

# Structural Analysis of the Generalized Collatz Tree (k-Tree)

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## Abstract

In earlier work [1], we introduced a refined and more structurally representative Collatz tree, within which we identified a singularity. A subsequent preprint [2] established a methodological generalization of the Collatz sequences that preserves this singularity by extending it to a generalized singularity. In the present paper, we investigate the structure of the generalized Collatz tree—referred to as the *k-Tree*—arising from this transformation. Our analysis focuses on the ordering, propagation, and interaction of branch beginnings across ranks, with particular attention to the structural sets  $\mathcal{B}_k$  and  $\mathcal{A}_k$ . This study aims to elucidate the internal architecture of the generalized tree and to clarify the extent to which the geometric and dynamical features of the classical Collatz tree persist under the generalization.

## 1 Introduction

The Collatz conjecture is traditionally explored through the inverse Collatz tree, whose root is the integer 1 and whose branches encode all trajectories eventually reaching this fixed point. Despite its apparent simplicity, the structural organization of this tree has long remained insufficiently understood.

In a previous preprint [1], we proposed a new formulation of the Collatz tree that more faithfully reflects the internal dynamics of the Collatz map. This construction revealed a recurring structural phenomenon—referred to as a *singularity*—that is represented by rank 1, branch beginnings. Building upon this discovery, a second preprint [2] introduced a methodological generalization of the Collatz sequences. The purpose of this generalization was to preserve

the intrinsic binary singularity of the classical system while extending it to a parameterized framework governed by a transformation of the form

$$(1 + 2^k)n + S_k(n) \quad \text{or} \quad \frac{n}{2^k}$$

depending on the congruence class of  $n \pmod{2^k}$ . The present work continues this program by examining the structure of the generalized Collatz tree denoted as the **k-Tree**. This tree is generated by parameter-dependent branch generators. Our objective is to analyze the behavior, ordering, and propagation of branch beginnings across the ranks of the k-Tree, and we are not aiming to provide any proof of Collatz conjecture.

## 2 Generalized Collatz sequences $(1+2^k)n+S_k(n), n/2^k$ Statement

In the previous work [1] We first defined the two auxiliary functions:

$$g_k : \mathbb{E} \rightarrow \mathbb{N}$$

$$g_k(n) = \frac{n}{2^k} \quad \text{where } \mathbb{E} \text{ is the set of strictly positive natural numbers that are} \\ \text{congruent to zero modulo } 2^k$$

$$f_k : \mathbb{L} \rightarrow \mathbb{N}$$

$$f_k(n) = (1 + 2^k)n + S_k(n) \quad \text{where } S_k(n) = 2^k - (n \pmod{2^k}) \\ \text{and } \mathbb{L} \text{ is the set of strictly positive natural numbers that are not} \\ \text{congruent to zero modulo } 2^k$$

**k** is called the **Exponent parameter**.

The generalized Collatz sequence is then defined as follows:

$$C_k : \mathbb{N}^* \rightarrow \mathbb{N}$$

$$C_0 \in \mathbb{N}^*$$

$$C_{n+1} = \begin{cases} f_k(C_n) & \text{if } C_n \not\equiv 0 \pmod{2^k} \\ g_k(C_n) & \text{if } C_n \equiv 0 \pmod{2^k} \end{cases}$$

## 3 Properties of the generalized Collatz Sequences

### 3.1 Trajectory in the generalized Collatz Sequences

A *trajectory* of a strictly positive natural number  $C_0$ , refers to the ordered sequence of terms obtained by successively applying the recurrence rule of the generalized Collatz sequences, until the value **1** is reached for the first time, if it ever is.

Thus, the trajectory includes the initial term  $C_0$ , all the intermediate terms, and stops as soon as **1** is reached for the first time.

