On the irrationality of some notable constants

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Abstract

We present the quantum Riemann sum (Q-sum) operator framework and use it to prove the irrationality of the Riemann-ζ function at odd integers, the Dirichlet-β function at at all positive integers $n \geq 2$, as well as that of the Euler-Mascheroni constant (γ). By establishing a recursive functional hierarchy, we circumvent the classical "parity barrier" that has traditionally isolated even and odd zeta-type constants. We utilize the p-adic Newton Polygon to demonstrate that the arithmetic complexity of the operator kernel is an invariant of the functional hierarchy. Therefore, the irrationality of the transcendental anchors $\zeta(2)$ and $\beta(1)$ necessitates the irrationality of the entire chain. This line of reasoning can be extended to incorporate γ, thereby substantiating its long-held irrationality.

1 Introduction

The arithmetic nature of mathematical constants has remained one of the most recalcitrant problems in number theory. While the irrationality of π and e has been known since the work of Lambert (1761) and Hermite (1873), other notable constants have largely resisted classification:

- *ζ at odd integers:* Though Apéry (1979) famously proved irrationality of $\zeta(3)$ using a specific recurrence, his method was widely regarded as a "miracle" that did not generalize to other odd zeta values.
- *Catalan's constant (G):* Defined as $\beta(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$, its irrationality has been a major open conjecture since the 19th century. Previous results were limited to proving the irrationality of at least one of several values in a set (Rivoal, 2000; Zudilin, 2001).
- *The Euler-Mascheroni constant (γ):* Even the basic question of whether γ is algebraic remained unanswered (Sondow, 2003). While it is conjectured to be transcendental, no proof existed to even rule out it being a simple fraction.

The study of these constants is inextricably linked to the summation of divergent or slowly convergent series. Traditionally, methods such as Borel summation or Abel regularization (Abel, 1826) have been used to assign values to such series. However, these methods often obscure the underlying arithmetic structure. The Euler-Mascheroni constant γ, defined as:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \tag{1}$$

can be viewed as a regularized sum of the harmonic series. Hardy (1949) noted that the regularization of such sums often requires a "subtraction of infinities," a process that in modern terms is handled by the zeta-function regularization of Hawking (1977) and others in the context of quantum field theory.

In this paper, we bypass the limitations of the previous proof attempts by employing an operator-based approach to regularization which acts as a p-adic isometry, mapping basis functions into a functional hierarchy where irrationality is a conserved property. The inspiration for our work comes from the ancient idea of building mathematics from knowledge of the physical world: specifically, we want the rules of calculus - that anticipated the discovery of quantum mechanics - to be reconciled with the quantized nature of reality. To this end, we will introduce the concept of *quantum Riemann sum*, in short Q-sum, to adapt the concept of Riemann integral in the context of quantized observables. In fact, in the continuum, the evaluation of for example a theormodynamic state function $S(x)$ along a path is given by the total change:

$$\mathcal{S} \;=\; \int_{\Gamma(x)} dS \;=\; \int_0^\infty \mathscr{D}S\; dx \tag{2}$$

where $\mathscr{D} = \frac{d}{dx}$. This "standard" Riemann Sum approach assumes that the derivative can be evaluated at an infinitesimal scale. In the quantum world, however, the path Γ is not a smooth flow but a sequence of discrete jumps between allowed states.

The Q-sum is introduced precisely to give meaning to this state change in a quantized domain. We assume that the total change of the function $S(x$ is the sum of the changes $\mathscr{D}S(k)$ at each of the allowed integer value (k) of the variable x. By identifying the translation operator $e^{\mathscr{D}}$ as the generator of our discrete path, we map the classical integral to the operational Q-sum:

$$\int_0^\infty \mathscr{D}S\,dx \;\overset{\mathrm{Q}}{=}\; \sum_{k=0}^{\infty} \mathscr{D}S(k) \;\overset{\mathrm{Q}}{=}\; \left(\frac{\mathscr{D}}{1-e^{\mathscr{D}}}\right) S(x)\Big|_{x=0} \tag{3}$$

The operator $\hat{h} = \frac{\mathscr{D}}{1-e^{\mathscr{D}}}$ can be formally associated to a Todd operator, $\mathrm{Td}(\mathscr{D})$, which serves as the generating function for the Bernoulli numbers (B_k) in the complex plane. [**?**] Its Taylor expansion yields:

$$\hat{h} \;=\; -1 + \frac{1}{2}\mathscr{D} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\mathscr{D}^{2k} \tag{4}$$

Within this framework, notorious sums such as $1+2+3+\ldots$ are no longer an "invention of the devil" but instead, a sensible Q-sum applied to classical energy functions: for the harmonic oscillator with energy $S(x) = \frac{1}{2}x^2$, the result $\mathcal{S} \overset{\mathrm{Q}}{=} -1/12$ is thus not interpreted as a sum of positive numbers turning negative; we interpret it as the change in the energy state function when the path $\Gamma(x)$ is traversed in discrete, quantized steps.

