} \right) + 36\pi - 144 \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right) = 4\pi \\
 &48\pi - 48 \sin^{-1} \left( \sqrt{\frac{3x^2 - 1}{4x^2 - 1}} \right) - 144 \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right) = 0 \\
 &\pi - \sin^{-1} \left( \sqrt{\frac{3x^2 - 1}{4x^2 - 1}} \right) - 3 \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right) = 0 \\
 &\pi - \sin^{-1} \left( \sqrt{\frac{3x^2 - 1}{4x^2 - 1}} \right) - \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right) = 2 \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right) \\
 &\pi - \left( \sin^{-1} \left( \sqrt{\frac{3x^2 - 1}{4x^2 - 1}} \right) + \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right) \right) = 2 \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right)
 \end{aligned}$$

Using following formula,  $\sin^{-1} x + \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$  &  $2 \sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2}) \forall |x|, |y| \in [0,1]$

$$\begin{aligned} \pi - \left( \sin^{-1} \left( \sqrt{\frac{3x^2-1}{4x^2-1}} \sqrt{1 - \frac{2x^2-1}{4x^2-1}} + \sqrt{\frac{2x^2-1}{4x^2-1}} \sqrt{1 - \frac{3x^2-1}{4x^2-1}} \right) \right) &= \sin^{-1} \left( 2 \sqrt{\frac{2x^2-1}{4x^2-1}} \sqrt{1 - \frac{2x^2-1}{4x^2-1}} \right) \\ \pi - \left( \sin^{-1} \left( \sqrt{\frac{3x^2-1}{4x^2-1}} \sqrt{\frac{2x^2}{4x^2-1}} + \sqrt{\frac{2x^2-1}{4x^2-1}} \sqrt{\frac{x^2}{4x^2-1}} \right) \right) &= \sin^{-1} \left( 2 \sqrt{\frac{2x^2-1}{4x^2-1}} \sqrt{\frac{2x^2}{4x^2-1}} \right) \\ \pi - \sin^{-1} \left( \frac{x\sqrt{2}\sqrt{3x^2-1}}{4x^2-1} + \frac{x\sqrt{2x^2-1}}{4x^2-1} \right) &= \sin^{-1} \left( \frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1} \right) \\ \pi - \sin^{-1} \left( \frac{x(\sqrt{6x^2-2} + \sqrt{2x^2-1})}{4x^2-1} \right) &= \sin^{-1} \left( \frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1} \right) \end{aligned}$$

Taking Sine on both the sides, we get

$$\begin{aligned} \sin \left( \pi - \sin^{-1} \left( \frac{x(\sqrt{6x^2-2} + \sqrt{2x^2-1})}{4x^2-1} \right) \right) &= \sin \left( \sin^{-1} \left( \frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1} \right) \right) \\ \sin \left( \sin^{-1} \left( \frac{x(\sqrt{6x^2-2} + \sqrt{2x^2-1})}{4x^2-1} \right) \right) &= \frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1} \\ \frac{x(\sqrt{6x^2-2} + \sqrt{2x^2-1})}{4x^2-1} &= \frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1} \\ \frac{x(\sqrt{6x^2-2} + \sqrt{2x^2-1})}{4x^2-1} - \frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1} &= 0 \\ \frac{x(\sqrt{6x^2-2} + \sqrt{2x^2-1} - 2\sqrt{2}\sqrt{2x^2-1})}{4x^2-1} &= 0 \\ \sqrt{6x^2-2} + \sqrt{2x^2-1} - 2\sqrt{2}\sqrt{2x^2-1} &= 0 \quad (\because x \neq 0, 4x^2-1 \neq 0) \\ \sqrt{6x^2-2} - (2\sqrt{2}-1)\sqrt{2x^2-1} &= 0 \\ \sqrt{6x^2-2} &= (2\sqrt{2}-1)\sqrt{2x^2-1} \\ (\sqrt{6x^2-2})^2 &= ((2\sqrt{2}-1)\sqrt{2x^2-1})^2 \\ 6x^2-2 &= (9-4\sqrt{2})(2x^2-1) \\ 6x^2-2 &= (18-8\sqrt{2})x^2-9+4\sqrt{2} \\ (18-8\sqrt{2})x^2-6x^2 &= -2+9-4\sqrt{2} \\ 4(3-2\sqrt{2})x^2 &= 7-4\sqrt{2} \\ x^2 &= \frac{7-4\sqrt{2}}{4(3-2\sqrt{2})} \\ x^2 &= \frac{(7-4\sqrt{2})(3+2\sqrt{2})}{4(3-2\sqrt{2})(3+2\sqrt{2})} \end{aligned}$$

$$x^2 = \frac{21 - 12\sqrt{2} + 14\sqrt{2} - 16}{4(3^2 - (2\sqrt{2})^2)}$$

$$x^2 = \frac{5 + 2\sqrt{2}}{4(9 - 8)} = \frac{5 + 2\sqrt{2}}{4}$$

$$x = \sqrt{\frac{5 + 2\sqrt{2}}{4}} = \frac{\sqrt{5 + 2\sqrt{2}}}{2} > 1 \quad \left( \text{Condition is satisfied i.e. } x = \frac{R}{a} > 1 \right)$$

$$\Rightarrow \frac{R}{a} = x = \frac{\sqrt{5 + 2\sqrt{2}}}{2} \Rightarrow R = \frac{a\sqrt{5 + 2\sqrt{2}}}{2}$$

Therefore, the radius ( $R$ ) of circumscribed spherical surface passing through all 24 identical vertices of a given rhombicuboctahedron with edge length  $a$  is given as follows

$$R = \frac{a}{2} \sqrt{5 + 2\sqrt{2}} \approx 1.398966326a \quad \dots \dots \dots (1)$$

**2.1. Normal distances  $H_T$  and  $H_S$  of equilateral triangular & square faces from the centre of rhombicuboctahedron**

Substituting the value of radius  $R$  of circumscribed sphere in the eq(I) (as derived above from fig-1), the normal distance  $OM = H_T$  of equilateral triangular face ABC from the centre O is

$$\begin{aligned} H_T &= \sqrt{\frac{3R^2 - a^2}{3}} = \sqrt{\frac{3\left(\frac{a\sqrt{5 + 2\sqrt{2}}}{2}\right)^2 - a^2}{3}} = \sqrt{\frac{3a^2(5 + 2\sqrt{2})}{4} - a^2} = \sqrt{\frac{a^2(11 + 6\sqrt{2})}{12}} = \sqrt{\frac{a^2(3 + \sqrt{2})^2}{12}} \\ &= \frac{a(3 + \sqrt{2})}{2\sqrt{3}} \end{aligned}$$

Similarly, substituting the value of radius  $R$  of circumscribed sphere in the eq(II) (as derived above from fig-1), the normal distance  $ON = H_S$  of square face ABDE from the centre O is

$$\begin{aligned} H_S &= \sqrt{\frac{2R^2 - a^2}{2}} = \sqrt{\frac{2\left(\frac{a\sqrt{5 + 2\sqrt{2}}}{2}\right)^2 - a^2}{2}} = \sqrt{\frac{2a^2(5 + 2\sqrt{2})}{4} - a^2} = \sqrt{\frac{a^2(3 + 2\sqrt{2})}{4}} = \sqrt{\frac{a^2(1 + \sqrt{2})^2}{4}} \\ &= \frac{a(1 + \sqrt{2})}{2} \end{aligned}$$