### 3.2 The Trivial Cycle

For all  $k \in \mathbb{N}^* \setminus \{2, 4\}$

We have:

$$\forall k \in \mathbb{N}^*, \forall n \in \mathbb{N}^*, \text{ if } n \bmod 2^k \neq 0$$

then the term following  $n$  is obtained by an iteration via the auxiliary function  $f_k$ , and it is equal to:

$$\text{Since } n \in [1, 2^k - 1] \text{ then } n \bmod 2^k = n, \text{ so } S_k(n) = 2^k - n$$

$$f_k(n) = (1 + 2^k) \cdot n + s_k(n) = n + n \cdot 2^k + 2^k - n = (n + 1) \cdot 2^k$$

We also have:

$$\forall k \in \mathbb{N}^*, \forall n \in \mathbb{N}^*$$

if  $(n + 1) \bmod 2^k = 0$ , then the term following  $(n + 1) \cdot 2^k$  is obtained via the second auxiliary function, and it is equal to:

$$\frac{(n + 1) \cdot 2^k}{2^k} = (n + 1)$$

So, from these two rules, and taking **1** as the first term of the generalized Collatz sequence, we get the following sequence:

$$\{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \dots, 2^k \cdot 2^k, 2^k\}$$

And since the term that succeeds the term  $2^k$  is:

$$\frac{2^k}{2^k} = 1$$

We conclude that there is a trivial generalized cycle of the form:

$$\{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \dots, 2^k \cdot 2^k, 2^k\}$$

Since **1** belongs to the cycle, then the set  $\{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \dots, 2^k \cdot 2^k, 2^k\}$  satisfies the generalized Collatz conjecture.

Moreover, Any  $a \in [1, 2^k - 1]$  came from  $g_k(a \cdot 2^k)$ , and we know that  $a \cdot 2^k$  such that  $a \in [1, 2^k]$  is an element of the trivial cycle. So all elements  $a \in [1, 2^k - 1]$  cannot be reached directly from the outside of the trivial cycle.

For  $a = 2^k$ , we know that it came from  $g_k(2^k \cdot 2^k)$ , so all elements  $a \in [1, 2^k]$  of the trivial cycle cannot be reached directly from the outside of the trivial cycle.

For the elements of the trivial cycle of the form  $a \cdot 2^k$  with  $a \in [2, 2^k]$ , we know that they came from  $f_k(a - 1)$ , which is an element of the trivial cycle, but it can also come from  $g_k(a \cdot 2^k \cdot 2^k)$  with  $a \cdot 2^k \cdot 2^k$  not an element of the trivial cycle.

So the trivial cycle has  $2^k - 1$  gateways of the form  $a \cdot 2^k$  such that  $a \in [2, 2^k]$ . These gateways are the only elements of the trivial cycle that can be reached directly from the outside of the cycle.

### 3.3 Property

It can be shown that every non-zero natural number converges to a gateway  $2^k \cdot 2^k$  of the trivial cycle of the form before reaching **1**. To demonstrate this, we define the *inverse generalized Collatz sequence*  $C_k^{-1}$ .

#### 3.4.1 Inverse Functions

- $g_k^{-1} : \mathbb{N} \rightarrow \mathbb{N}$

$$g_k^{-1}(n) = n \cdot 2^k$$

- Let  $k \in \mathbb{N}^* \setminus \{2, 4\}$  be the exponent parameter, so:

$$f_k(m) = (1 + 2^k) \cdot m + S_k(m)$$

where  $S_k(m) = 2^k - (m \bmod 2^k)$  is the smallest non-negative additive inverse of  $m \bmod 2^k$ , satisfying:

$$0 < S_k(m) < 2^k$$

We define the inverse function  $f_k^{-1}$  as follows:

$$f_k^{-1}(n) = \frac{n - t}{1 + 2^k}$$

where  $t = S_k(f_k^{-1}(n))$  is the smallest strictly positive integer such that:

$$(1 + 2^k)f_k^{-1}(n) + t \equiv 0 \pmod{2^k}$$

**Domain of the Inverse.** The inverse function  $f_k^{-1}$  is defined for all  $n \in \mathbb{N}^*$  such that:

$$\begin{cases} n \equiv 0 \pmod{2^k} \\ \exists t \in \mathbb{N}^*, 0 < t < 2^k, \text{ such that } n - t \equiv 0 \pmod{1 + 2^k} \end{cases}$$