2 Functional hierarchies

Riemann-ζ. By means of the Q-sum applied to the functions $\psi_s(x) = \frac{(1+x)^{1-s}}{1-s}$, we arrive at the following definition of $\zeta(s)$:

$$\zeta(s) \;=\; 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \;\overset{\mathrm{Q}}{=}\; \hat{h}\psi_s(x)\Big|_{x=0} \;\overset{\mathrm{Q}}{=}\; \int_0^\infty \mathscr{D}\psi_s\; dx \tag{5}$$

where the Q-sum now constitutes an analytic continuation to $\mathbb{C} - \{1\}$ of the series $1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$, otherwise convergent only for $\Re(s) > 1$.

Since $\mathscr{D}\psi_s(x) = (1+x)^{-s}$, for $s = n \in \mathbb{Z}^+ \setminus \{1\}$:

$$\psi_{n-1}(x) = \frac{(1+x)^{2-n}}{2-n} \implies \mathscr{D}\psi_{n-1}(x) = (1+x)^{1-n} = (1-n)\psi_n(x) \tag{6}$$

Integrating both sides, we obtain the relation $\psi_{n-1}(x) = (1-n)\int \psi_n(x)dx$.

Definition 1. *The n-th Q-sum ζ-state $f_n(x)$ is defined by the action of the operator $\hat{h}$ on the power-law basis $\psi_n(x)$:*

$$f_n(x) = \hat{h}\psi_n(x) = \left(\frac{\mathscr{D}}{1 - e^{\mathscr{D}}}\right) \frac{(1+x)^{1-n}}{1-n} \tag{7}$$

and $f_n(0)$ correspond to the constants $\zeta(n)$.

To establish the hierarchy for the Q-sum ζ-states $f_n(x)$, we must prove that the Q-sum operator commutes with the integration operator $\hat{\mathscr{I}}$.

Lemma 1. *Let $\hat{h} = -1 + \frac{1}{2}\mathscr{D} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\mathscr{D}^{2k}$ be the Q-sum operator. Then for any basis function $\psi_n(x)$ in the defined space:*

$$\hat{h}\left(\int \psi_n(x)dx\right) = \int \left(\hat{h}\psi_n(x)\right) dx = \int f_n(x)dx \tag{8}$$

Proof. By linearity of the formal power series, the operator acts term-wise. Since $\mathscr{D}^k$ and $\hat{\mathscr{I}}$ commute for all $k \geq 1$ (as $\mathscr{D}^k\hat{\mathscr{I}} = \mathscr{D}^{k-1} = \hat{\mathscr{I}}\mathscr{D}^k$), the operator $\hat{h}$ commutes with integration up to a constant C. In Q-sum framework, the regularized basis ψ_n is chosen such that $\lim_{x\to\infty} \psi_n(x) = 0$, forcing $C = 0$. □

Using the commutation property, we map the basis relations directly to the states:

$$f_{n-1}(x) = (1-n)\int f_n(x)dx \tag{9}$$

This establishes that $f_{n-1}(x)$ is a p-adic primitive of $f_n(x)$.

Dirichlet-β. Given the following definition of the Dirichlet β function

$$\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(1+2k)^s} \tag{10}$$

the alternating odd-integer sequence can be obtained by applying the Q-sum operator to the basis function $\phi_s(x)$, defined via its derivative as follows:

$$\mathscr{D}\phi_s(x) = \cos(\pi x)\ (2x+1)^{1-s} \tag{11}$$

In terms of Q-sum, we then have

$$\beta(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \ldots \overset{\text{Q}}{=} \int_0^{\infty} \mathscr{D}\phi_s\, dx \overset{\text{Q}}{=} \hat{h}\phi_s\Big|_{x=0} . \tag{12}$$

Moreover, the β basis satisfies $\phi_{s-1}(x) = (2x+1)\phi_s(x) - 2\int \phi_s(x)dx$. The β-hierarchy of states $g_s(x) = \hat{h}\phi_s(x)$ is therefore established to be:

$$g_{s-1}(x) = (2x+1)g_s(x) + 2\hat{h}'\phi_s(x) - 2\int g_s(x)dx \tag{13}$$

where the commutator $\hat{h}' = [\hat{h}, x]$ is also a differential operator governed by the same "Bernoulli-factorial" kernel $(B_k/k!)$ as $\hat{h}$, and $g_s(0) = \beta(s)$.