Hence, the normal distances  $H_T$  and  $H_S$  of equilateral triangular & square faces respectively from the centre of rhombicuboctahedron with edge length  $a$ , are given as follows

$$H_T = \frac{a(3 + \sqrt{2})}{2\sqrt{3}} \approx 1.274273694 a \quad \dots \dots \dots (2)$$

$$H_S = \frac{a(1 + \sqrt{2})}{2} \approx 1.207106781 a \quad \dots \dots \dots (3)$$

It is clear from above values of normal distances that the equilateral triangular faces are farther from the centre while square faces are closer to the centre. For finite value of edge length  $a \Rightarrow H_S < H_T < R$

## 2.2. Surface Area ( $A_s$ ) of rhombicuboctahedron

The surface of a rhombicuboctahedron consists of 8 congruent equilateral triangular faces each of side  $a$  & 18 congruent square faces each with side  $a$ . Therefore, the (total) surface area of rhombicuboctahedron is given as follows

$$\begin{aligned} A_s &= 8(\text{Area of equilateral triangular face}) + 18(\text{Area of square face}) \\ &= 8\left(\frac{\sqrt{3}}{4}a^2\right) + 18(a^2) = 2a^2\sqrt{3} + 18a^2 = 2a^2(9 + \sqrt{3}) \\ \therefore \text{Surface area, } A_s &= 2a^2(9 + \sqrt{3}) \approx 21.46410162 a^2 \quad \dots \dots \dots (4) \end{aligned}$$

## 2.3. Volume ( $V$ ) of rhombicuboctahedron

The surface of a rhombicuboctahedron consists of 8 congruent equilateral triangular faces each of side  $a$  & 18 congruent square faces each with side  $a$ . Thus a solid rhombicuboctahedron can be assumed to consist of 8 congruent right pyramids with equilateral triangular base with side  $a$  & normal height  $H_T$ , and 18 congruent right pyramids with square base of side  $a$  & normal height  $H_S$  (as shown in above fig-1). Therefore, the volume of rhombicuboctahedron is given as

$$\begin{aligned} V &= 8(\text{Volume of equilateral triangular right pyramid}) + 18(\text{Volume of square right pyramid}) \\ &= 8\left(\frac{1}{3}\left(\frac{\sqrt{3}}{4}a^2\right) \cdot \frac{a(3 + \sqrt{2})}{2\sqrt{3}}\right) + 18\left(\frac{1}{3}a^2 \cdot \frac{a(1 + \sqrt{2})}{2}\right) \\ &= \frac{a^3(3 + \sqrt{2})}{3} + 3a^3(1 + \sqrt{2}) \\ &= \frac{a^3(3 + \sqrt{2}) + 9a^3(1 + \sqrt{2})}{3} = \frac{a^3(12 + 10\sqrt{2})}{3} = \frac{2a^3(6 + 5\sqrt{2})}{3} \\ \therefore \text{Volume, } V &= \frac{2a^3(6 + 5\sqrt{2})}{3} \approx 8.714045208 a^3 \quad \dots \dots \dots (5) \end{aligned}$$

## 2.4. Mean radius ( $R_m$ ) of rhombicuboctahedron

It is the radius of the sphere having a volume equal to that of a given rhombicuboctahedron with edge lengths  $a$ . It is computed as follows

volume of sphere with mean radius  $R_m$  = volume of rhombicuboctahedron with edge  $a$

$$\begin{aligned} \frac{4}{3}\pi(R_m)^3 &= \frac{2a^3(6 + 5\sqrt{2})}{3} \\ (R_m)^3 &= \frac{a^3(6 + 5\sqrt{2})}{2\pi} \\ R_m &= \left(\frac{a^3(6 + 5\sqrt{2})}{2\pi}\right)^{1/3} \end{aligned}$$

$$R_m = a \left( \frac{6 + 5\sqrt{2}}{2\pi} \right)^{1/3}$$

∴ **Mean radius,  $R_m = a \left( \frac{6 + 5\sqrt{2}}{2\pi} \right)^{1/3} \approx 1.276567352 a$**  ..... (6)

It's worth noticing that for a finite value of edge length  $a \Rightarrow H_S < H_T < R_m < R$ .

**3. Dihedral angle between any two faces meeting at a vertex of rhombicuboctahedron**

A rhombicuboctahedron has 24 identical vertices at each of which one equilateral triangular and three square faces meet together. Consider a vertex A at which an equilateral triangular face ABC and three square faces ABDE, AFGH & AHIJ meet together (as shown in fig-2). Therefore, we will consider each pair of two faces meeting at the vertex A, and find the dihedral angle measured internally between each two faces. For ease of understanding the following symbols for different dihedral angles will be used.

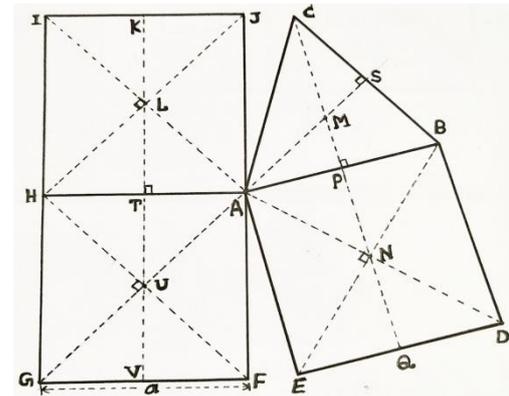


Figure 2: One equilateral triangular & three square faces meeting at vertex A, are flattened onto the plane of paper (projected view of faces).

$\theta_{TS}$  → Dihedral angle between equilateral triangular & square faces with common side. The subscript T stands for **Triangle** & S stands for **Side common**.

$\theta_{TV}$  → Dihedral angle between equilateral triangular & square faces with common vertex. Subscript T stands for **Triangle** & V stands for **Vertex common**.

$\theta_{SS}$  → Dihedral angle between two square faces with common side. The subscript T stands for **Triangle** & S stands for **Side common**.

$\theta_{SV}$  → Dihedral angle between two square faces with common vertex only. The subscript T stands for **Triangle** & S stands for **Vertex common**.

The above symbols show that there are four different types of dihedral angles (all are measured internally) between all possible distinct pairs of the faces meeting at a vertex of rhombicuboctahedron.

**3.1. Angle between adjacent triangular and square faces:** Consider an equilateral triangular face ABC and a square face ABDE having a common side AB & meeting each other at the vertex A such that they are inclined at a dihedral angle  $\theta_{TS}$ . The line CP shows the altitude of equilateral triangular face ABC & line PQ shows the line-segment joining the mid-points of opposite sides AB & DE of square face ABDE (see fig-2 above). Now, drop the perpendiculars OM & ON from the centre O of rhombicuboctahedron to the centres M & N of triangular & square faces respectively (as shown in fig-3).

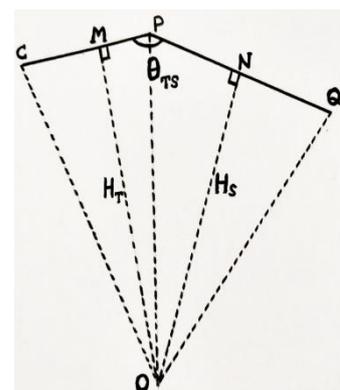


Figure 3: Dihedral angle  $\angle CPQ = \theta_{TS}$  between equi. triangular & square faces shown by the lines CP & PQ  $\perp$  to the plane of paper.

