ON THE FIXED DEGREE SCHOLZ-BRAUER PROBLEM

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Abstract. Denote the minimal length of a fixed degree $d \geq 2$ addition chain that leads to n by $\ell^d(n)$. We introduce the concept of a *strong Brauer* number of *rank* $d \geq 2$ and show that all numbers belonging to this class satisfy the inequality

$$\ell^d(d^n - 1) \leq n - 1 + \ell^d(n).$$

This extends the concept of a *Brauer* number in standard addition chain theory to the fixed degree $d \geq 2$ framework.

1. Introduction

Addition chains are a classical combinatorial tool that describes efficient exponentiation procedures: an addition chain for a positive integer n is a sequence of integers beginning with 1 and ending with n in which each term after the first is the sum of two (not necessarily distinct) earlier terms. The length of a shortest such sequence (the length of the addition chain) measures the minimal number of multiplications required to compute x^n from x, and therefore the addition chain theory lies at the intersection of number theory, combinatorics, and algorithmic arithmetic; for background and algorithmic context. See, e.g., [1, 6].

A central conjecture in the theory classically referred to as the Scholz-Brauer (or Brauer-Scholz) conjecture proposes the inequality

$$\ell(2^n - 1) \leq n - 1 + \ell(n),$$

relating the minimal length of the addition chain $\ell(\cdot)$ for n and for $2^n - 1$. This conjecture, originating in the work of Scholz and early investigations by Brauer, has motivated extensive structural and computational studies: Brauer introduced the star (Brauer) chain viewpoint and proved a related bound, and subsequent authors (both theoretical and computational) have verified the conjecture in many special cases and developed rich structural classifications of chains (star/Brauer chains, Hansen chains, etc.). See, e.g., [3, 2, 4, 6, 7].

When the restriction that each step is a sum of *two* earlier terms is relaxed, new phenomena appear, and constructions that work in the classical setting may require additional hypotheses. In this paper, we consider *fixed-degree* addition chains: for a fixed integer $d \geq 2$ every step is allowed to be the sum of at most d previous terms (repetition allowed) [5]. This model interpolates between the classical binary-summand model ($d = 2$) and fully general additive models, natural from both the viewpoint of algebraic exponentiation (where one may permit more than

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two summands per step) and from combinatorial classification problems for constrained chains [6].

Our main contribution is the introduction of Brauer-type structural notions in the fixed-degree framework and a proof that the natural Scholz-type inequality holds for a robust structural class. Concretely, after defining Brauer and *strong Brauer* chains of rank $d \geq 2$ and the corresponding Brauer-type numbers, we prove:

The main theorem. *For every fixed integer $d \geq 2$, every strong Brauer number n of rank $d \geq 2$ satisfies*

$$\ell^d(d^n - 1) \leq n - 1 + \ell^d(n).$$

The proof adapts the classical Brauer seed-and-dilate construction to the fixed-degree setting; however, the passage $d > 2$ introduces combinatorial complications that force us to strengthen the standard Brauer/star condition to the *strong Brauer* hypothesis. The main construction and proof are shown in Section 3; Section 2 fixes the notation and recalls standard definitions (including the star/Brauer and Hansen notions). We conclude with a remark on directions for future work (for example, speculations on whether analogous inequalities hold for Hansen or closed chains in the fixed-degree framework).

2. Preliminaries and setup

Definition 2.1. Let $n \geq 3$ and $d \geq 2$ be a fixed positive integer. We say that the sequence of positive integers

$$s_0 = 1 < s_1 < \cdots < s_h = n$$

is an addition chain of fixed degree $d \geq 2$ leading to n of length h if for each $1 \leq i \leq h$ the representation

$$s_i = \sum_{\substack{i_j \in [0, i-1] \cap \mathbb{Z} \\ j \in [1, d] \cap \mathbb{N}}} s_{i_j}, \quad (s_{i_j} < s_i)$$

holds.

In other words, each term term in an addition chain with a fixed degree $d \geq 2$ is the sum of at most d previous terms in the chain, with repetition allowed. We call the shortest fixed degree $d \geq 2$ addition chain that leads to a target an *optimal* degree d addition chain. We denote the length of an optimal addition chain with fixed degree $d \geq 2$ leading to a target n with $\ell^d(n)$. We denote the length of the fixed degree chain ($d \geq 2$) - whether or not it is optimal - by $l^d(n)$. The special case where the fixed degree $d = 2$ recovers the well-known concept of an addition chain first introduced in [3].