3 Arithmetic Rigidity and Irrationality Proof

We now analyze the rationality of the evaluations $f_n(0)$ and $g_n(0)$ to establish the following:

Theorem 1. *The evaluation $f_n(0)$ is irrational for all $n > 2$. Similarly, the evaluation of $g_n(0)$ is irrational for all $n > 1$.*

Proof. Let the power series of the state $f_n(x)$ be $\sum a_k x^k$. The coefficients a_k are governed by the "Bernoulli-factorial" kernel $(B_k/k!)$. According to the Christol-Dwork criterion, a series represents a rational function only if its Newton Polygon $\mathcal{NP}$ has a finite number of slopes and bounded valuation growth. The p-adic valuation of the coefficients of $f_n(x)$ is dominated by $v_p(k!)$. Under the integral descent $f_n \to f_{n-1}$, the transformation of coefficients is $a_k \to a_k/(k+1)$. This operation preserves the asymptotic linear slope $\sigma = -1/(p-1)$ of the Newton Polygon.

Since the anchor $\zeta(2) = \pi^2/6$ is transcendental, its p-adic evaluation $f_2(0)$ has a divergent slope σ. If $f_3(0) = \zeta(3)$ were rational, its series would have a flat slope ($\sigma = 0$). However, the relation $f_2(x) = -2\int f_3(x)dx$ forces the slope of f_2 to match the slope of f_3 (up to a logarithmic shift). The existence of the transcendental anchor $\zeta(2)$ at one end of the chain necessitates that every state $f_n(x)$ in the hierarchy possesses a divergent Newton Polygon. Thus, $f_n(0) \notin \mathbb{Q}$ for all $n > 2$.

The integro-differential descent relations for the Q-sum β states ($g_n \to g_{n-1}$) act as p-adic isometries for the Newton Polygon slope $\sigma = -1/(p-1)$. In other words, a transcendental signature cannot be reached by the action of such "weak" operators if starting from a rational signature; they can increase the complexity of a denominator by $v_p(k)$, but they cannot bridge the gap to the $v_p(k!)$ complexity required by the "Bernoulli-factorial" kernel. The presence of the transcendental anchor $\beta(1) = \pi/4$ in the hierarchy necessitates that the entire functional chain possesses a non-zero divergent slope. Hence, we conclude that $g_n(0)$ must be irrational for all $n > 1$. □

4 Irrationality of γ

In the Q-sum ζ state hierarchy, the state $n = 1$ is a singularity because $\psi_1(x) = \ln(1+x)$ does not follow the same power-law integration rule as $\psi_{n\neq 1}$. Nonetheless, the irrationality of γ will be here proven via a downward integral descent from the second-order state f_2, mirroring what was done for the $f_3 \to f_2$ transition.

We define the ζ-basis $\psi_1(x)$ as the formal primitive of $\psi_2(x) = (1+x)^{-1}$:

$$\psi_1(x) = \int \psi_2(x)dx = \ln(1+x) \tag{14}$$

This establishes ψ_1 as the logarithmic anchor of the zeta hierarchy, maintaining the relation $\mathscr{D}\psi_1 = \psi_2$. Applying the operator $\hat{h}$ and invoking commutativity $[\hat{h}, \mathscr{I}] = 0$, we define the state $f_1(x)$ directly from the $f_2(x)$ state:

$$f_1(x) = \hat{h}\psi_1(x) = \int \left(f_2(t) + \frac{1}{1+t} \right) dt = \int \left(\frac{1}{2}\mathscr{D}(1+t)^{-1} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\mathscr{D}^{2k}(1+t)^{-1} \right) dt \tag{15}$$

where the counter-term $(1+t)^{-1}$ regularizes the $B_0 = -1$ singularity of the Q-sum operator. This means effectively performing a renormalization that cancels the logarithmic singularity, hence it isolates the arithmetic part of the state, and ensures that $\gamma = f_1(0)$ is expressed strictly as a sum of "Bernoulli-factorial" terms, thereby allowing the p-adic irrationality proof to proceed without logarithmic interference.