In right  $\triangle OMP$  (fig-3),

$$\tan \angle MPO = \frac{OM}{PM} = \frac{H_T}{\left(\frac{PC}{3}\right)} = \frac{\left(\frac{a(3 + \sqrt{2})}{2\sqrt{3}}\right)}{\left(\frac{a\sqrt{3}}{3}\right)} = 3 + \sqrt{2} \quad \left( \because PC = a \sin 60^\circ = \frac{a\sqrt{3}}{2} \right)$$

$$\angle MPO = \tan^{-1}(3 + \sqrt{2})$$

In right  $\triangle ONP$  (fig-3),

$$\tan \angle NPO = \frac{ON}{PN} = \frac{H_s}{\left(\frac{PQ}{2}\right)} = \frac{\left(\frac{a(1 + \sqrt{2})}{2}\right)}{\left(\frac{a}{2}\right)} = 1 + \sqrt{2} \quad (\because PQ = AE = a)$$

$$\angle NPO = \tan^{-1}(1 + \sqrt{2})$$

$$\Rightarrow \angle CPQ = \angle MPO + \angle NPO$$

$$\theta_{TS} = \tan^{-1}(3 + \sqrt{2}) + \tan^{-1}(1 + \sqrt{2})$$

$$= \pi + \tan^{-1}\left(\frac{3 + \sqrt{2} + 1 + \sqrt{2}}{1 - (3 + \sqrt{2})(1 + \sqrt{2})}\right) \quad \left(\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x + y}{1 - xy}\right) \forall xy > 1\right)$$

$$= \pi + \tan^{-1}\left(\frac{4 + 2\sqrt{2}}{-4 - 4\sqrt{2}}\right) = \pi + \tan^{-1}\left(\frac{2\sqrt{2}(\sqrt{2} + 1)}{-4(\sqrt{2} + 1)}\right) = \pi + \tan^{-1}\left(\frac{1}{-\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Hence, the dihedral angle  $\theta_{TS}$  (measured internally) between any two adjacent equilateral triangular & square faces of a rhombicuboctahedron, is given as follows

$$\theta_{TS} = \pi - \tan^{-1}\frac{1}{\sqrt{2}} \approx 144.73^\circ \quad \dots \dots \dots (7)$$

**3.2. Angle between triangular and square faces with common vertex but no common side:**

Consider an equilateral triangular face ABC and a square face AFGH meeting each other at the vertex A & having no common side such that they are inclined at a dihedral angle  $\theta_{TV}$ . The line AS shows the altitude of equilateral triangular face ABC & line AG shows the diagonal of square face AFGH (see fig-2 above). Now, drop the perpendiculars OM & OU from the centre O of rhombicuboctahedron to the centres M & U of triangular & square faces respectively (as shown in fig-4).

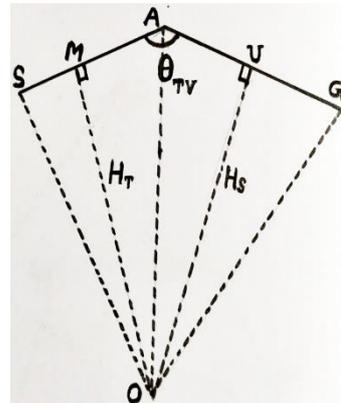


Figure 4: Dihedral angle  $\angle SAG = \theta_{TV}$  between equi. triangular & square faces shown by the lines AS & AG  $\perp$  to the plane of paper.

In right  $\triangle OMA$  (fig-4),

$$\tan \angle MAO = \frac{OM}{AM} = \frac{H_T}{\left(\frac{2AS}{3}\right)} = \frac{\left(\frac{a(3 + \sqrt{2})}{2\sqrt{3}}\right)}{\left(\frac{2 \cdot \frac{a\sqrt{3}}{2}}{3}\right)} = \frac{3 + \sqrt{2}}{2} \quad \left(\because AS = a \sin 60^\circ = \frac{a\sqrt{3}}{2}\right)$$

$$\angle MAO = \tan^{-1}\left(\frac{3 + \sqrt{2}}{2}\right)$$

In right  $\triangle OUA$  (fig-4),

$$\tan \angle UAO = \frac{OU}{AU} = \frac{H_s}{\left(\frac{AG}{2}\right)} = \frac{\left(\frac{a(1 + \sqrt{2})}{2}\right)}{\left(\frac{a\sqrt{2}}{2}\right)} = \frac{1 + \sqrt{2}}{\sqrt{2}} \quad (\because AG = \text{diagonal} = a\sqrt{2})$$

$$\begin{aligned} \angle UAO &= \tan^{-1}\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right) \\ \Rightarrow \angle SAG &= \angle MAO + \angle UAO \\ \theta_{TV} &= \tan^{-1}\left(\frac{3+\sqrt{2}}{2}\right) + \tan^{-1}\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right) \\ &= \pi + \tan^{-1}\left(\frac{\frac{3+\sqrt{2}}{2} + \frac{1+\sqrt{2}}{\sqrt{2}}}{1 - \left(\frac{3+\sqrt{2}}{2}\right)\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)}\right) \quad \left(\because \tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \forall xy > 1\right) \\ &= \pi + \tan^{-1}\left(\frac{\frac{5+2\sqrt{2}}{2}}{1 - \frac{5+4\sqrt{2}}{2\sqrt{2}}}\right) = \pi + \tan^{-1}\left(\frac{\sqrt{2}(5+2\sqrt{2})}{-(5+2\sqrt{2})}\right) = \pi + \tan^{-1}(-\sqrt{2}) = \pi - \tan^{-1}(\sqrt{2}) \end{aligned}$$

Hence, the dihedral angle  $\theta_{TV}$  (measured internally) between equilateral triangular & square faces (with no common side) meeting at a vertex of rhombicuboctahedron, is given as follows

$$\theta_{TV} = \pi - \tan^{-1}\sqrt{2} \approx 125.26^\circ \quad \dots \dots \dots (8)$$

**3.3. Angle between two adjacent square faces with a common side:** Consider two congruent square faces AFGH & AHIJ having a common side AH & meeting each other at the vertex A (see above fig-2) such that they are inclined at a dihedral angle  $\theta_{SS}$ . The line TK joins the mid-points of sides AH & IJ of square face AHIJ & the line TV joins the mid-points of sides AH & FG of square face AFGH. Drop the perpendiculars OL & OU from the centre O of rhombicuboctahedron to the centres L & U of two square faces AHIJ & AFGH respectively (see fig-5).

In right  $\triangle OLT$  (see fig-5),

$$\tan \angle LTO = \frac{OL}{TL} = \frac{H_s}{\left(\frac{AJ}{2}\right)} = \frac{\left(\frac{a(1+\sqrt{2})}{2}\right)}{\left(\frac{a}{2}\right)} = 1 + \sqrt{2} \quad (\because AJ = \text{side of square} = a)$$

$$\angle LTO = \tan^{-1}(1 + \sqrt{2})$$

$$\Rightarrow \angle UTO = \angle LTO = \tan^{-1}(1 + \sqrt{2})$$

$$\therefore \angle KTV = \angle LTO + \angle UTO$$

$$\Rightarrow \theta_{RR} = \tan^{-1}(1 + \sqrt{2}) + \tan^{-1}(1 + \sqrt{2})$$

$$= 2 \tan^{-1}(1 + \sqrt{2})$$

$$= \pi + \tan^{-1}\left(\frac{2(1+\sqrt{2})}{1 - (1+\sqrt{2})^2}\right)$$

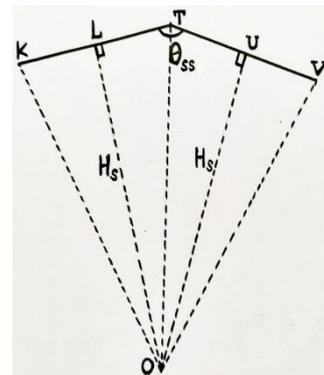


Figure 5: Dihedral angle  $\angle KTV = \theta_{SS}$  between two square faces with a common side AH shown by lines TK & TV  $\perp$  to the plane of paper.

$$= \pi + \tan^{-1}\left(\frac{2 + 2\sqrt{2}}{-2 - 2\sqrt{2}}\right) = \pi + \tan^{-1}(-1) = \pi + \left(-\frac{\pi}{4}\right) = \frac{3\pi}{4}$$

Hence, the dihedral angle  $\theta_{SS}$  between any two adjacent square faces of rhombicuboctahedron, is given as follows

$$\theta_{SS} = \frac{3\pi}{4} = 135^\circ \quad \dots \dots \dots (9)$$

It is worth noticing that the dihedral angles given from the above equations (7), (8) and (9) are the same as previously derived angles [8].

