An infinite sum in vol. 1 “Integrals and Series - Elementary Functions”(1986) by Prudnikov, Brychkov and Marychev

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ABSTRACT. In this note, we study the sum

$$S=\sum_{n=1}^{\infty}\frac{1}{n\left(n^2+1\right)}=\frac{1}{2}+\frac{1}{10}+\frac{1}{30}+\frac{1}{68}+\ldots$$

Keywords: Euler’s sums, Digamma function, Euler-Mascheroni constant, series, integrals.

1. Introduction

In this note, we study the sum

$$\sum_{n=1}^{\infty}\frac{1}{n\left(n^2+1\right)}=\frac{1}{2}\left(\psi(1+i)+\psi(1-i)-2\,\psi(1)\right)=0.6718659855\ldots \quad (1)$$

where $\psi(x)$ is the Digamma function. This formula appears in Prudnikov’s book [12] (Chapter 5, p.686, 5.1.25.13, 5.1.25.14).

Remark: $i=\sqrt{-1}$, $\psi(1)=-\gamma$ (γ is the Euler-mascheroni constant).

In this note, we give some representations for

$$S=\sum_{n=1}^{\infty}\frac{1}{n\left(n^2+1\right)}=\sum_{n=1}^{\infty}\frac{1}{n^3+n}=\frac{1}{1^3+1}+\frac{1}{2^3+2}+\frac{1}{3^3+3}+\ldots \quad (2)$$

Remark: (1) appears in [3,p.3] and [8,p.27].

Notations

- Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\ ,\ s>1 \quad (3)$$

- Euler’s constant

$$\gamma=\lim_{n\to\infty}\left(H_n-\ln(n)\right) \quad (4)$$

where $H_n=\sum_{k=1}^{n}\frac{1}{k}$ is the Harmonic number.

- Natural logarithm of 2

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \tag{5}$$

■ Number Pi

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1} \tag{6}$$

■ Complex numbers (Real and Imaginary Parts)

$$z = x + i\, y\,,\ \ \mathrm{x},\ y \in \mathbb{R}\ ;\ \ \mathrm{Re}(z) = x\,,\ \ \mathrm{Im}(z) = y\,,\ \ i = \sqrt{-1} \tag{7}$$

■ Digamma function

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n\,(n+z)}\ \ ,\ z \neq 0,\ -1,\ -2,\ \ldots \tag{8}$$

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right),\ \ z \neq 0,\ -1,\ -2,\ \ldots \tag{9}$$

■ Pochhammer symbol

$$(x)_n = x\,(x+1)\,(x+2) \ldots (x+n-1)\ \ ,\ (x)_0 = 1 \tag{10}$$

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2. Main formulas

Entry 1.

$$\mathrm{S} = 1 - \sum_{n=1}^{\infty} \frac{n}{(n+2)\left((n+1)^2+1\right)} = 1 - \sum_{n=2}^{\infty} \frac{n-1}{(n+1)\left(n^2+1\right)} \tag{11}$$

Entry 2.

$$\mathrm{S} = \sum_{n=1}^{\infty} \frac{1}{n\left(n^3+1\right)} + \sum_{n=2}^{\infty} \frac{(n-1)\,n}{\left(n^2+1\right)\left(n^3+1\right)} \tag{12}$$

$$\mathrm{S} = \sum_{n=1}^{\infty} \frac{1}{(2\,n-1)\left((n-1)^2+n^2\right)} - \sum_{n=1}^{\infty} \frac{3\,n}{\left(n^2+1\right)\left(4\,n^2+1\right)} \tag{13}$$

$$\mathrm{S} = \sum_{n=1}^{\infty} \frac{n^2+1}{n^3\left(n^2+2\right)} - \sum_{n=2}^{\infty} \frac{1}{n^3\left(n^2+1\right)\left(n^2+2\right)} \tag{14}$$