### 3.4.2 Inverse generalized Collatz Sequence

$$C_0^{-1} \in \mathbb{N},$$

$$C_{n+1}^{-1} = \begin{cases} f_k^{-1}(C_n^{-1}) \text{ or } g_k^{-1}(C_n^{-1}) & \text{if } C_n^{-1} \equiv 0 \pmod{2^k}, \\ g_k^{-1}(C_n^{-1}) & \text{if } C_n^{-1} \not\equiv 0 \pmod{2^k} \end{cases}$$

$$\exists j \in \mathbb{N}^* : C_j = 1$$

$$\text{So } C_{j-1} = g_k^{-1}(C_j) = g_k^{-1}(1) = 2^k \cdot 1 = 2^k$$

We have  $C_{j-1} = 2^k$  is not a gateway of the trivial cycle, then can only be an element of the trivial cycle.

$$C_{j-2} = g_k^{-1}(C_{j-1}) = g_k^{-1}(2^k) = 2^k \cdot 2^k$$

We conclude that:

$$\forall C_0 \in \mathbb{N}^* \setminus \{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \dots, 2^k \cdot 2^k, 2^k\} :$$

$$\exists j \in \mathbb{N}^* : C_j = 1 \Rightarrow \exists m < j : C_m = 2^k \cdot 2^k$$

### 3.4.3 Theorem

Every natural number  $C_0 \in \mathbb{N}^* \setminus \{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \dots, 2^k \cdot 2^k, 2^k\}$ , if it converges to 1, then it first converges to  $2^k \cdot 2^k$

Formal statement:

$\forall k \in \mathbb{N}^* \setminus \{2, 4\}$  then:

$\forall C_0 \in \mathbb{N}^* \setminus \{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \dots, 2^k \cdot 2^k, 2^k\}$  :

$\exists j \in \mathbb{N}^*$  such that  $C_j = 1$

$\Rightarrow \exists m \in \mathbb{N}^*$  such that  $C_m = 2^k \cdot 2^k$  with  $m = j - 2$

### 3.5 Property

- $\forall k \in \mathbb{N}^*, \forall C_0 \in \mathbb{N}^*, \forall j \in \mathbb{N}$  we have

$$C_j \not\equiv 0 \pmod{2^k} \Rightarrow C_{j+1} \equiv 0 \pmod{2^k} \text{ with } C_{j+1} = f_k(C_j)$$

That is to say, every non-congruent to zero modulo  $2^k$  term is followed by a term that is congruent to zero modulo  $2^k$ .

- $\forall k \in \mathbb{N}^*, \forall C_0^{-1} \in \mathbb{N}^*, \forall j \in \mathbb{N}$  we have:

$$C_j^{-1} \not\equiv 0 \pmod{2^k} \Rightarrow C_{j+1}^{-1} \equiv 0 \pmod{2^k} \text{ with } C_{j+1}^{-1} = g_k(C_j^{-1})$$

That is to say, every non-congruent to zero modulo  $2^k$  term in the inverse generalized Collatz sequence is followed by a term that is congruent to zero modulo  $2^k$ .

- $\forall k \in \mathbb{N}^*, \forall C_0^{-1} \in \mathbb{N}^*, \forall j \in \mathbb{N}$  we have:

$$C_j^{-1} \equiv 0 \pmod{2^k} \Rightarrow \begin{cases} C_{j+1}^{-1} = g_k^{-1}(C_j^{-1}) \\ C_{j+1}^{-1} = f_k^{-1}(C_j^{-1}), \text{ if } C_j^{-1} \text{ is a branch generator} \end{cases}$$

That is to say, every congruent to zero modulo  $2^k$  term in the inverse generalized Collatz sequence is followed by a congruent to zero modulo  $2^k$  term, but it can be followed by a non-congruent to zero modulo  $2^k$  term if  $C_j^{-1}$  is a branch generator.

- $\forall k \in \mathbb{N}^*, \forall C_0 \in \mathbb{N}^*, \forall j \in \mathbb{N}$  we have :

$$C_j \equiv 0 \pmod{2^k} \Rightarrow \begin{cases} C_{j+1} = g_k(C_j) \equiv 0 \pmod{2^k} \\ C_{j+1} = g_k(C_j) \not\equiv 0 \pmod{2^k} \text{ if } C_j \not\equiv 0 \pmod{2^{2k}} \end{cases}$$

Every congruent to zero modulo  $2^k$  term in the generalized Collatz sequence is followed by either a congruent to zero modulo  $2^k$  or a non-congruent to zero modulo  $2^k$  term.

## 4 Construction of the generalized Collatz Tree

### 4.1 Division of $\mathbb{N}^*$ into Subsets

According to the fundamental theorem of arithmetic, every strictly positive natural number can be written as a product of prime numbers in a unique way.