**3.4. Angle between two adjacent square faces with a common vertex but no common side:**

Consider two congruent square faces ABDE & AHIJ meeting each other at the vertex A (with no common side as shown in fig-6(a)) such that they are inclined at a dihedral angle  $\theta_{SV}$ . It is worth noticing that the diagonals AD & AI of square faces ABDE & AHIJ respectively are not collinear. Therefore, draw a straight line from common vertex A which intersect the diagonals BE & HJ at the points N' & L' respectively such that the angle between AN' & side AB is  $\theta_p$  and similarly angle between AL' & side AJ is  $\theta_p$ .

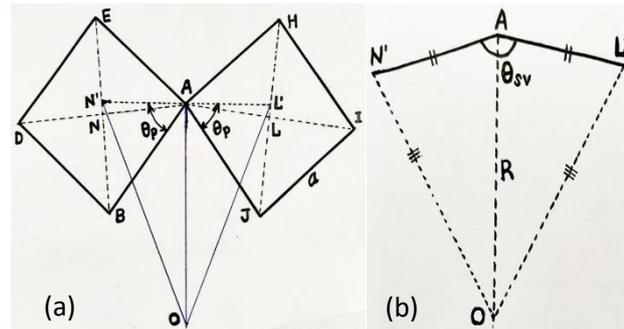


Figure 6: Dihedral angle  $\angle N'AL' = \theta_{SV}$  between two square faces ABDE & AHIJ meeting at the vertex A & having no common side (a) top-view.  $ON = H_S$ , and (b) front view.

The angle  $\theta_p$  is computed from HCR's Cosine Formula [9] as follows

$$\theta_p = \cos^{-1}\left(\frac{\sqrt{1532 - 368\sqrt{2} + 16\sqrt{34(71 - 8\sqrt{2})}}}{68}\right) \approx 52.26^\circ$$

In right  $\Delta ANN'$  (fig-6(a)),

$$\tan \angle N'AN = \frac{NN'}{AN} \Rightarrow \tan\left(\frac{\pi}{4} - \theta_p\right) = \frac{NN'}{\frac{a\sqrt{2}}{2}} \Rightarrow NN' = \frac{a}{\sqrt{2}} \tan\left(\frac{\pi}{4} - \theta_p\right) \quad \& \quad AN' = \frac{a}{\sqrt{2}} \sec\left(\frac{\pi}{4} - \theta_p\right)$$

Using Pythagorean theorem in right  $\Delta ONN'$  (fig-6(a)), as follows

$$(ON')^2 = (ON)^2 + (NN')^2 = \left(\frac{a(1 + \sqrt{2})}{2}\right)^2 + \left(\frac{a}{\sqrt{2}} \tan\left(\frac{\pi}{4} - \theta_p\right)\right)^2 = \frac{a^2(3 + 2\sqrt{2})}{4} + \frac{a^2}{2} \tan^2\left(\frac{\pi}{4} - \theta_p\right)$$

Now, using Cosine rule in  $\Delta AN'O$  (fig-6(b)) as follows

$$\cos \angle N'AO = \frac{(AO)^2 + (AN')^2 - (ON')^2}{2(AO)(AN')}$$

$$\cos \frac{\theta_{SV}}{2} = \frac{\left(\frac{a}{2}\sqrt{5 + 2\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}} \sec\left(\frac{\pi}{4} - \theta_p\right)\right)^2 - \left(\frac{a^2(3 + 2\sqrt{2})}{4} + \frac{a^2}{2} \tan^2\left(\frac{\pi}{4} - \theta_p\right)\right)}{2\left(\frac{a}{2}\sqrt{5 + 2\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}} \sec\left(\frac{\pi}{4} - \theta_p\right)\right)}$$

$$\begin{aligned}\cos \frac{\theta_{SV}}{2} &= \frac{\frac{a^2(5+2\sqrt{2})}{4} + \frac{a^2}{2} \sec^2\left(\frac{\pi}{4} - \theta_P\right) - \frac{a^2(3+2\sqrt{2})}{4} - \frac{a^2}{2} \tan^2\left(\frac{\pi}{4} - \theta_P\right)}{\frac{a^2}{\sqrt{2}} \sqrt{5+2\sqrt{2}} \sec\left(\frac{\pi}{4} - \theta_P\right)} \\ \cos \frac{\theta_{SV}}{2} &= \frac{\frac{5+2\sqrt{2}-3-2\sqrt{2}}{4} + \frac{1}{2} \left(\sec^2\left(\frac{\pi}{4} - \theta_P\right) - \tan^2\left(\frac{\pi}{4} - \theta_P\right)\right)}{\frac{1}{\sqrt{2}} \sqrt{5+2\sqrt{2}} \sec\left(\frac{\pi}{4} - \theta_P\right)} \\ \cos \frac{\theta_{SV}}{2} &= \frac{\frac{1}{2} + \frac{1}{2}}{\frac{1}{\sqrt{2}} \sqrt{5+2\sqrt{2}} \sec\left(\frac{\pi}{4} - \theta_P\right)} = \frac{\sqrt{2} \cos\left(\frac{\pi}{4} - \theta_P\right)}{\sqrt{5+2\sqrt{2}}} = \frac{\sqrt{2} \cos\left(\frac{\pi}{4} - \theta_P\right) \sqrt{5-2\sqrt{2}}}{\sqrt{5+2\sqrt{2}} \sqrt{5-2\sqrt{2}}} \\ \cos \frac{\theta_{SV}}{2} &= \sqrt{\frac{10-4\sqrt{2}}{17}} \cos\left(\frac{\pi}{4} - \theta_P\right) \Rightarrow \theta_{SV} = 2 \cos^{-1} \left( \sqrt{\frac{10-4\sqrt{2}}{17}} \cos\left(\frac{\pi}{4} - \theta_P\right) \right)\end{aligned}$$

Hence, the dihedral angle  $\theta_{SV}$  between any two square faces, meeting at a vertex with no common side in a rhombicuboctahedron, is given as follows

$$\theta_{SV} = 2 \cos^{-1} \left( \sqrt{\frac{10-4\sqrt{2}}{17}} \cos\left(\frac{\pi}{4} - \theta_P\right) \right) \approx 119.815^\circ \quad \dots \dots \dots (10)$$

Where,  $\theta_P \approx 52.26^\circ$  (as given above from Cosine Formula [9])

#### 4. Solid angles $\omega_T$ & $\omega_S$ subtended by equilateral triangular & square faces respectively at the centre of rhombicuboctahedron

**4.1. Solid angle of equilateral triangular face:** The solid angle subtended by an equilateral triangular face at the centre of rhombicuboctahedron, is given by above Eq.(III) as follows

$$\omega_T = 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{3x^2 - 1}{4x^2 - 1}} \right)$$