$$\mathrm{S} = \sum_{n=1}^{\infty} \frac{1}{n^3+n+1} + \sum_{n=2}^{\infty} \frac{1}{n\left(n^2+1\right)\left(n^3+n+1\right)} \tag{15}$$

$$\mathrm{S} = \sum_{n=1}^{\infty} \frac{a}{n\,(n+a)} - \sum_{n=1}^{\infty} \frac{a\,n-1}{(n+a)\left(n^2+1\right)}\ \ ,\ a > 0 \tag{16}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^3+n+a} + \sum_{n=1}^{\infty} \frac{a}{\left(n^3+n\right)\left(n^3+n+a\right)}\ \ ,\ a \geq 0 \tag{17}$$

$$S=\sum_{n=1}^{\infty}\frac{1}{n^3+a\,n}+\sum_{n=1}^{\infty}\frac{a-1}{\left(n^3+n\right)\left(n^2+a\right)}\ ,\ a\geq 0 \tag{18}$$

Entry 3.

$$\mathrm{S}=\gamma+\sum_{n=1}^{\infty}\left(\ln\left(\frac{n+1}{n}\right)-\frac{n}{n^2+1}\right) \tag{19}$$

$$\mathrm{S}=\gamma+\sum_{n=1}^{\infty}\left(\ln\left(\frac{2\,n+1}{2\,n-1}\right)-\frac{2\,n-1}{(2\,n-1)^2+1}-\frac{2\,n}{(2\,n)^2+1}\right) \tag{20}$$

$$S=\gamma+\ln\left(\prod_{n=1}^{m}\left(\frac{2\,n+1}{2\,n-1}\right)\right)-\sum_{n=1}^{2\,m}\frac{n}{n^2+1}+\sum_{n=m+1}^{\infty}\left(\ln\left(\frac{2\,n+1}{2\,n-1}\right)-\frac{2\,n-1}{(2\,n-1)^2+1}-\frac{2\,n}{(2\,n)^2+1}\right)\ ,\ m\in\{1,\,2,\,3,\,\ldots\} \tag{21}$$

Entry 4.

$$\mathrm{S}=\gamma+\frac{\ln 2}{2}-\sum_{n=1}^{\infty}|G_n|\,(n-1)!\,\mathrm{Re}\left(\frac{1}{(1+i)_n}\right) \tag{22}$$

Remark: G_n are the Gregory numbers

$$G_1=\frac{1}{2},\ \ G_n=\frac{(-1)^{n+1}}{n+1}-\sum_{k=1}^{n-1}\frac{(-1)^{n-k}\,G_k}{n+1-k}\ ,\ n=2,3,4,\ \ldots \tag{23}$$

$$G_n=\frac{1}{n!}\int_0^1 x\,(x-1)\,(x-2)\ldots(x-n+1)\,dx=\int_0^1\binom{x}{n}dx \tag{24}$$

$$G_n=\left\{\frac{1}{2},\,-\frac{1}{12},\,\frac{1}{24},\,-\frac{19}{720},\,\frac{3}{160},\,-\frac{863}{60\,480},\ \ldots\right\} \tag{25}$$

Entry 5.

$$\mathrm{S}=\frac{5}{2}-\frac{\pi^2}{6}-\sum_{n=2}^{\infty}\frac{n+1}{(n-1)\,n^2\left(n^2+1\right)} \tag{26}$$

$$\mathrm{S}=\frac{\pi\coth(\pi)-1}{2}-\sum_{n=1}^{\infty}\frac{n}{(n+1)\left((n+1)^2+1\right)} \tag{27}$$

$$\mathrm{S}=\frac{\pi^2}{6}-\frac{\pi\coth(\pi)-1}{2}+\sum_{n=2}^{\infty}\frac{n-1}{n^2\left(n^2+1\right)} \tag{28}$$

Entry 6.