Example 2.2. Choose the target $n = 21$ and fix the degree $d = 3$. The sequence

$$s_0 = 1, s_1 = 2, s_2 = 4, s_3 = 8, s_4 = 16, s_5 = 21$$

is an addition chain with fixed degree $d = 3$, because $s_2 = 2s_1, s_3 = 2s_2, s_4 = 2s_3, s_5 = s_4 + s_2 + s_0$. However, it is not of minimal length. An example chain of fixed degree 3 and of minimal length is

$$s_0 = 1, s_1 = 3, s_2 = 9, s_3 = 21.$$

Example 2.3. Choose the target $n = 63$ and fix the degree $d = 4$. The sequence

$$s_0 = 1, s_1 = 2, s_2 = 4, s_3 = 8, s_4 = 16, s_5 = 32, s_6 = 56, s_7 = 63$$

is an addition chain with fixed degree $d = 4$, because $s_2 = 2s_1, s_3 = 2s_2, s_4 = 2s_3, s_5 = 2s_4, s_6 = s_5 + s_4 + s_3, s_7 = s_6 + s_2 + s_1 + s_0$. A chain of fixed degree 4 and of minimal length leading to 63 is given by

$$s_0 = 1, s_1 = 4, s_2 = 16, s_3 = 52, s_4 = 61, s_5 = 63.$$

We now provide definitions that adapt the notions of big steps and small steps for fixed degree d addition chains.

Definition 2.4. Let $d \geq 2$ be a fixed integer and

$$s_0 = 1 < s_1 < \cdots < s_h = n$$

be an addition chain of fixed degree d. We define the sets

$$G := \{j \ : \ s_j = d \cdot s_{j-1}, 1 \leq j \leq h\}$$

and

$$K := \{j \ : \ s_j < d \cdot s_{j-1}, 1 \leq j \leq h\}.$$

We call G the *d-dilate steps* and K the *non d-dilate steps*, respectively, in the chain. We denote by $|G|$ and $|K|$ the number of d-dilate and non-d-dilate steps in the addition chain with fixed degree.

Example 2.5. Fix $d = 3$ and choose the target $n = 20$. We construct the addition chain with fixed degree $d = 3$ as follows

$$s_0 = 1, s_1 = 2, s_2 = 6, s_3 = 18, s_4 = 20$$

with $s_1 = 1 + 1, s_2 = 2 + 2 + 2, s_3 = 6 + 6 + 6, s_4 = 18 + 2$. In this case, we have

$$G = \{2, 3\} \quad \text{and} \quad K = \{1, 4\}$$

as the 3-dilate and non 3-dilate steps, respectively.

Definition 2.6 (The fixed degree chain index function)**.** Let $d \geq 2$ be a fixed integer and

$$s_0 = 1 < s_1 < \cdots < s_h = n$$

be an addition chain of fixed degree d. For each $1 \leq i \leq h$, we define recursively the function

$$\lambda(s_{i-2}) = \begin{cases} 0 & \text{if} \quad s_i - s_{i-1} - s_{i-2} < 0 \\ 1 & \text{otherwise,} \end{cases}$$

and for $m \geq 2$

$$\lambda(s_{i-m}) = \begin{cases} 0 & \text{if} \quad s_i - \sum\limits_{v=1}^{m} \lambda(s_{i-v}) s_{i-v} < 0 \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.7 (Brauer and strongly Brauer fixed degree chains)**.** Let $d \geq 2$ be a fixed integer and

$$1 = s_0 < s_1 < \cdots < s_h = n$$

be a fixed degree $d \geq 2$ addition chain leading to n. We say that the chain is *Brauer* if for each $1 \leq i \leq h$ with

$$s_i := \sum_{\substack{i_j \in [0, i-1] \cap \mathbb{Z} \\ j \in [1,d] \cap \mathbb{N}}} s_{i_j}, \quad (s_{i_j} < s_i)$$

we have

$$\max_{\substack{i_j \in [0, i-1] \cap \mathbb{Z} \\ j \in [1,d] \cap \mathbb{N}}} \{s_{i_j}\} = s_{i-1}.$$