Substituting the value of  $x$  (as derived above using HCR's Theory of Polygon) in above formula, we get

$$\begin{aligned}\omega_T &= 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{3 \left( \frac{\sqrt{5+2\sqrt{2}}}{2} \right)^2 - 1}{4 \left( \frac{\sqrt{5+2\sqrt{2}}}{2} \right)^2 - 1}} \right) = 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{15+6\sqrt{2}-4}{20+8\sqrt{2}-4}} \right) \\ &= 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{11+6\sqrt{2}}{16+8\sqrt{2}}} \right) = 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{(11+6\sqrt{2})(2-\sqrt{2})}{8(2+\sqrt{2})(2-\sqrt{2})}} \right) \\ &= 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{22+12\sqrt{2}-11\sqrt{2}-12}{8(4-2)}} \right) = 2\pi - 6 \sin^{-1} \left( \sqrt{\frac{10+\sqrt{2}}{16}} \right) = 2\pi - 6 \sin^{-1} \left( \frac{\sqrt{10+\sqrt{2}}}{4} \right)\end{aligned}$$

Hence, the solid angle  $\omega_T$  subtended by any of 8 congruent equilateral triangular faces at the centre of a rhombicuboctahedron, is given as follows

$$\omega_T = 2\pi - 6 \sin^{-1} \left( \frac{\sqrt{10 + \sqrt{2}}}{4} \right) \text{ sr} \approx 0.24801961 \text{ sr} \quad \dots \dots \dots (11)$$

**4.2. Solid angle of square face:** The solid angle subtended by an equilateral triangular face at the centre of rhombicuboctahedron is given by above Eq.(IV) as follows

$$\omega_S = 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{2x^2 - 1}{4x^2 - 1}} \right)$$

Substituting the value of  $x$  (as derived above using Theory of Polygon) in above formula, we get

$$\begin{aligned} \omega_S &= 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{2 \left( \frac{\sqrt{5 + 2\sqrt{2}}}{2} \right)^2 - 1}{4 \left( \frac{\sqrt{5 + 2\sqrt{2}}}{2} \right)^2 - 1}} \right) = 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{10 + 4\sqrt{2} - 4}{20 + 8\sqrt{2} - 4}} \right) \\ &= 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{6 + 4\sqrt{2}}{16 + 8\sqrt{2}}} \right) = 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{(6 + 4\sqrt{2})(2 - \sqrt{2})}{8(2 + \sqrt{2})(2 - \sqrt{2})}} \right) \\ &= 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{12 + 8\sqrt{2} - 6\sqrt{2} - 8}{8(4 - 2)}} \right) = 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{4 + 2\sqrt{2}}{16}} \right) = 2\pi - 8 \sin^{-1} \left( \sqrt{\frac{\sqrt{2} + 1}{4\sqrt{2}}} \right) \\ &= 4 \left( \frac{\pi}{2} - 2 \sin^{-1} \left( \sqrt{\frac{\sqrt{2} + 1}{4\sqrt{2}}} \right) \right) = 4 \left( \frac{\pi}{2} - \sin^{-1} \left( 2 \sqrt{\frac{\sqrt{2} + 1}{4\sqrt{2}}} \sqrt{1 - \frac{\sqrt{2} + 1}{4\sqrt{2}}} \right) \right) \\ &= 4 \left( \frac{\pi}{2} - \sin^{-1} \left( \sqrt{\frac{5 + 2\sqrt{2}}{8}} \right) \right) = 4 \cos^{-1} \left( \sqrt{\frac{5 + 2\sqrt{2}}{8}} \right) = 4 \sin^{-1} \left( \sqrt{1 - \frac{5 + 2\sqrt{2}}{8}} \right) = 4 \sin^{-1} \left( \sqrt{\frac{3 - 2\sqrt{2}}{8}} \right) \\ &= 4 \sin^{-1} \left( \sqrt{\frac{6 - 4\sqrt{2}}{16}} \right) = 4 \sin^{-1} \left( \sqrt{\frac{(2 - \sqrt{2})^2}{4^2}} \right) = 4 \sin^{-1} \left( \frac{2 - \sqrt{2}}{4} \right) \end{aligned}$$

Hence, the solid angle  $\omega_S$  subtended by any of 18 congruent square faces at the centre of a rhombicuboctahedron, is given as follows

$$\omega_S = 4 \sin^{-1} \left( \frac{2 - \sqrt{2}}{4} \right) \text{ sr} \approx 0.587900762 \text{ sr} \quad \dots \dots \dots (12)$$

**4.3. Solid angle subtended by rhombicuboctahedron at any of its 24 identical vertices**

Consider any of 24 identical vertices say vertex P of rhombicuboctahedron with edge length  $a$ . Join the end points A, B, C & D of the edges AP, BP, CP & DP, meeting at vertex P, to get a (plane) trapezium ABCD of sides  $AB = a, BC = CD = AD = a\sqrt{2}$  (as shown in fig-7).

Join the foot point Q of perpendicular PQ drawn from vertex P to the plane of trapezium ABCD, to the vertices A, B, C & D. Drop the perpendiculars MN & QE from midpoint M & foot point Q to the sides CD & BC respectively of trapezium ABCD (as shown in the fig-8).

We have found out that dihedral angle between equilateral triangular & square faces is  $\angle MPN = \pi - \tan^{-1} \sqrt{2}$ .

The perpendicular PQ dropped from the vertex P to the plane of trapezium ABCD will fall at the point Q lying on the line MN (as shown in fig-9).

In  $\Delta MPN$  (see fig-9), using cosine formula as follows

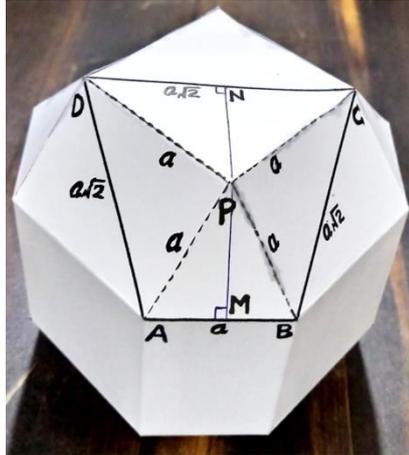


Figure 7: A trapezium ABCD is formed by joining the endpoints A, B, C & D of edges AP, BP, CP & DP meeting at the vertex P of given polyhedron.

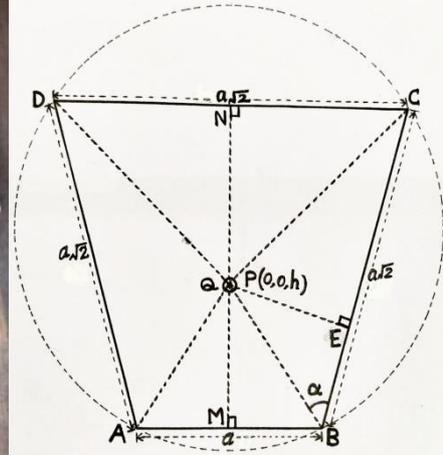


Figure 8: Point Q is the foot of perpendicular PQ drawn from the vertex P to the plane of trapezium ABCD. Point P is lying at a normal height  $h$  from the point foot Q ( $\perp$  to the plane of paper).

