Inversion Identity for The k-th Root Function

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Abstract

In this paper, we derive a formal inversion identity from the Taylor expansion of $\sqrt[k]{x}$ to get x as an infinite series of the function. Along with the derivation, we also give a proof of the identity by justifying some crucial mathematically rigorous statements regarding analyticity, validity of Cauchy's convolution, and the convergence, and also derive a trivial infinite series for π, e (Euler's constant) and a formal infinite series identity of γ (Euler-Mascheroni constant) in terms of their k-th roots.

1 Taylor Series for a Smooth Analytical Function $f(x)$

As shown in [1], we know that a smooth analytical function $f(x)$ can be Taylor expanded by resolving $x = a + b$, where b is a small perturbation as follows,

$$f(x) = f(a+b) = \sum_{n=0}^{\infty} \frac{b^n}{n!}\frac{d^n f}{da^n}. \tag{1}$$

We know the very famous Taylor expansion for e^x as follows,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{2}$$

From Equation (2) one can derive the Euler's identity $e^{ix} = \cos x + i\sin x$ where $i = \sqrt{-1}$ by making the exponent as ix instead of just x as follows,

$$\begin{aligned} e^{ix} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \cdots &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \cos x + i\sin x. \end{aligned} \tag{3}$$

In [3], a general formal existence of Taylor expansions of functions over a field of generalized series has been worked out.

2 Cauchy's Product Convolution

We can evaluate the exponents of infinite summations by using Cauchy's product convolution.

Let,

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then the Cauchy product of these two series gives a new series $C(x)$ as follows,

$$(A \cdot B)(x) = C(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n = \sum_{n=0}^{\infty} c_n x^n.$$

3 Taylor Expansion for k-th Root of x

To Taylor-expand $f(x) = \sqrt[k]{x} = x^{\frac{1}{k}}$, we first resolve $x = a + b$, now we use Equation (1) and get the Taylor expansion for $f(x)$ as follows,

$$f(x) = f(a+b) = \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{d^n}{da^n} (\sqrt[k]{a}). \tag{4}$$

3.1 Inverting $f(x)$ to get x as an Infinite Series

From Equation (4) we know,

$$x^{\frac{1}{k}} = \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{d^n}{da^n} (\sqrt[k]{a}).$$

To get x from $f(x)$, we raise it to the k-th power and use Cauchy's product convolution as follows,

$$\begin{aligned} (x^{\frac{1}{k}})^k = x &= \Big(\sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{d^n}{da^n} (\sqrt[k]{a}) \Big)^k \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \Big(\prod_{j=1}^{m} \frac{b^{n_j}}{n_j!} \frac{d^{n_j}}{da^{n_j}} (\sqrt[k]{a}) \Big). \end{aligned} \tag{5}$$

Since $\sum_{i=1}^{m} n_i = n_1 + n_2 + \cdots n_m = n$, we derive the final infinite series of x from $\sqrt[k]{x}$ as follows,

$$\begin{aligned} (\sqrt[k]{x})^k = x &= \sum_{n=0}^{\infty} \Big(\sum_{n_1+n_2+\cdots n_m = n} \frac{b^{n_1} b^{n_2} b^{n_3} \cdots b^{n_m}}{n_1! n_2! n_3! \cdots n_m!} (\prod_{j=1}^{m} \frac{d^{n_j}}{da^{n_j}} (\sqrt[k]{a})) \Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{n_1+n_2+\cdots n_m = n} \frac{b^{n_1+n_2+n_3+\cdots n_m}}{n_1! n_2! n_3! \cdots n_m!} (\prod_{j=1}^{m} \frac{d^{n_j}}{da^{n_j}} (\sqrt[k]{a})) \Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{n_1+n_2+\cdots n_m = n} \frac{b^{n}}{n_1! n_2! n_3! \cdots n_m!} (\prod_{j=1}^{m} \frac{d^{n_j}}{da^{n_j}} (\sqrt[k]{a})) \Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{n_1+n_2+\cdots n_m = n} b^n (\prod_{j=1}^{m} \frac{1}{n_j!} \frac{d^{n_j}}{da^{n_j}} (\sqrt[k]{a})) \Big) \end{aligned}$$

Since $b = x - a$ we have,

$$\boxed{\therefore (\sqrt[k]{x})^k = x = \sum_{n=0}^{\infty} \Big(\sum_{n_1+n_2+\cdots n_k = n} (\prod_{j=1}^{k} \frac{(x-a)^{n_j}}{n_j!} \frac{d^{n_j}}{da^{n_j}} (\sqrt[k]{a})) \Big).} \tag{6}$$

Since m is a dummy index, $m = k$.