For a given  $k \in \mathbb{N}^* \setminus \{2, 4\}$  of generalized Collatz sequence,

We define a partition of  $\mathbb{N}^*$  into subsets based on their prime factorization.

- If the factorization of the natural number  $n \in \mathbb{N}^*$  gives factors that are all equal to **2**, and the number of these factors equal to 2 is a multiple of  $k$ , then  $n$  belongs to the subset of natural numbers that are powers of  $2^k$  :

$$\mathcal{A}_k = \{n \in \mathbb{N}^* \mid \exists j \in \mathbb{N}, n = (2^k)^j\}$$

That is to say  $\mathcal{A}_k$  is the set of  $n \in \mathbb{N}^*$  that are a pure power of  $2^k$ .

Here, **1** is included in  $\mathcal{A}_k$ , because  $1 = (2^k)^0$ .

- If the factorization of the natural number  $n \in \mathbb{N}^*$ , gives factors so that the number of prime factors equal to 2 is less than  $k$ , then  $n$  belongs to the subset:

$$\mathcal{B}_k = \{n \in \mathbb{N}^* \mid n \not\equiv 0 \pmod{2^k}\}$$

That is to say  $\mathcal{B}_k$  is the set of  $n \in \mathbb{N}^*$  that are not divisible by  $2^k$ .

we have  $\mathcal{A}_k \cap \mathcal{B}_k = \{1\}$  if  $k = 1$ , otherwise  $\mathcal{A}_k \cap \mathcal{B}_k = \emptyset$

Because for  $k = 1$ , 1 is represented as the beginning of the Trunk but also as the smallest singularity (branch beginning with rank 1).

- If the factorization of  $n \in \mathbb{N}^*$ , gives prime factors (not all equal to 2) so that the number of prime factors equal to **2** is greater than or equal to  $k$ .

Or the factorization of  $n \in \mathbb{N}^*$ , gives prime factors (all equal to 2) so that the number of prime factors equal to **2** is greater than or equal to  $k$ , but not a multiple of  $k$ , then  $\mathbf{n}$  belongs to the subset:

$$\mathcal{M}_k = \{n \in \mathbb{N}^* \mid n = p \cdot (2^k)^j, p \not\equiv 0 \pmod{2^k}, j \geq 1\}$$

That is to say  $\mathcal{M}_k$  is the set of  $n \in \mathbb{N}^*$  that are divisible by  $2^k$ , but not a pure power of  $2^k$ .

$$\text{we have } \mathcal{M}_k \cap \mathcal{A}_k = \emptyset$$

Thus, we have the following partition:

$$\mathbb{N}^* = \mathcal{A}_k \cup \mathcal{B}_k \cup \mathcal{M}_k$$

## 4.2 Description of the Generalized Collatz $k$ -Tree to Construct

We define the  $k$ -Tree as the tree structure generated by the generalized Collatz function  $(1 + 2^k)n + S_k(n), n/2^k, k \in \mathbb{N}^* \setminus \{2, 4\}$  is the exponent parameter.

The Collatz  $k$ -Tree we are about to construct represents the set of strictly positive natural numbers that converge to **1** with  $k \in \mathbb{N}^* \setminus \{2, 4\}$ , the representation of these numbers will be based on the **rank** of the integer.

The **rank** of a natural number  $\mathbf{n}$ , denoted  $\mathbf{rank}(\mathbf{n})$ , is defined as the number of  $f_k^{-1}$  iterations required to reach a value equal to  $\mathbf{n}$ , using the inverse Collatz sequence  $C_k^{-1}$ , and starting from  $C_0^{-1} = 1$  as the initial term.