$$\mathrm{S}=\ln(2)+\frac{1}{60}-\sum_{n=4}^{\infty}\frac{(n-3)\,(n-1)}{(2\,n-1)\,2\,n\left(n^2+1\right)} \tag{29}$$

$$\mathrm{S}=6-8\ln(2)+\sum_{n=1}^{\infty}\frac{3\,n-1}{n(n+1)\,(2\,n+1)\left(n^2+1\right)} \tag{30}$$

$$\mathrm{S}=48\ln(2)-32-\sum_{n=1}^{\infty}\frac{17}{\left(n^2+1\right)\left(16\,n^3-n\right)} \tag{31}$$

$$S = 8\ln(2) - 4 - \sum_{n=1}^{\infty} \frac{5}{(n^2+1)(4n^3-n)} \tag{32}$$

Entry 7. for $m \in \{1, 2, 3, \ldots\}$ we have

$$S = \sum_{n=1}^{m} \frac{1}{n^3+n} + \sum_{n=0}^{\infty} (-1)^n \left(\zeta(2n+3) - \sum_{k=1}^{m} k^{-2n-3}\right) \tag{33}$$

Entry 8.

$$S = \gamma + \frac{\ln(2)}{2} - \int_0^{\infty} \left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right) e^{-x} \cos(x)\, dx \tag{34}$$

$$S = \gamma + \frac{\ln(2)}{2} - \frac{1}{4} - 2\int_0^{\infty} \frac{x^3}{(4+x^4)(e^{2\pi x}-1)}\, dx \tag{35}$$

$$S = \frac{1}{4} + \frac{\ln(2)}{2} - 2\int_0^{\infty} \frac{x(4-x^2)}{(e^{2\pi x}-1)(4+4x^2+x^4+x^6)}\, dx \tag{36}$$

Entry 9.

$$S = \frac{1}{3} + \sum_{n=0}^{\infty} (\zeta(n+3)-1) \sum_{k=\lfloor n/3 \rfloor}^{\lfloor n/2 \rfloor} (-1)^k \binom{k}{n-2k} + \sum_{n=1}^{\infty} \frac{1}{n(n^2+1)(n^3+n+1)} \tag{37}$$

Entry 10. for $k \in \{2, 3, 4, \ldots\}$ we have

$$S = \sum_{n=1}^{\infty} \frac{1}{n^3+kn} + \sum_{n=1}^{\infty} \sum_{m=2}^{k} \frac{1}{n(n^2+m-1)(n^2+m)} \tag{38}$$

Entry 11.

$$S = \sum_{n=1}^{\infty} \frac{2n+1}{(n^2+1)((n+1)^2+1)} H_n \tag{39}$$

Remark: $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the Harmonic number.

Entry 12.

$$S = 1 - \sum_{n=1}^{\infty} \sum_{k=n^2+1}^{(n+1)^2-1} \frac{1}{k(k+1)} + \sum_{n=1}^{\infty} \frac{n}{(n+1)^2((n+1)^2+1)} \tag{40}$$

Entry 13.

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{n}{n^2+1}\right) \tag{41}$$

$$S = H_m - \sum_{n=1}^{m} \frac{n}{n^2+1} + \sum_{n=m+1}^{\infty} \frac{1}{n(n^2+1)}, \quad m \in \mathbb{N} \tag{42}$$

$$S = \gamma + \lim_{m \to \infty} \left(\ln(m) - \sum_{n=1}^{m} \frac{n}{n^2+1}\right) \tag{43}$$

Remark: $H_m = \sum_{k=1}^{m} \frac{1}{k}$ is the Harmonic number.