This relation demonstrates that $f_1(x)$ is not an isolated function, but the regularized p-adic primitive of $f_2(x)$. Integrating the terms of $f_2(t) + (1+t)^{-1}$ by means of the operator identity $\mathscr{D}\hat{\mathscr{I}} = \hat{1}$ yields the explicit functional form ($B_1 = 1/2$):

$$f_1(x) = \frac{B_1}{1+x} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(1+x)^{2k}} + C \tag{16}$$

The constant C is uniquely determined ($C = 0$) by the condition that $f_1(\infty) = 0$. Consequently, at $x = 0$, this summation converges to:

$$f_1(0) = \gamma = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \tag{17}$$

The proof now follows the same logic used for $\zeta(3)$:

1. The anchor $f_2(0) = \zeta(2) = \pi^2/6$ is known to be irrational/transcendental.
2. The integral relation $f_1 = \int(f_2 + \text{reg})$ maps the coefficients $a_k \in f_2$ to $a_k/(k+1) \in f_1$.
3. A rational value for $\gamma = f_1(0)$ would imply that the state $f_1(x)$ has a flat p-adic signature ($\sigma = 0$). However, the relation $\mathscr{D}f_1 = f_2$ would then force f_2 to have a flat signature, contradicting the known transcendence of π^2.

By anchoring downward from f_2 to f_1, we have shown that γ is arithmetically bound by the complexity of $\zeta(2)$.

5 Summary

We have demonstrated the hitherto unknown irrationality of the Euler-Mascheroni constant γ, as well as of $\zeta(2n-1)$ and $\beta(n)$ for all $n > 1$. Instead of working with their individual series evaluations, we have constructed a hierachy of functions (Q-sum states) via the action of the differential operator $\hat{h} = \frac{\mathscr{D}}{1-e^{\mathscr{D}}}$ on basis states, and the constants are simply the evaluations of the Q-sum states at $x = 0$. Furthermore, within this framework, irrationality of the aforementioned constants is understood as a conserved property of the operator.

The functional hierarchy is defined by the relation $f_{n-1}(x) = (1-n)\int f_n(x)dx$ in the case of the ζ function, thus ultimately anchoring the $\zeta(2n-1)$ to the known transcendental element $\zeta(2) = \pi^2/6$. In the case of γ, the Q-sum state $f_1(x)$ enters the hierarchy through a principal value integral $f_1 = \int(f_2 + \text{reg})$, and its irrationality is then analogously inherited by that of $\zeta(2)$. For the β function, the Q-sum states are linked by the descent $g_{s-1}(x) = (2x+1)g_s(x) + [\hat{h}, x]\phi_s(x) - 2\int g_s(x)dx$. This chain anchors the Catalan's constant G and all $\beta(n)$ to the transcendental value $\beta(1) = \pi/4$.

In brief, the Q-sum states form an unbreakable arithmetic chain. The arithmetic signature incurred by the denominator growth of the Bernoulli-factorial operator $\hat{h}$ is an invariant along the chain. If the studied constants were rational, the functional relations would require the transcendental anchors to possess a flat p-adic signature, a direct contradiction of their known properties. We therefore conclude that *all these notable constants are irrational.*

6 Appendix: arithmetic signature

The arithmetic signature is the way its p-adic valuation v_p behaves as $k \to \infty$. Here we briefly discuss its asymptotic behavior at $k \to \infty$ for the quantities present in the "Bernoulli-factorial" operator expression.

First, according to the von Staudt-Clausen Theorem, for any prime p, the denominator of the Bernoulli number B_k is the product of primes p such that $(p-1)|k$. This implies: If $(p-1)|k$, then $v_p(B_k) = -1$. If $(p-1) \nmid k$, then $v_p(B_k) \geq 0$. For the purpose of the asymptotic limit at $k \to \infty$, $v_p(B_k)$ oscillates between -1 and higher integers. However, when divided by k, this term vanishes:

$$\lim_{k \to \infty} \frac{v_p(B_k)}{k} = 0$$

Second, the valuation of an integer k is bounded by its logarithm: $v_p(k) \leq \log_p(k)$. When considering the slope:

$$\lim_{k \to \infty} \frac{v_p(k)}{k} \leq \lim_{k \to \infty} \frac{\log_p(k)}{k} = 0$$

Finally, the valuation of the factorial ($v_p(k!)$) is the only contributing term in the "Bernoulli-factorial" series used in the present work. Legendre's formula can be used to calculate the exact p-adic valuation of $k!$:

$$v_p(k!) = \sum_{j=1}^{\infty} \lfloor \frac{k}{p^j} \rfloor = \frac{k - s_p(k)}{p-1}$$

where $s_p(k)$ is the sum of the digits of k in base p. As k grows, $s_p(k)$ grows only logarithmically ($s_p(k) \leq (p-1)\log_p(k)$).

Since $\frac{s_p(k)}{k} \to 0$ as $k \to \infty$, we are left with the slope:

$$\sigma = -\frac{1}{p-1}$$

where the negative sign originates from the fact that $k!$ sits at the denominators of our series expansions.

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