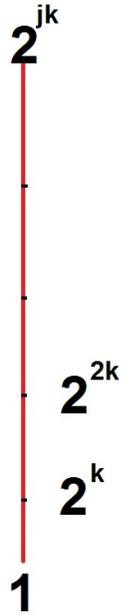
To determine the **rank** of a natural number and its representation in the Collatz  $k$ -Tree, it is not possible to deduce it directly from  $\mathbf{n}$  using  $C_k^{-1}$ , Instead, it is necessary to:

- Verify whether  $\mathbf{n}$  converges to **1** (otherwise, it cannot be represented in the  $k$ -Tree);

- Determine the **rank** of  $\mathbf{n}$ ,  $\mathbf{rank}(\mathbf{n})$ , as well as the number of successive  $g_k$  iterations required to go from one non-congruent to zero modulo  $2^k$  term to the next non-congruent to zero modulo  $2^k$  term.
- We agree that when referring to the notions of  $k$ -Tree, **rank**, or generalized Collatz sequence, we are considering a **fixed** exponent parameter  $k \in \mathbb{N}^* \setminus \{2, 4\}$ .

According to the generalized Collatz sequence, for each term, there is only one possible next term. This implies that every natural number has a unique trajectory in the generalized Collatz sequence  $C_k$ . Therefore, every natural number that converges to  $\mathbf{1}$  for a given  $C_k$  can only have one unique representation in the Collatz  $k$ -Tree).

#### 4.2.1 Properties of the Elements in Subset $\mathcal{A}_k$



**Figure1:** Trunk of the Collatz  $k$ -Tree

$$k \in \mathbb{N}^* \setminus \{2, 4\}, \forall C_0 \in \mathcal{A}_k : C_0 > 1 \quad \text{then} \quad C_0 = (2^k)^j : \quad j \in \mathbb{N}^*$$

$$\Rightarrow C_1 = g_k(C_0) = \frac{C_0}{2^k} = \frac{(2^k)^j}{2^k} = (2^k)^{j-1}$$

$$\Rightarrow C_j = \frac{2^j}{2^j} = 1$$

Therefore, for any  $C_0 \in \mathcal{A}_k : C_0 > 1$ ,  $C_0$  converges to 1.

For  $C_0 = 1$ , we have previously shown that 1 converges to 1 since it is an element of the generalized trivial cycle.

Thus:

$$\forall C_0 \in \mathcal{A}_k, C_0 \text{ converges to } 1.$$

The elements of subset  $\mathcal{A}_k$  will be represented by a vertical line, which we will call the *Trunk* of the Collatz  $k$ -Tree (Figure 1).

### 4.2.2 Remark:

The elements of subset  $\mathcal{A}_k$  have rank zero.

Let  $n \in \mathcal{M}_k$

$$\mathcal{M}_k = \{n \in \mathbb{N}^* \mid n = p \cdot (2^k)^j, p \not\equiv 0 \pmod{2^k}, j \geq 1\}$$

Thus,

$$g_k^j(n) = g_k^j(p \cdot (2^k)^j) = p$$

where  $g_k$  is the division by  $2^k$  function.

$p \in \mathcal{B}_k$ , so any natural number  $n \in \mathcal{M}_k$  will be transformed into an element of  $\mathcal{B}_k$  in the Collatz sequence by applying  $j$  times the  $g_k$  iteration.

Therefore, if all the elements of  $\mathcal{B}_k$  converges to **1**, then all the elements of  $\mathcal{M}_k$  also converges to **1**.

This is expressed as:

$$\forall n \in \mathcal{B}_k, \exists j \in \mathbb{N}^*, C_j = 1 \Rightarrow \forall m \in \mathcal{M}_k, \exists i \in \mathbb{N}^*, C_i = 1$$

### 4.2.3 Conclusion

We have shown that for all  $k \in \mathbb{N}^* \setminus \{2, 4\}$ :

- All elements of the subset  $\mathcal{A}_k$  converge to **1**.
- If all elements of the subset  $\mathcal{B}_k$  converge to **1**, then all elements of the subset  $\mathcal{M}_k$  also converges to **1**.

And since:

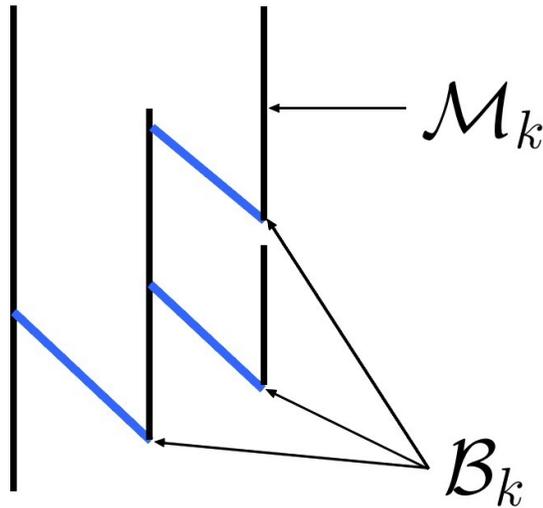
$$\mathcal{A}_k \cup \mathcal{B}_k \cup \mathcal{M}_k = \mathbb{N}^*$$

we derive the following Theorem:

#### 4.2.4 Theorem

The convergence to **1** for all  $n \in \mathbb{N}^*$  is equivalent to the convergence to **1** for all element of subset  $\mathcal{B}_k$  of non-congruent to zero modulo  $2^k$ .

#### 4.2.5 Representation of Elements from $\mathcal{B}_k$ and $\mathcal{M}_k$



**Figure2:** Diagram of a Branch

Each element  $n \in \mathcal{B}_k$  that converge to 1 will be represented by a distinct point on the Collatz  $k$ -tree.

All elements  $m \in \mathcal{M}_k$  Reducible to an  $n \in \mathcal{B}_k$  which means  $m = n \cdot (2^k)^j$  with  $n > 1$ ,  $n \not\equiv 0 \pmod{2^k}$ ,  $j \geq 1$ , will be represented on a half-line starting from it's original point in  $\mathbf{n}$ .

This half-line is called a **Branch** (Figure 2) and the origin point  $\mathbf{n}$  is called the **Beginning of a branch**.

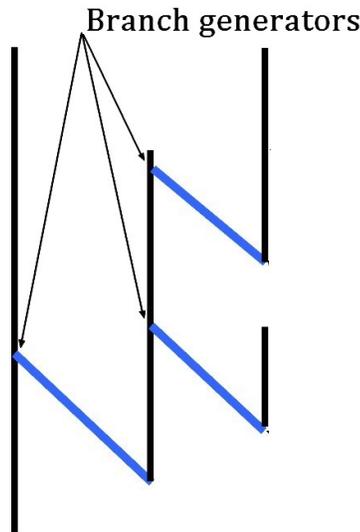
A branch can be:

- A **mother branch** (emerging directly from the trunk or another branch);
- Or a **daughter branch** (emerging from a mother branch or from the trunk).

### 4.2.6 Exception

For the number **1**, it will be represented on the trunk, and represented as a branch beginning, without representing the  $\mathcal{M}_k$  elements otherwise it will reproduce the Trunk another time; from the trivial cycle  $\{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \dots, 2^k \cdot 2^k, 2^k\}$  we can deduce that  $\text{rank}(1) = 2^k - 1$ .

### 4.2.7 Branch Generator



**Figure3:** Branch generator

A **branch generator** (Figure 3) is a natural number that belongs to a mother branch or the trunk, which means, it is an element of  $\mathcal{A}_k$  or  $\mathcal{M}_k$ .

A non-zero natural number **g** is a branch generator if and only if  $f_k^{-1}(g) = n$ , with  $n$  being a branch beginning.

$$f_k^{-1}(g) = n \quad \Rightarrow \quad n = \frac{g - S_k(n)}{1 + 2^k}$$

$$\Rightarrow g \equiv 0 \pmod{2^k} \quad \text{and} \quad g - S_k(n) \equiv 0 \pmod{1 + 2^k}$$

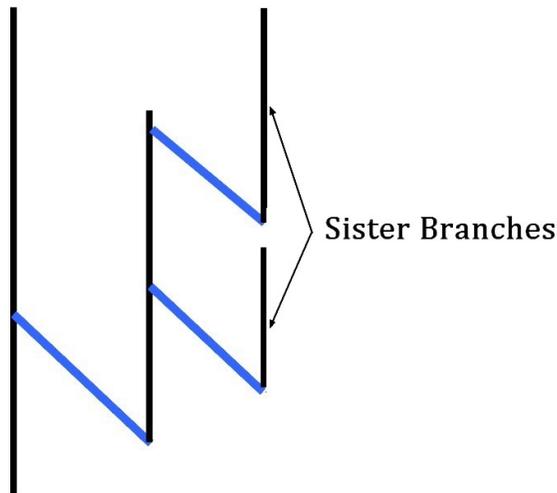
Therefore,  $\mathbf{g}$  is a branch generator if and only if:

$$\exists n \in \mathcal{B}_k, g = (1 + 2^k)n + S_k(n)$$

We represent the connection between a branch and its mother branch (or the trunk) by an inclined **line segment**, running from the top (the branch generator) down to the bottom (the branch's beginning).