Entry 14.

$$S = \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} (\zeta(3n) - 1) - \sum_{n=2}^{\infty} \frac{n-1}{n(n^2+1)(n^3+1)} \tag{44}$$

Entry 15.

$$S = \frac{1 - 2\ln(2)}{2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3 + 2n - 1} + \sum_{n=1}^{\infty} \frac{4n}{16n^4 - 1} \tag{45}$$

$$S = \frac{2\ln(2) - 1}{2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3 + 2n - 1} - \sum_{n=1}^{\infty} \frac{1}{16n^5 - n} \tag{46}$$

Entry 16.

$$S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n^2 - 2n + 1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 + n} \tag{47}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{4n^3 + n} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 + n} \tag{48}$$

Entry 17. for $m \in \{1, 2, 3, \ldots\}$ we have

$$S = \sum_{n=1}^{m} (-1)^{n-1} \zeta(2n+1) + (-1)^m \sum_{n=1}^{\infty} \frac{1}{n^{2m+1}(n^2+1)} \tag{49}$$

Entry 18.

$$S = \lim_{m \to \infty} \left(\frac{(-1)^m}{2} + \sum_{n=1}^{m} (-1)^{n-1} \zeta(2n+1) \right) \tag{50}$$

Entry 19.

$$S = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{n^5}{n^8 - 1} - \sum_{n=2}^{\infty} \frac{1}{n^5 - n} + \sum_{n=2}^{\infty} \frac{n}{n^8 - 1} \tag{51}$$

$$S = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{n^3}{n^6 + 1} - \sum_{n=2}^{\infty} \frac{n}{n^6 + 1} + \sum_{n=2}^{\infty} \frac{1}{n^7 + n} \tag{52}$$

$$S = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{n^5}{n^8 - 1} - \sum_{n=2}^{\infty} \frac{n^3}{n^8 - 1} + \sum_{n=2}^{\infty} \frac{n}{n^8 - 1} - \sum_{n=2}^{\infty} \frac{1}{n^9 - n} \tag{53}$$

Entry 20.

$$S = \lim_{m \to \infty} \sum_{n=m+1}^{\infty} \frac{6mn^2 + 2m^3 + 2m}{((n+m)^3 + n + m)((n-m)^3 + n - m)} \tag{54}$$

$$S = \sum_{n=m+1}^{\infty} \frac{6mn^2 + 2m^3 + 2m}{((n+m)^3 + n + m)((n-m)^3 + n - m)} + \sum_{n=2m+1}^{\infty} \frac{1}{n^3 + n}, \quad m \in \{1, 2, 3, \ldots\} \tag{55}$$

Entry 21.

$$S = \zeta(3) - \sum_{n=1}^{\infty} \frac{1}{n^5 + n^3} \tag{56}$$

$$S = \frac{7}{8}\,\zeta(3) - \sum_{n=1}^{\infty} \frac{1}{(2\,n-1)^5 + (2\,n-1)^3} + \sum_{n=1}^{\infty} \frac{1}{(2\,n)^3 + 2\,n} \tag{57}$$

Entry 22.

$$S = \int_0^{\infty} \frac{1 - \cos(x)}{e^x - 1}\, dx \tag{58}$$

$$S = \int_{-\infty}^{\infty} \frac{1 - \cos(\ln(1 + e^{-x}))}{1 + e^{-x}}\, dx \tag{59}$$

$$S = \int_0^{2\pi} \frac{1 - \cos(x)}{e^x - 1}\, dx + \sum_{n=1}^{\infty} \frac{e^{-2\pi n}}{n^3 + n} \tag{60}$$

$$S = 4 \int_0^{\pi} \frac{(\sin(x))^2}{e^{2x} - 1}\, dx + \sum_{n=1}^{\infty} \frac{e^{-2\pi n}}{n^3 + n} \tag{61}$$

$$S = -\int_0^{\infty} \ln(1 - e^{-x}) \sin(x)\, dx \tag{62}$$

$$S = -\int_0^{\pi/2} \ln(1 - e^{-x}) \sin(x)\, dx + \sum_{n=1}^{\infty} \frac{2 \cosh(n\pi/2)}{(n^2 + 1)(e^{n\pi} + 1)} \tag{63}$$

$$S = \frac{1}{2} \int_0^{\infty} \frac{2\,x - \sin(2\,x)}{(\sinh(x))^2}\, dx \tag{64}$$

$$S = -\frac{4}{\pi} \int_0^{\pi/2} x \sinh(x) \sin(\ln(2 \cos(x)))\, dx \tag{65}$$