Otherwise, we say that the fixed degree $d \geq 2$ chain is not *Brauer*. We say that a chain of fixed degree $d \geq 2$ is *strongly Brauer* if it is *Brauer*, and for each i with $1 \leq i \leq h$ such that

$$s_i := \sum_{\substack{i_j \in [0, i-1] \cap \mathbb{Z} \\ j \in [1,d] \cap \mathbb{N}}} s_{i_j}, \quad (s_{i_j} < s_i)$$

the inequality

$$0 \leq \sum_{\substack{i_j \in [0, i-1] \cap \mathbb{Z} \\ j \in [1,d] \cap \mathbb{N}}} s_{i_j} - \sum_{1 \leq m \leq k} \lambda(s_{i-m}) s_{i-m} \leq s_{i-k}$$

holds for each integer k satisfying $i - 1 \geq k \geq 1$ with $\lambda(s_{i-k}) = 1$. Furthermore, each such integer $k \geq 1$ satisfies $\lambda(s_{i-k-1}) = 1$.

We denote the smallest number h for which there exists a fixed degree $d \geq 2$ *Brauer* chain of length h that leads to n by $(\ell^d)^*(n)$ and call it the minimal (shortest) length of the fixed degree $d \geq 2$ *Brauer* chain. Similarly, we denote the smallest number h for which there exists a *strong Brauer* chain of fixed degree $d \geq 2$ that leads to n by $(\ell^d)^{**}(n)$ and call it the minimal (shortest) length of a *strong Brauer* chain of fixed degree $d \geq 2$.

Example 2.8. Choose the target $n = 63$ and fix the degree $d = 4$. The sequence

$$s_0 = 1, s_1 = 2, s_2 = 4, s_3 = 8, s_4 = 16, s_5 = 32, s_6 = 56, s_7 = 63$$

is a fixed degree ($d = 4$) *Brauer* addition chain, because $s_2 = 2s_1, s_3 = 2s_2, s_4 = 2s_3, s_5 = 2s_4, s_6 = s_5 + s_4 + s_3, s_7 = s_6 + s_2 + s_1 + s_0$.

Example 2.9. Choose the target $n = 15$ and fix the degree $d = 3$. The sequence

$$s_0 = 1, s_1 = 2, s_2 = 4, s_3 = 7, s_4 = 13, s_5 = 15$$

with $2 = 1 + 1$, $4 = 2 + 2$, $7 = 1 + 2 + 4$, $13 = 2 + 4 + 7$, $15 = 13 + 2$ is a *strong Brauer* addition chain of fixed degree $d = 3$, because it is *Brauer* and

$$s_5 - \sum_{1 \leq m \leq 4} \lambda(s_{5-m}) s_{5-m} = s_1 = 2$$

with $\lambda(s_0)=1$,

$$s_4-\sum_{1\leq m\leq 2}\lambda(s_{4-m})s_{4-m}=s_1=2$$

with $\lambda(s_0)=1$,

$$s_3-\sum_{1\leq m\leq 2}\lambda(s_{3-m})s_{3-m}\leq s_1$$

with $\lambda(s_0)=1$,

$$s_2-s_1=2$$

with $\lambda(s_0)=1$.

Example 2.10. Choose the target and fix the degree $d=3$. The fixed degree $d=3$ minimal length chain below is *Brauer* but not *strongly Brauer*

$$s_0=1, s_1=3=1+1+1, s_2=9=3+3+3, s_3=21=9+9+3.$$

Here, we observe that $s_2-s_1=6>s_1$ and $s_2-s_1-s_0=5>s_0$.

Definition 2.11 (Brauer and strong Brauer numbers of rank $d\geq 2$)**.** We call a number n for which there exists a minimal length fixed degree $d\geq 2$ chain leading to n that is a *Brauer* chain a *Brauer* number of *rank* $d\geq 2$. Similarly, we say that a number n is a *strong Brauer* number of *rank* $d\geq 2$ is there exists a minimal length fixed degree $d\geq 2$ addition chain leading to n that is a *strong Brauer* chain.

Remark 2.12. If we fix the degree $d=2$, then the concept of the Brauer chain and the strongly Brauer chain is indistinguishable. Consequently, every Brauer number is a strongly Brauer number and vise-versa.