4 Proof of the Inversion Identity

We know,

$$\sqrt[k]{x} = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \frac{d^n}{da^n}(\sqrt[k]{a})$$

And from this, we derived

$$x = \sum_{n=0}^{\infty} \Big(\sum_{n_1+n_2+\cdots n_k=n} (\prod_{j=1}^{k} \frac{(x-a)^{n_j}}{n_j!} \frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{a}))\Big).$$

We can verify this identity as follows,

$$\begin{aligned} x &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\cdots\sum_{n_k=0}^{\infty}\Big(\prod_{j=1}^{k}\frac{(x-a)^{n_j}}{n_j!}\frac{d^{n_j}}{da^{n_j}}(a^{\frac{1}{k}})\Big) \\ &= \Big(\sum_{n_1=0}^{\infty}\frac{(x-a)^{n_1}}{n_1!}\frac{d^{n_1}(a^{\frac{1}{k}})}{da^{n_1}}\Big)\cdot\Big(\sum_{n_2=0}^{\infty}\frac{(x-a)^{n_2}}{n_2!}\frac{d^{n_2}(a^{\frac{1}{k}})}{da^{n_2}}\Big)\cdots\Big(\sum_{n_k=0}^{\infty}\frac{(x-a)^{n_k}}{n_k!}\frac{d^{n_k}(a^{\frac{1}{k}})}{da^{n_k}}\Big) \\ &= \prod_{\alpha=1}^{k}\Big(\sum_{n_\alpha=0}^{\infty}\frac{(x-a)^{n_\alpha}}{n_\alpha!}\frac{d^{n_\alpha}(a^{\frac{1}{k}})}{da^{n_\alpha}}\Big) \\ &= \underbrace{\sqrt[k]{x}\cdot\sqrt[k]{x}\cdots\sqrt[k]{x}}_{\text{k times}} = (\sqrt[k]{x})^k = x. \end{aligned}$$

Thus the identity holds.

Regarding the mathematically rigorous statements, we justify them as follows,

- Analyticity - The Taylor expansion of the function $f(x) = \sqrt[k]{x}$ is analytical in some neighbourhood of a, $\forall a \in \mathbb{R}^+$ and $a > 0$.
- Cauchy's Convolution Validity - Since the Taylor series converges absolutely within its radius of convergence, applying Cauchy's convolution is valid in the same domain of convergence.
- Convergence and Domain of The Identity - Since the Taylor expansion works for analytically smooth functions like $\sqrt[k]{x}$, in both $\mathbb{R}$ and $\mathbb{C}$, it also implies that multiplying k copies of that same Taylor expansion gives a multinomial expansion which also works for both real and complex numbers.

5 Formal Definition of the Identity

Proposition: Multinomial Taylor Expansion of x in terms of $\sqrt[k]{x}$

Let $f(x) = x^{\frac{1}{k}} = \sqrt[k]{x}$ be the principal branch of the k-th root function, analytic in some neighbourhood of a, where $a \in \mathbb{R}^+$ or in a suitable domain in $\mathbb{C}$ avoiding branch points and singularities and $a > 0$. Then for x sufficiently close[1] to a, the identity

$$x = \sum_{n=0}^{\infty} \Big(\sum_{\substack{n_1,n_2,\ldots,n_k \geq 0 \\ n_1+n_2+\cdots+n_k=n}} (\prod_{j=1}^{k} \frac{(x-a)^{n_j}}{n_j!} \frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{a}))\Big),$$

where $f(a) = \sqrt[k]{a}$, holds, and the right-hand side converges absolutely to x in the same domain where the Taylor series of $f(x)$ converges.[2]

[1] The series converges for $|x-a| < R$ where R is the radius of convergence depending on a.

[2] Thus the domain of this identity becomes $\mathbb{R}^+ \cup (\mathbb{C} \setminus \mathfrak{S})$, where $\mathfrak{S}$ is the set of branch points and singularities in the complex numbers. $\mathfrak{S} \subset \mathbb{C}$, and $\mathfrak{S} \cap \mathbb{R}^+ = \emptyset$.

6 Examples and Conclusion

As stated in the abstract, we can trivially approximate the value of π and e in terms of their k-th roots as follows,

$$\pi = \sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\ldots,n_k\geq 0\\ n_1+n_2+\cdots+n_k=n}}(\prod_{j=1}^{k}\frac{(\pi-a)^{n_j}}{n_j!}\frac{d^{n_j}(a^{\frac{1}{k}})}{da^{n_j}})\Big),$$

$$e = \sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\ldots,n_k\geq 0\\ n_1+n_2+\cdots+n_k=n}}(\prod_{j=1}^{k}\frac{(e-a)^{n_j}}{n_j!}\frac{d^{n_j}(a^{\frac{1}{k}})}{da^{n_j}})\Big).$$

These two illustrative examples are a quite simple case of this identity, for a more difficult case, we shall try finding an infinite series regarding this identity for the Euler-Mascheroni constant (γ).