### 4.2.9 Sister Branches

Two branches are **sister branches** (Figure 6) if and only if they have the same mother branch or they are daughter branches of the trunk.



**Figure4:** Sister Branches

## 5 Recognition of Structural Patterns in the Dynamics of generalized Collatz

### 5.1 The Form of a Branch Generator

Let  $g \in \mathcal{M}_k$ ,  $g$  is a branch generator if and only if:

$$\exists n \in \mathcal{B}_k : g = f_k(n) \quad \text{and this is equivalent to} \quad \exists n \in \mathcal{B}_k : n = f_k^{-1}(g)$$

We have:

$$\exists n \in \mathcal{B}_k : g = f_k(n) \Rightarrow g \equiv 0 \pmod{2^k}$$

The smallest branch generator must be superior than any gateway, witch means:

$$g = 2^k \cdot 2^k$$

And we have:

$$\exists n \in \mathcal{B}_k : n = f_k^{-1}(g) \Rightarrow \exists 0 < S_k(n) < 2^k \quad \text{such that} \quad \frac{g - S_k(n)}{1 + 2^k} = n$$

with

$$S_k(n) = 2^k - (n \bmod 2^k)$$

is always a number that is not congruent to zero modulo  $2^k$

#### 5.1.1 Theorem

A natural number  $g$  is a branch generator if and only if:

- $g \in \mathcal{M}_k$
- $g \geq 2^k \cdot 2^k$
- $\exists n \in \mathcal{B}_k : \frac{g - S_k(n)}{1 + 2^k} = n$

### 5.2 The First Branch Generator

Now we ask the question: how many times must we multiply by  $2^k$  a branch beginning or the beginning of the trunk to obtain the first branch generator?

We must first prove that if we have two branch beginnings  $d_1, d_2$  that come respectively from the branch generators  $g_1, g_2$ , then

$$d_1 < d_2 \implies g_1 < g_2$$

To prove this, let  $d_1 < d_2$  be two branch beginnings generated respectively from the branch generators  $g_1, g_2$ . Then we have:

$$d_1(1 + 2^k) < d_2(1 + 2^k)$$

Since the smallest value of  $d_2$  is  $(d_1 + 1)$ , we get:

$$d_2(1 + 2^k) \geq d_1(1 + 2^k) + (1 + 2^k)$$

Thus,

$$d_2(1 + 2^k) > d_1(1 + 2^k) + 2^k$$

We also have:

$$|S_k(d_1) - S_k(d_2)| < 2^k$$

Therefore,

$$d_2(1 + 2^k) + S_k(d_2) > d_1(1 + 2^k) + 2^k + S_k(d_1) - 2^k$$

which simplifies to:

$$d_2(1 + 2^k) + S_k(d_2) > d_1(1 + 2^k) + S_k(d_1)$$

Hence:

$$d_1 < d_2 \implies g_1 < g_2$$

### 5.2.1 Representation of a Branch Beginning

Let  $\mathbf{n}$  be a branch beginning, meaning  $n \in \mathcal{B}_k$ , or the beginning of the trunk, where  $n = 1$ . Then:

$$n \in \mathcal{B}_k \cup \{1\}.$$

We have, for all  $\mathbf{n} \in \mathcal{B}_k \cup \{1\}$ ,

$$n = \begin{cases} (1 + 2^k)j, \\ or \\ (1 + 2^k)j + 1, \\ or \\ \cdot \\ \cdot \\ or \\ (1 + 2^k)j + (2^k - 1), \\ or \\ (1 + 2^k)j + 2^k, \end{cases} \quad j \in \mathbb{N}.$$

Hence, for all  $\mathbf{n} \in \mathcal{B}_k \cup \{1\}$ ,

$$n = (1 + 2^k)j + \varphi, \quad \varphi \in [0, 2^k], \quad (j = 0 \Rightarrow \varphi > 0).$$