3. Main construction: fixed degree $d\geq 2$ Brauer-type construction

In previous investigations, we conjectured the inequality

$$\ell^d(d^n-1)\leq n-1+\ell^d(n)$$

which may be regarded as a generalization of the unproven Scholz-Brauer inequality [3]. In this section, we prove this generalized inequality for a class of fixed degree $d\geq 2$ chains that are *strongly Brauer* and a class of numbers that are *strongly Brauer.*

Theorem 3.1. *Let $d\geq 2$ be a fixed integer. All strongly Brauer numbers $n\geq 2$ satisfy the inequality*

$$\ell^d(d^n-1)\leq n-1+\ell^d(n).$$

Proof. Suppose that $n\geq 2$ is a *strong Brauer* number of *rank* $d\geq 2$. There exists, therefore, a *strongly Brauer* chain

$$1=s_0<s_1<\cdots<s_h=n$$

with $h:=(\ell^d)^{**}(n)=\ell^d(n)$. Now, define $m_i:=d^{s_i}-1$ for each $0\leq i\leq h$, and enumerate

$$1\leq d^{s_0}-1<d^{s_1}-1<\cdots<d^{s_h}-1:=d^n-1.$$

We call each term in the sequence (m_i) a *seed* of a chain leading to d^n-1. For each $1\leq i\leq h$, we perform a repeated d-dilation on each seed in the following way

$$\begin{aligned}
&d(d^{s_{i-1}}-1)\\
&d^2(d^{s_{i-1}}-1)\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
&d^{s_i-s_{i-1}}(d^{s_{i-1}}-1)
\end{aligned}$$

and include these new terms in the sequence of *seeds* of the chain that leads to d^n-1. The resulting sequence is therefore a fixed degree $d\geq 2$ addition chain observing that the representation

$$d^{s_i}-1=d^{s_i-s_{i-1}}(d^{s_{i-1}}-1)+d^{s_i-s_{i-1}}-1$$

holds for each $1\leq i\leq h$ and that

$$d^{s_0}-1=d-1=\overbrace{1+\cdots+1}^{\text{d-1 times}}.$$

For each $1\leq i\leq h$, we define recursively the function

$$\lambda(s_{i-2})=\begin{cases}0 & \text{if} \quad s_i-s_{i-1}-s_{i-2}<0\\ 1 & \text{otherwise,}\end{cases}$$

and for $m\geq 2$

$$\lambda(s_{i-m})=\begin{cases}0 & \text{if} \quad s_i-\sum\limits_{v=1}^{m}\lambda(s_{i-v})s_{i-v}<0\\ 1 & \text{otherwise.}\end{cases}$$

Because every *strongly Brauer* chain of fixed degree $d\geq 2$ is also a *Brauer* chain of fixed degree $d\geq 2$, we can write

$$s_i-s_{i-1}:=\sum_{\substack{i_j\in[0,i-1]\cap\mathbb{Z}\\ j\in[1,d-1]\cap\mathbb{N}}}s_{i_j}.$$

Under the strongly Brauer hypothesis, we may assume that $\lambda(s_{i-2})=1$. We obtain the inequality

$$0\leq s_i-s_{i-1}-\lambda(s_{i-2})s_{i-2}:=-s_{i-2}+\sum_{\substack{i_j\in[0,i-1]\cap\mathbb{Z}\\ j\in[1,d-1]\cap\mathbb{N}}}s_{i_j}\leq s_{i-1}-s_{i-2}.$$

We can therefore write

$$d^{s_i-s_{i-1}}-1:=d^{s_{i-2}}d^{s_i-s_{i-1}-s_{i-2}}-1=d^{s_i-s_{i-1}-s_{i-2}}(d^{s_{i-2}}-1)+d^{s_i-s_{i-1}-s_{i-2}}-1.$$