We know,

$$\gamma = -\psi(1). \tag{7}$$

Where $\psi(x)$ is the digamma function defined as,

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x). \tag{8}$$

To apply the identity of Section 5, we can express the natural logarithm in terms of its k-th root. Firstly, we expand $x = a + b$, then,

$$\ln x = \ln(a+b),$$

$$\Rightarrow x = a + b, \tag{9}$$

$$\Rightarrow b = x - a. \tag{10}$$

Now we expand the natural logarithm using Taylor expansion,

$$\sqrt[k]{\ln x} = \sum_{n=0}^{\infty}\frac{(x-a)^n}{n!}\frac{d}{da}(\sqrt[k]{\ln a}). \tag{11}$$

Now we can apply the identity and the expansion for $\ln x$ becomes as follows,

$$\ln x = \sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\ldots,n_k\geq 0\\ n_1+n_2+\cdots+n_k=n}}(\prod_{j=1}^{k}\frac{(x-a)^{n_j}}{n_j!}\frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{\ln a}))\Big). \tag{12}$$

Now, to find the natural logarithm of the gamma function, we will take the argument of the natural logarithm as the gamma function, and for the sake of this expansion, we shall consider the gamma function itself as the argument x. Thus, we can expand it as follows,

$$\ln(\Gamma(z)) = \sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\ldots,n_k\geq 0\\ n_1+n_2+\cdots+n_k=n}}(\prod_{j=1}^{k}\frac{(\Gamma(z)-a)^{n_j}}{n_j!}\frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{\ln a}))\Big) \tag{13}$$

Now we can apply Equation (8) on Equation (13) as follows,

$$\begin{aligned}
\frac{d}{dz}\ln(\Gamma(z)) = \psi(z) &= \frac{d}{dz}\sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\dots,n_k\geq 0\\ n_1+n_2+\dots+n_k=n}}(\prod_{j=1}^{k}\frac{(\Gamma(z)-a)^{n_j}}{n_j!}\frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{\ln a}))\Big)\\
&= \psi(z) = \sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\dots,n_k\geq 0\\ n_1+n_2+\dots+n_k=n}}\frac{d}{dz}\Big[\prod_{j=1}^{k}\frac{(\Gamma(z)-a)^{n_j}}{n_j!}\frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{\ln a})\Big]\Big)\\
&= \Gamma'(z)\sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\dots,n_k\geq 0\\ n_1+n_2+\dots+n_k=n}}\sum_{p=1}^{k}\frac{(\Gamma(z)-a)^{n_p-1}}{(n_p-1)!}(\prod_{\substack{j=1\\ j\neq p}}^{k}\frac{(\Gamma(z)-a)^{n_j}}{n_j!}\frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{\ln a}))\Big).
\end{aligned}$$

Assuming uniform convergence in a neighbourhood of z, differentiation term-by-term is justified here.

Hence, the digamma function derived as per the identity is

$$\psi(z) = \Gamma'(z)\sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\dots,n_k\geq 0\\ n_1+n_2+\dots+n_k=n}}\sum_{p=1}^{k}\frac{(\Gamma(z)-a)^{n_p-1}}{(n_p-1)!}(\prod_{\substack{j=1\\ j\neq p}}^{k}\frac{(\Gamma(z)-a)^{n_j}}{n_j!}\frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{\ln a}))\Big). \tag{14}$$

Now, to find the Euler-Mascheroni constant, we will let $z = 1$. Thus the digamma function becomes,

$$\psi(1) = \Gamma'(1)\sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\dots,n_k\geq 0\\ n_1+n_2+\dots+n_k=n}}\sum_{p=1}^{k}\frac{(1-a)^{n_p-1}}{(n_p-1)!}(\prod_{\substack{j=1\\ j\neq p}}^{k}\frac{(1-a)^{n_j}}{n_j!}\frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{\ln a}))\Big) = -\gamma. \tag{15}$$

Hence the infinite series for the Euler-Mascheroni constant becomes,

$$\gamma = -\Gamma'(1)\sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2,\dots,n_k\geq 0\\ n_1+n_2+\dots+n_k=n}}\sum_{p=1}^{k}\frac{(1-a)^{n_p-1}}{(n_p-1)!}(\prod_{\substack{j=1\\ j\neq p}}^{k}\frac{(1-a)^{n_j}}{n_j!}\frac{d^{n_j}}{da^{n_j}}(\sqrt[k]{\ln a}))\Big). \tag{16}$$

Hence, Equation (16) can be considered a formal infinite series expansion for the Euler-Mascheroni constant using infinite series. Similar work has been done in [2] by using limits and E-Harmonic Functions and its linearizations, using which, the author obtained a family of new formulas for it.