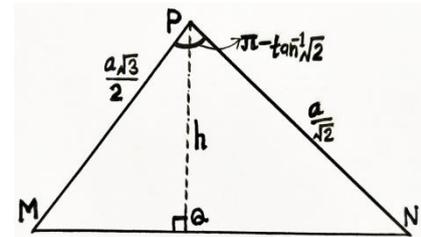


Figure 9: Perpendicular PQ dropped from vertex P to the plane of trapezium ABCD falls at the point Q on the line MN.

$$\begin{aligned} \cos \angle MPN &= \frac{(PM)^2 + (PN)^2 - (MN)^2}{2(PM)(PN)} \\ \Rightarrow \cos(\pi - \tan^{-1} \sqrt{2}) &= \frac{\left(\frac{a\sqrt{3}}{2}\right)^2 + \left(\frac{a}{\sqrt{2}}\right)^2 - (MN)^2}{2\left(\frac{a\sqrt{3}}{2}\right)\left(\frac{a}{\sqrt{2}}\right)} \\ -\cos(\tan^{-1} \sqrt{2}) &= \frac{\frac{3a^2}{4} + \frac{a^2}{2} - MN^2}{a^2 \sqrt{\frac{3}{2}}} \\ -\cos\left(\cos^{-1} \frac{1}{\sqrt{3}}\right) &= \frac{\frac{5a^2}{4} - MN^2}{a^2 \sqrt{\frac{3}{2}}} \Rightarrow -\frac{1}{\sqrt{3}} = \frac{\frac{5a^2}{4} - MN^2}{a^2 \sqrt{\frac{3}{2}}} \Rightarrow -a^2 \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} = \frac{5a^2}{4} - MN^2 \\ \Rightarrow MN^2 &= \frac{5a^2}{4} + \frac{a^2}{\sqrt{2}} \Rightarrow MN^2 = \frac{a^2(5 + 2\sqrt{2})}{4} \Rightarrow MN = \frac{a}{2} \sqrt{5 + 2\sqrt{2}} \end{aligned}$$

Now, the area of  $\Delta MPN$  (see above fig-9), is given as follows

$$\begin{aligned} \frac{1}{2}(MN)(PQ) &= \frac{1}{2}(PM)(PN) \sin(\pi - \tan^{-1} \sqrt{2}) \\ \left(\frac{a}{2} \sqrt{5 + 2\sqrt{2}}\right)(PQ) &= \left(\frac{a\sqrt{3}}{2}\right)\left(\frac{a}{\sqrt{2}}\right) \sin(\tan^{-1} \sqrt{2}) \end{aligned}$$

$$\sqrt{5 + 2\sqrt{2}} PQ = a \sqrt{\frac{3}{2}} \sin \left( \sin^{-1} \sqrt{\frac{2}{3}} \right) \Rightarrow PQ = \frac{1}{\sqrt{5 + 2\sqrt{2}}} a \sqrt{\frac{3}{2}} \sqrt{\frac{2}{3}} = a \sqrt{\frac{5 - 2\sqrt{2}}{17}}$$

Using Pythagorean theorem in right  $\Delta PQM$  (see above fig-9), we get

$$\begin{aligned} MQ &= \sqrt{(PM)^2 - (PQ)^2} = \sqrt{\left(\frac{a\sqrt{3}}{2}\right)^2 - \left(a \sqrt{\frac{5 - 2\sqrt{2}}{17}}\right)^2} = \sqrt{\frac{3a^2}{4} - \frac{a^2(5 - 2\sqrt{2})}{17}} = \frac{a}{2} \sqrt{\frac{31 + 8\sqrt{2}}{17}} \\ \Rightarrow QN &= MN - MQ = \frac{a}{2} \sqrt{5 + 2\sqrt{2}} - \frac{a}{2} \sqrt{\frac{31 + 8\sqrt{2}}{17}} = \frac{a}{2} \left( \sqrt{5 + 2\sqrt{2}} - \sqrt{\frac{31 + 8\sqrt{2}}{17}} \right) \\ &= \frac{a}{2} \sqrt{\left( \sqrt{5 + 2\sqrt{2}} - \sqrt{\frac{31 + 8\sqrt{2}}{17}} \right)^2} = \frac{a}{2} \sqrt{\left( \frac{\sqrt{85 + 34\sqrt{2}} - \sqrt{31 + 8\sqrt{2}}}{\sqrt{17}} \right)^2} \\ &= \frac{a}{2} \sqrt{\frac{(\sqrt{85 + 34\sqrt{2}} - \sqrt{31 + 8\sqrt{2}})^2}{17}} = \frac{a}{2} \sqrt{\frac{85 + 34\sqrt{2} + 31 + 8\sqrt{2} - 2\sqrt{(85 + 34\sqrt{2})(31 + 8\sqrt{2})}}{17}} \\ &= \frac{a}{2} \sqrt{\frac{116 + 42\sqrt{2} - 2\sqrt{289(11 + 6\sqrt{2})}}{17}} = \frac{a}{2} \sqrt{\frac{116 + 42\sqrt{2} - 2\sqrt{17^2(3 + \sqrt{2})^2}}{17}} \\ &= \frac{a}{2} \sqrt{\frac{116 + 42\sqrt{2} - 34(3 + \sqrt{2})}{17}} = \frac{a}{2} \sqrt{\frac{14 + 8\sqrt{2}}{17}} = a \sqrt{\frac{7 + 4\sqrt{2}}{34}} \end{aligned}$$

Using Pythagorean theorem in right  $\Delta QMB$  (see above fig-8), we get

$$BQ = \sqrt{(MQ)^2 + (MB)^2} = \sqrt{\left(\frac{a}{2} \sqrt{\frac{31 + 8\sqrt{2}}{17}}\right)^2 + \left(\frac{a}{2}\right)^2} = \sqrt{\frac{a^2(31 + 8\sqrt{2} + 17)}{68}} = a \sqrt{\frac{12 + 2\sqrt{2}}{17}}$$

Using Pythagorean theorem in right  $\Delta PQC$  (see fig-10), we get

$$\begin{aligned} QC &= \sqrt{(PC)^2 - (PQ)^2} = \sqrt{(a)^2 - \left(a \sqrt{\frac{5 - 2\sqrt{2}}{17}}\right)^2} = \sqrt{a^2 - \frac{a^2(5 - 2\sqrt{2})}{17}} \\ &= a \sqrt{\frac{12 + 2\sqrt{2}}{17}} \end{aligned}$$

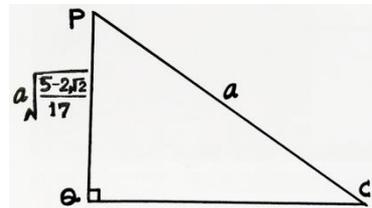


Figure 10: Right  $\Delta PQC$  is obtained by dropping  $\perp$  to the plane of trapezium ABCD.

Using Pythagorean theorem in right  $\Delta BEQ$  (see above fig-11), we get

$$\begin{aligned}
 QE &= \sqrt{(BQ)^2 - (BE)^2} = \sqrt{\left(a\sqrt{\frac{12+2\sqrt{2}}{17}}\right)^2 - \left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{\frac{a^2(12+2\sqrt{2})}{17} - \frac{a^2}{2}} = \sqrt{\frac{a^2(24+4\sqrt{2}-17)}{34}} \\
 &= \sqrt{\frac{a^2(7+4\sqrt{2})}{34}} = a\sqrt{\frac{7+4\sqrt{2}}{34}}
 \end{aligned}$$

From computed values from fig-8 above, it is clear that  $BQ = QC = QD = QA$  &  $QE = QN$ . Therefore  $\Delta BQC$ ,  $\Delta CQD$  &  $\Delta AQD$  are congruent isosceles triangles.