Because $s_i-s_{i-1}-s_{i-2}\leq s_{i-1}-s_{i-2}$ the term $d^{s_i-s_{i-1}-s_{i-2}}(d^{s_{i-2}}-1)$ has already been generated in the repeated d-dilation of the *seed* $d^{s_{i-2}}-1$. Without loss of generality, we may assume that $\lambda(s_{i-3})=1$ and write under the *strongly Brauer* hypothesis

$$0\leq s_i-s_{i-1}-s_{i-2}:=-s_{i-1}-s_{i-2}+\sum_{\substack{i_j\in[0,i-1]\cap\mathbb{Z}\\ j\in[1,d]\cap\mathbb{N}}}s_{i_j}\leq s_{i-2}$$

and we get

$$0 \leq s_i - s_{i-1} - s_{i-2} - s_{i-3} \leq s_{i-2} - s_{i-3}.$$

We can further write

$$d^{s_i - s_{i-1} - s_{i-2}} - 1 = d^{s_i - s_{i-1} - s_{i-2} - s_{i-3}}(d^{s_{i-3}} - 1) + d^{s_i - s_{i-1} - s_{i-2} - s_{i-3}} - 1.$$

Because $0 \leq s_i - s_{i-1} - s_{i-2} - s_{i-3} \leq s_{i-2} - s_{i-3}$ the term

$$d^{s_i - s_{i-1} - s_{i-2} - s_{i-3}}(d^{s_{i-3}} - 1)$$

has already been generated from the repeated d-dilation of the *seed* $d^{s_{i-3}} - 1$. By induction, we can repeat the process by decomposing

$$d^{s_i - s_{i-1} - s_{i-2} - s_{i-3}} - 1$$

in a similar way. Under the *strongly Brauer* hypothesis

$$s_i - s_{i-1} - \lambda(s_{i-2})s_{i-2} - \cdots - \lambda(s_{i-k})s_{i-k} = 0$$

for $2 \leq k \leq d$. Hence, each term $d^{s_i} - 1$ for $1 \leq i \leq h$ is the sum of at most d terms from the sequence of *seeds* and their repeated d-dilates. We deduce the following

$$\begin{aligned}
\ell^d(d^n - 1) &\leq (\ell^d)^*(d^n - 1) \\
&\leq \sum_{i=1}^{h}(s_i - s_{i-1}) + (\ell^d)^{**}(n) \\
&= \sum_{i=1}^{h}(s_i - s_{i-1}) + \ell^d(n) \\
&= n - 1 + \ell^d(n).
\end{aligned}$$

□

We illustrate how this construction works in the following example. The fixed degree $d \geq 2$ *strong Brauer* addition chain in use may not be of minimal length, but has been used to demonstrate how the construction works.

Example 3.2. Choose the target $n = 15$ and fix the degree $d = 3$. We have already shown that the fixed degree $d = 3$ chain

$$1 = s_0, s_1 = 2, s_2 = 4, s_3 = 7, s_4 = 13, s_5 = 15$$

is *strongly Brauer*. According to the proof, we define $m_i = 3^{s_i} - 1$ and enumerate

$$1 \leq 3^1 - 1 < 3^2 - 1 < 3^4 - 1 < 3^7 - 1, 3^{13} - 1 < 3^{15} - 1 := 3^{15} - 1.$$

Using the construction in the proof of the main theorem, we obtain the sequence

$$1, 3^1 - 1, 3(3^1 - 1), 3^2 - 1, 3(3^2 - 1), 3^2(3^2 - 1), 3^4 - 1, 3(3^4 - 1), 3^2(3^4 - 1),$$

$$3^3(3^4 - 1), 3^7 - 1, 3(3^7 - 1), 3^2(3^7 - 1), 3^3(3^7 - 1), 3^4(3^7 - 1), 3^5(3^7 - 1), 3^6(3^7 - 1),$$

$$3^{13} - 1, 3(3^{13} - 1), 3^2(3^{13} - 1), 3^{15} - 1.$$

One can check that this construction yields an addition chain of fixed degree $d = 3$ that leads to $3^{15} - 1$.

Remark 3.3. Our construction works because of the *strongly Brauer* hypothesis. In this fixed degree $d > 2$ framework one cannot directly adapt the Brauer construction without enforcing the *strongly Brauer* condition. It will be interesting to investigate the validity of the conjectured inequality

$$\ell^d(d^n - 1) \leq n - 1 + \ell^d(n)$$

for other special classes of fixed degree $d \geq 2$ chain that are not strongly *Brauer*. For example, one may investigate the validity of this inequality for *Hansen* chains [4] or *closed* addition chains and identify a structural hypothesis that makes these constructions work in the fixed degree $d \geq 2$ framework.

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