If we multiply  $\mathbf{n}$  by  $2^{ik}$ , we obtain:

$$g = (1 + 2^k)j \cdot 2^{ik} + \varphi \cdot 2^{ik}.$$

If  $\mathbf{g}$  is a branch generator, then for some  $d \in \mathcal{B}_k$ :

$$d = \frac{g - S_k(d)}{1 + 2^k} = \frac{(1 + 2^k)j \cdot 2^{ik} + \varphi \cdot 2^{ik} - S_k(d)}{1 + 2^k}.$$

Thus:

$$d = j \cdot 2^{ik} + \frac{\varphi \cdot 2^{ik} - S_k(d)}{1 + 2^k}.$$

The smallest number of times we must multiply  $\mathbf{n}$  by  $2^k$  to become a branch generator is the smallest natural number  $\mathbf{i} > \mathbf{0}$  such that:

$$\begin{cases} j \in \mathbb{N}, \\ \varphi \in [0, 2^k], \\ j = 0 \Rightarrow \varphi > 0, \\ d \in \mathcal{B}_k, \\ S_k(d) \in [1, 2^k - 1]. \end{cases}$$

### 5.3 The smallest successor of a branch generator

Let  $d_1, d_2$  be two branch beginnings that are generated respectively from  $g_1, g_2$  such that:  $g_1 < g_2$

$$\begin{aligned} \text{We have } g_1 &= (1 + 2^k) d_1 + S_k(d_1) \\ g_2 &= (1 + 2^k) d_2 + S_k(d_2) \end{aligned}$$

$$\text{We also have } g_2 = g_1 \cdot 2^{ik} = (1 + 2^k) d_1 \cdot 2^{ik} + S_k(d_1) \cdot 2^{ik}$$

What is the smallest  $i \in \mathbb{N}^*$  such that

$$\begin{cases} d_1 \cdot 2^{ik} \neq d_2, & \text{because } d_2 \not\equiv 0 \pmod{2^k}, \\ S_k(d_1) \cdot 2^{ik} \neq S_k(d_2), & \text{because } S_k(d_2) \in [1, 2^k - 1] \end{cases}$$

Let  $\varphi \in \mathbb{N}^*$ .

$$\begin{aligned} g_2 &= (1 + 2^k) d_1 \cdot 2^{ik} + S_k(d_1) \cdot 2^{ik} - (1 + 2^k) \varphi + (1 + 2^k) \varphi \\ &= (1 + 2^k) (d_1 \cdot 2^{ik} + \varphi) + [S_k(d_1) \cdot 2^{ik} - (1 + 2^k) \cdot \varphi] \end{aligned}$$

$$\begin{cases} d_1 \cdot 2^{ik} + \varphi = d_2, \\ [S_k(d_1) \cdot 2^{ik} - (1 + 2^k) \cdot \varphi] = S_k(d_2) \end{cases}$$

So we must find the smallest  $i \in \mathbb{N}^*$  such that we can find  $\varphi \in \mathbb{N}^*$  satisfying

$$\begin{cases} S_k(d_1) \cdot 2^{ik} - \varphi(1 + 2^k) \in [1, 2^k - 1], \\ d_1 \cdot 2^{ik} + \varphi \not\equiv 0 \pmod{2^k} \end{cases}$$

The second condition is equivalent to  $d_1 \cdot 2^{ik} + \varphi \not\equiv 0 \pmod{2^k}$

## 6 Conclusion

The structural analysis of the generalized Collatz tree, or k-Tree, has revealed that its organization does not replicate the exact structural formulas observed in the classical Collatz tree. While several analogies persist—particularly in the behavior of branch beginnings and their hierarchical propagation—the generalized framework displays distinctions that require deeper investigation before a complete structural correspondence can be established.

In this preprint, our exploration of the k-Tree remained intentionally limited. A full structural characterization of the generalized tree, including closed-form descriptions of ranks, branch generators, and their asymptotic behavior, has not yet been undertaken.

## References

- [1] Ammar Hamdous, *Revealing a Singularity in Collatz Sequences*, Zenodo, 2025.
- [2] Ammar Hamdous, *Methodological Generalization of the Collatz Sequences to  $(1 + 2^k)n + S_k(n)$  if  $n \bmod 2^k \neq 0$ , and  $n/2^k$  if  $n \bmod 2^k = 0$ , with Computational Verification for  $k = 1$  up to  $k = 20$* , Zenodo, 2025.