We know from HCR's Theory of Polygon that the solid angle ( $\omega$ ), subtended by a right triangle OGH having perpendicular  $p$  & base  $b$  at any point P at a normal distance  $h$  on the vertical axis passing through the vertex O (as shown in the fig-11), is given by the Standard Formula-1 [5,6] as follows

$$\omega = \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) - \sin^{-1}\left(\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right)$$

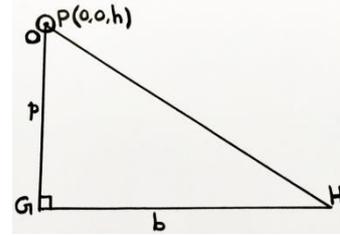


Figure-11: Point P lies at normal height  $h$  from vertex O of right  $\Delta OGH$  ( $\perp$  to plane of paper).

Now, the solid angle  $\omega_{\Delta QMB}$  subtended by right  $\Delta QMB$  at the vertex P (see above fig-8) is obtained by substituting the corresponding values (as derived above) in above standard-1

formula i.e. base  $b = MB = \frac{a}{2}$ , perpendicular  $p = MQ = \frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}$  & normal height  $h = PQ = a\sqrt{\frac{5-2\sqrt{2}}{17}}$  as follows

$$\omega_{\Delta QMB} = \sin^{-1}\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}\right)^2}}\right) - \sin^{-1}\left(\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}\right)^2}}\right)\left(\frac{a\sqrt{\frac{5-2\sqrt{2}}{17}}}{\sqrt{\left(a\sqrt{\frac{5-2\sqrt{2}}{17}}\right)^2 + \left(\frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}\right)^2}}\right)\right)$$

$$\omega_{\Delta QMB} = \sin^{-1}\left(\frac{\frac{a}{2}}{a\sqrt{\frac{12+2\sqrt{2}}{17}}}\right) - \sin^{-1}\left(\left(\frac{\frac{a}{2}}{a\sqrt{\frac{12+2\sqrt{2}}{17}}}\right)\left(\frac{a\sqrt{\frac{5-2\sqrt{2}}{17}}}{\frac{a\sqrt{3}}{2}}\right)\right)$$

$$\omega_{\Delta QMB} = \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{17}{12+2\sqrt{2}}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{3}}\sqrt{\frac{5-2\sqrt{2}}{12+2\sqrt{2}}}\right) = \sin^{-1}\left(\frac{\sqrt{6-\sqrt{2}}}{4}\right) - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2-\sqrt{2}}{3}}\right)$$

Similarly, the solid angle  $\omega_{\Delta BEQ}$  subtended by right  $\Delta BEQ$  at the vertex P (see above fig-8) is obtained by substituting the corresponding values (as derived above) in above standard formula-1 i.e. base  $b = BE = \frac{a}{\sqrt{2}}$ ,

perpendicular  $p = QE = a\sqrt{\frac{7+4\sqrt{2}}{34}}$  & normal height  $h = PQ = a\sqrt{\frac{5-2\sqrt{2}}{17}}$  as follows

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{\frac{a}{\sqrt{2}}}{\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(a\sqrt{\frac{7+4\sqrt{2}}{34}}\right)^2}}\right) - \sin^{-1}\left(\left(\frac{\frac{a}{\sqrt{2}}}{\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(a\sqrt{\frac{7+4\sqrt{2}}{34}}\right)^2}}\right)\left(\frac{a\sqrt{\frac{5-2\sqrt{2}}{17}}}{\sqrt{\left(a\sqrt{\frac{5-2\sqrt{2}}{17}}\right)^2 + \left(a\sqrt{\frac{7+4\sqrt{2}}{34}}\right)^2}}\right)\right)$$

$$\omega_{\Delta BEQ} = \sin^{-1} \left( \frac{\frac{a}{\sqrt{2}}}{a\sqrt{\frac{12+2\sqrt{2}}{17}}} \right) - \sin^{-1} \left( \left( \frac{\frac{a}{\sqrt{2}}}{a\sqrt{\frac{12+2\sqrt{2}}{17}}} \right) \left( \frac{a\sqrt{\frac{5-2\sqrt{2}}{17}}}{\frac{a}{\sqrt{2}}} \right) \right)$$

$$\omega_{\Delta BEQ} = \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{17}{6+\sqrt{2}}} \right) - \sin^{-1} \left( \sqrt{\frac{5-2\sqrt{2}}{12+2\sqrt{2}}} \right) = \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{6-\sqrt{2}}{2}} \right) - \sin^{-1} \left( \frac{\sqrt{2-\sqrt{2}}}{2} \right)$$

Thus, using symmetry in trapezium ABCD (see above fig-8), the solid angle  $\omega_{ABCD}$  subtended by the trapezium ABCD at the vertex P of rhombicuboctahedron will be twice the solid angle  $\omega_{MBCN}$  subtended by the trapezium MBCN at the vertex P, as follows

$$\begin{aligned} \omega_{ABCD} &= 2\omega_{MBCN} = 2(\omega_{\Delta QMB} + \omega_{\Delta BEQ} + \omega_{\Delta CEQ} + \omega_{\Delta CNQ}) = 2(\omega_{\Delta QMB} + 3\omega_{\Delta BEQ}) \\ &= 2 \left\{ \sin^{-1} \left( \frac{\sqrt{6-\sqrt{2}}}{4} \right) - \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{2-\sqrt{2}}{3}} \right) + 3 \left( \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{6-\sqrt{2}}{2}} \right) - \sin^{-1} \left( \frac{\sqrt{2-\sqrt{2}}}{2} \right) \right) \right\} \\ &= 2 \left\{ \sin^{-1} \left( \frac{\sqrt{6-\sqrt{2}}}{4} \right) - \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{2-\sqrt{2}}{3}} \right) + 3 \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{6-\sqrt{2}}{2}} \right) - 3 \sin^{-1} \left( \frac{\sqrt{2-\sqrt{2}}}{2} \right) \right\} \\ &= 2 \left\{ \left( \sin^{-1} \left( \frac{\sqrt{6-\sqrt{2}}}{4} \right) + \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{6-\sqrt{2}}{2}} \right) \right) - \left( \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{2-\sqrt{2}}{3}} \right) + \sin^{-1} \left( \frac{\sqrt{2-\sqrt{2}}}{2} \right) \right) \right. \\ &\quad \left. + 2 \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{6-\sqrt{2}}{2}} \right) - 2 \sin^{-1} \left( \frac{\sqrt{2-\sqrt{2}}}{2} \right) \right\} \end{aligned}$$