Entry 23.

$$S = \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{(i)_n}{n\,(1 + i)_n}\right) \tag{66}$$

$$S = \operatorname{Re}\left(-\sum_{n=1}^{\infty} \frac{(i)_n}{n\, n!}\right) \tag{67}$$

$$S = \operatorname{Re}\left(2 \sum_{n=0}^{\infty} \frac{1}{n + 1 + i} \left({}_2F_1(1, -i, 2 + n + i, -1) - 1\right)\right) \tag{68}$$

$$S = \operatorname{Re}\left(1 - \frac{1 - i}{2}\, {}_3F_2(1, 1, 2 - i; 2, 3; 1)\right) \tag{69}$$

Remark: ${}_2F_1$ is the Gauss hypergeometric function, and ${}_3F_2$ is the generalized hypergeometric function.

Entry 24.

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\zeta(2\,n+1, 2\,n+1) + \frac{(2\,n-1)^{-(2\,n-1)}}{(2\,n-1)^2 + 1} + \frac{(2\,n)^{-2\,n+1}}{(2\,n)^2 + 1}\right) \tag{70}$$

Remark: $\zeta(2n+1, 2n+1)$ is the Hurwitz zeta function

$$\zeta(2n+1, 2n+1) = \sum_{k=2n+1}^{\infty} \frac{1}{k^{2n+1}} \tag{71}$$

Entry 25.

$$S \sim S(n) \;,\; n \longrightarrow \infty \tag{72}$$

$$S(n) = \gamma - \sum_{k=1}^{n} \frac{k}{k^2+1} + \frac{\ln\left((n+1)^2+1\right)}{2} - \frac{n+1}{2\left((n+1)^2+1\right)} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\,\mathrm{Re}\left((n+1+i)^{-2k}\right)}{2k} B_k \tag{73}$$

Remark: B_k are the Bernoulli numbers

$$B_k = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \ldots\right\} \tag{74}$$

Entry 26.

$$\mathrm{S} = \sum_{n=1}^{\infty} \frac{1}{n^3 - (-1)^n n} - \sum_{n=1}^{\infty} \frac{1}{n\left((2n)^4 - 1\right)} \tag{75}$$

$$\mathrm{S} = \frac{1}{2} + \sum_{n=0}^{\infty} \left(Li_{2n+3}\left((-1)^n\right) - (-1)^n\right) - \sum_{n=1}^{\infty} \frac{\zeta(4n+1)}{2^{4n}} \tag{76}$$

Remark: $\mathrm{Li}_s(z)$ is the Polylogarithm function

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \;,\; |z| < 1 \tag{77}$$

Entry 27.

$$S + \frac{\pi}{4} = \frac{5}{4} + \sum_{n=1}^{\infty} \frac{1}{(2n)^3 + 2n} + \sum_{n=1}^{\infty} \frac{2(4n-1)}{(4n-1)^4 - 1} - \sum_{n=1}^{\infty} \frac{2}{(4n+1)^5 - (4n+1)} \tag{78}$$

$$S - \frac{\pi}{4} + \frac{1}{4} = \sum_{n=1}^{\infty} \frac{1}{(2n)^3 + 2n} + \sum_{n=1}^{\infty} \frac{2(4n+1)}{(4n+1)^4 - 1} - \sum_{n=1}^{\infty} \frac{2}{(4n-1)^5 - (4n-1)} \tag{79}$$

Entry 28. for $m \in \{1, 2, 3, \ldots\}$ we have

$$\mathrm{S} = \sum_{n=1}^{\infty} \sum_{k=1}^{m} \frac{1}{2^{3k-3}(2n-1)^3 + 2^{k-1}(2n-1)} + \sum_{n=1}^{\infty} \frac{1}{2^{3m} n^3 + 2^m n} \tag{80}$$