Using formula:  $\sin^{-1} x + \sin^{-1} y = \cos^{-1}(\sqrt{1-x^2}\sqrt{1-y^2} - xy)$  &  $2 \sin^{-1} x = \cos^{-1}(1-2x^2) \quad \forall 0 \leq |x|, |y| \leq 1$

$$\begin{aligned} &= 2 \left\{ \cos^{-1} \left( \frac{\sqrt{10+\sqrt{2}}}{4} \cdot \frac{1}{2} \sqrt{\frac{2+\sqrt{2}}{2}} - \frac{\sqrt{6-\sqrt{2}}}{4} \cdot \frac{1}{2} \sqrt{\frac{6-\sqrt{2}}{2}} \right) - \cos^{-1} \left( \frac{1}{2} \sqrt{\frac{10+\sqrt{2}}{3}} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{1}{2} \sqrt{\frac{2-\sqrt{2}}{3}} \cdot \frac{\sqrt{2-\sqrt{2}}}{2} \right) \right. \\ &\quad \left. + \cos^{-1} \left( 1 - 2 \left( \frac{1}{2} \sqrt{\frac{6-\sqrt{2}}{2}} \right)^2 \right) - \cos^{-1} \left( 1 - 2 \left( \frac{\sqrt{2-\sqrt{2}}}{2} \right)^2 \right) \right\} \\ &= 2 \left\{ \cos^{-1} \left( \frac{\sqrt{22+12\sqrt{2}}}{8\sqrt{2}} - \frac{6-\sqrt{2}}{8\sqrt{2}} \right) - \cos^{-1} \left( \frac{\sqrt{22+12\sqrt{2}}}{4\sqrt{3}} - \frac{2-\sqrt{2}}{4\sqrt{3}} \right) + \cos^{-1} \left( 1 - \frac{6-\sqrt{2}}{4} \right) - \cos^{-1} \left( 1 - \frac{2-\sqrt{2}}{2} \right) \right\} \\ &= 2 \left\{ \cos^{-1} \left( \frac{\sqrt{(3\sqrt{2}+2)^2 - 6 + \sqrt{2}}}{8\sqrt{2}} \right) - \cos^{-1} \left( \frac{\sqrt{(3\sqrt{2}+2)^2 - 2 + \sqrt{2}}}{4\sqrt{3}} \right) + \cos^{-1} \left( \frac{-2+\sqrt{2}}{4} \right) - \cos^{-1} \left( \frac{\sqrt{2}}{2} \right) \right\} \\ &= 2 \left\{ \cos^{-1} \left( \frac{3\sqrt{2}+2-6+\sqrt{2}}{8\sqrt{2}} \right) - \cos^{-1} \left( \frac{3\sqrt{2}+2-2+\sqrt{2}}{4\sqrt{3}} \right) + \cos^{-1} \left( -\frac{2-\sqrt{2}}{4} \right) - \cos^{-1} \left( \frac{1}{\sqrt{2}} \right) \right\} \\ &= 2 \left\{ \cos^{-1} \left( \frac{4\sqrt{2}-4}{8\sqrt{2}} \right) - \cos^{-1} \left( \frac{4\sqrt{2}}{4\sqrt{3}} \right) + \pi - \cos^{-1} \left( \frac{2-\sqrt{2}}{4} \right) - \frac{\pi}{4} \right\} \end{aligned}$$

$$\begin{aligned}
 &= 2 \left\{ \frac{3\pi}{4} + \cos^{-1} \left( \frac{2 - \sqrt{2}}{4} \right) - \cos^{-1} \left( \sqrt{\frac{2}{3}} \right) - \cos^{-1} \left( \frac{2 - \sqrt{2}}{4} \right) \right\} \\
 &= 2 \left\{ \frac{3\pi}{4} - \cos^{-1} \left( \sqrt{\frac{2}{3}} \right) \right\} \\
 &= \frac{3\pi}{2} - 2 \cos^{-1} \left( \sqrt{\frac{2}{3}} \right) = \frac{3\pi}{2} - \cos^{-1} \left( 2 \left( \sqrt{\frac{2}{3}} \right)^2 - 1 \right) = \frac{3\pi}{2} - \cos^{-1} \left( \frac{4}{3} - 1 \right) = \frac{3\pi}{2} - \cos^{-1} \left( \frac{1}{3} \right) \\
 &= \pi + \left( \frac{\pi}{2} - \cos^{-1} \left( \frac{1}{3} \right) \right) = \pi + \left( \sin^{-1} \left( \frac{1}{3} \right) \right) = \pi + \sin^{-1} \left( \frac{1}{3} \right)
 \end{aligned}$$

It's worth noticing that the solid angle  $\omega_V$  subtended by rhombicuboctahedron at its vertex P will be equal to the solid angle  $\omega_{ABCD}$  subtended by the trapezium ABCD at the vertex P i.e.  $\omega_V = \omega_{ABCD} = \pi + \sin^{-1} \left( \frac{1}{3} \right)$

Hence, the solid angles  $\omega_V$  subtended by a rhombicuboctahedron at any of its 24 identical vertices (at each of which one regular triangular and three square faces meet), is given as follows

$$\omega_V = \pi + \sin^{-1} \left( \frac{1}{3} \right) \text{ sr} \approx 3.481429563 \text{ sr} \quad \dots \dots \dots (13)$$

**Conclusions:** Let there be a rhombicuboctahedron (small rhombicuboctahedron) having 8 congruent equilateral triangular faces and 18 congruent square faces, 48 edges each of length  $a$ , and 24 identical vertices then all its important parameters are determined as tabulated below.

Radius( $R$ ) of circumscribed sphere passing through all 24 vertices	$R = \frac{a}{2} \sqrt{5 + 2\sqrt{2}} \approx 1.398966326a$
Normal distances $H_T$ & $H_S$ of regular triangular & square faces from the centre of rhombicuboctahedron	$H_T = \frac{a(3 + \sqrt{2})}{2\sqrt{3}} \approx 1.274273694 a$ & $H_S = \frac{a(1 + \sqrt{2})}{2} \approx 1.207106781 a$
Surface area ( $A_s$ )	$A_s = 2a^2(9 + \sqrt{3}) \approx 21.46410162 a^2$
Volume ( $V$ )	$V = \frac{2a^3(6 + 5\sqrt{2})}{3} \approx 8.714045208 a^3$
Mean radius ( $R_m$ ) or radius of sphere having volume equal to that of the rhombicuboctahedron	$R_m = a \left( \frac{6 + 5\sqrt{2}}{2\pi} \right)^{\frac{1}{3}} \approx 1.276567352 a$
Mid-radius ( $R_{md}$ ) or radius of midsphere touching all 48 edges of the rhombicuboctahedron	$R_{md} = \sqrt{R^2 - \left( \frac{a}{2} \right)^2} = a \sqrt{\frac{2 + \sqrt{2}}{2}} \approx 1.306562965 a$
Dihedral angle $\theta_{TS}$ between adjacent equilateral triangular & square faces (i.e. with a common side)	$\theta_{TS} = \pi - \tan^{-1} \frac{1}{\sqrt{2}} \approx 144.73^\circ$

Dihedral angle $\theta_{TV}$ between equilateral triangular & square faces with common vertex but no common side	$\theta_{TV} = \pi - \tan^{-1} \sqrt{2} \approx 125.26^\circ$
Dihedral angle $\theta_{SS}$ between any two adjacent square faces (i.e. having common side)	$\theta_{SS} = \frac{3\pi}{4} = 135^\circ$
Dihedral angle $\theta_{SV}$ between any two square faces meeting at the same vertex but no common side	$\theta_{SV} = 2 \cos^{-1} \left( \sqrt{\frac{10 - 4\sqrt{2}}{17}} \cos 7.26^\circ \right) \approx 119.815^\circ$
Solid angles $\omega_T$ & $\omega_S$ subtended by equilateral triangular & square faces at the centre of rhombicuboctahedron	$\omega_T = 2\pi - 6 \sin^{-1} \left( \frac{\sqrt{10 + \sqrt{2}}}{4} \right) \text{ sr} \approx 0.24801961 \text{ sr}$ $\omega_S = 4 \sin^{-1} \left( \frac{2 - \sqrt{2}}{4} \right) \text{ sr} \approx 0.587900762 \text{ sr}$
Solid angle $\omega_V$ subtended by a rhombicuboctahedron at its vertex	$\omega_V = \pi + \sin^{-1} \left( \frac{1}{3} \right) \text{ sr} \approx 3.481429563 \text{ sr}$

**Note:** Above articles had been *derived & illustrated* by Mr H.C. Rajpoot (M Tech, Production Engineering)

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