Entry 29. for $m \in \{1, 2, 3, \ldots\}$ we have

$$\mathrm{S} = \sum_{n=1}^{m} \frac{1}{n^3+n} + \sum_{n=0}^{\infty} \left(2(m+1)^2+1\right)^{-n-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(2(m+1)^2\right)^{k+1} \left(\zeta(2k+3) - \sum_{r=1}^{m} r^{-2k-3}\right) \tag{81}$$

Entry 30.

$$s_n = \frac{a(n)}{\prod_{k=1}^{n}\left(k^3+k\right)} = \frac{a(n)}{n!\,\prod_{k=1}^{n}\left(k^2+1\right)} \Longrightarrow s_n \longrightarrow S \tag{82}$$

where

$$a(n) = (n^3 + n)\, a(n-1) + \prod_{k=1}^{n-1} (k^3 + k) \quad , \quad n = 2, 3, 4, \ldots; \; a(1) = 1 \tag{83}$$

Entry 31.

$$\frac{1}{S} = u_1 - \frac{u_1^2}{u_1 + u_2 -} \; \frac{u_2^2}{u_2 + u_3 -} \; \frac{u_3^2}{u_3 + u_4 -} \; \ldots = 2 - \cfrac{2^2}{12 - \cfrac{10^2}{40 - \cfrac{30^2}{98 - \ldots}}} \tag{84}$$

where

$$u_n = n^3 + n \quad , \quad n = 1, 2, 3, \ldots \tag{85}$$

Entry 32.

$$\mathrm{S} = \sum_{n=1}^{\infty} \frac{1}{(n - i)(n + 1 - i)} \sum_{k=1}^{n} \frac{1}{k (k + i)} \tag{86}$$

$$\mathrm{S} = \sum_{n=1}^{\infty} \frac{1}{(n + i)(n + 1 + i)} \sum_{k=1}^{n} \frac{1}{k (k - i)} \tag{87}$$

Entry 33.

$$\mathrm{S} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n \left((n^2 + 1)^{-k} - (n^2 + 2)^{-k} \right) \tag{88}$$

Entry 34.

$$4 \; {}_3F_2\left(\frac{1}{2} - i, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; 1 \right) = (4 \ln(2) + 2\, S + (\pi \coth(\pi) - 1)\, i) \; {}_2F_1\left(\frac{1}{2} - i, \frac{1}{2}; \frac{3}{2}; 1 \right) \tag{89}$$

Remark: ${}_2F_1$ is the Gauss hypergeometric function, and ${}_3F_2$ is the generalized hypergeometric function.

Entry 35.

$$F(u) = \int_u^{\infty} \frac{1 - \cos(x)}{e^x - 1} \, dx \tag{90}$$

$$\mathrm{u} = F(u) \Longrightarrow u = 0.599642 \ldots \tag{91}$$

$$u_1 = \frac{3}{5} \quad , \quad u_{n+1} = F(u_n) \; , \; n = 1, 2, 3, \ldots \Longrightarrow u_n \longrightarrow u = 0.599642 \ldots \tag{92}$$

$$\mathrm{S} = u + \sum_{n=0}^{\infty} \frac{u^{n+2}}{n + 2} \, c(n) \tag{93}$$

$$c(0) = \frac{1}{2} \quad , \quad c(n) = \frac{(-1)^{\lfloor n/2 \rfloor}}{(2 \lfloor n/2 \rfloor + 2)!} \left(\frac{1 + (-1)^n}{2} \right) - \sum_{k=1}^{n} \frac{c(n-k)}{(k+1)!} \tag{94}$$

$$c(n) = \left\{ \frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{48}, -\frac{1}{360}, -\frac{1}{1440}, \frac{1}{6048}, \frac{1}{80\,640}, -\frac{1}{181\,440}, \ldots \right\} \tag{95}$$

Entry 36. For $0 < m < 2$ we have

$$S = \sum_{n=1}^{\infty} \frac{1}{n\left(n^2+1+m\right)} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k-1} m^k}{n\left(n^2+1\right)^{k+1}} \tag{96}$$

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