Thus, we can conclude that this identity works explicitly when there is another function inside the k-th root since we successfully derived a series for the Euler-Mascheroni constant by using both the natural logarithm and the gamma function. Thus, this identity can have many uses in the fields of *Number Theory*, *Real Analysis*, *Complex Analysis*, *Computational Methods*, *Functional Analysis*, and specifically *Approximation Theory* and give new ways of approximating functions and values in a relatively novel way.

A An Example: Calculating The Square Root of 2 Using The Taylor Expansion for k-th root of x in Python

We approximate $\sqrt{2}$ by calculating the first 1000 terms of Equation (4) using the following Python code by considering $a = 1.0$,

```python
import sympy as sp

# Define symbols
x, a, b, k, n = sp.symbols("x a b k n")
f = a ** (1 / k)  # f(a) = a^{1/k}

# n-th derivative of f(a)
def nth_derivative(n_val, a_val, k_val):
    f_sym = a ** (1 / k_val)
    derivative = sp.diff(f_sym, a, n_val)
    return derivative.subs(a, a_val)


# Taylor series expansion: f(a + b) = sum_{n=0}^{N} (b^n / n!) * f^{(n)
    }(a)
def taylor_expansion(a_val, b_val, k_val, terms):
    series_sum = 0
    for n_val in range(terms):
        deriv = nth_derivative(n_val, a_val, k_val)
        term = (b_val ** n_val / sp.factorial(n_val)) * deriv
        series_sum += term
    return series_sum


# Usage
a_val = 1.0  # Expansion point
x_val = 2  # Point of evaluation
k_val = 2  # Root degree
b_val = x_val - a_val

num_terms = 1000  # Set number of terms

# Compute the Taylor expansion sum
approx_value = taylor_expansion(a_val, b_val, k_val, terms=num_terms)
exact_value = x_val ** (1 / k_val)

print(f"Exact value: {exact_value}")
print(f"Absolute error: {abs(approx_value.evalf() - exact_value)}")
```

Listing 1: Python code to compute the Taylor expansion of $\sqrt{2}$

We get the output values as,

```
Exact value: 1.4142135623730951
Absolute error: 0.0000044653311013208
```

B Evaluating The Equation Derived for γ in Python

We can evaluate Equation (16) by the following Python code by considering $a = 2.0$, $k = 2$ (square root), $n = 50$ (first 50 terms) as follows,

```python
import mpmath as mp

def gamma_series_numeric(a_val=2.0, k_val=2, N=100, prec=50):
    """
    Fast numerical approximation of Euler-Mascheroni constant gamma
    using Equation (16) up to N terms.
```

```
    Parameters:
        a_val : expansion point (float > 1 recommended)
        k_val : root index (int)
        N     : number of terms
        prec  : decimal precision
    """
    mp.mp.dps = prec  # set precision
    total = mp.mpf('0')

    # Precompute derivatives of (ln a)^(1/k) up to N
    derivs = []
    for n in range(N+1):
        f = lambda x: mp.log(x)**(1/k_val)
        derivs.append(mp.diff(f, a_val, n))  # mpmath can do numerical
            derivatives

    # Main summation
    for n in range(1, N+1):
        # Generate all partitions (n1,...,nk) with sum=n
        def generate_tuples(n, k, prefix=[]):
            if k == 1:
                yield prefix + [n]
            else:
                for i in range(n+1):
                    yield from generate_tuples(n-i, k-1, prefix+[i])

        for n_tuple in generate_tuples(n, k_val):
            for p in range(k_val):
                if n_tuple[p] == 0:
                    continue
                term = ( (1-a_val)**(n_tuple[p]-1) /
                         mp.factorial(n_tuple[p]-1) )
                for j in range(k_val):
                    if j == p:
                        continue
                    term *= (1-a_val)**n_tuple[j] / mp.factorial(
                        n_tuple[j])
                # multiply by derivative factors
                for j in range(k_val):
                    term *= derivs[n_tuple[j]]
                total += term

    # gamma = -psi(1) = -Gamma'(1) * series_sum
    gamma_prime_1 = mp.diff(mp.gamma, 1, 1)  # Gamma'(1)
    return -gamma_prime_1 * total


# Example: 50 terms
approx_gamma = gamma_series_numeric(a_val=2.0, k_val=2, N=50, prec=50)
print("Approximated Value =", approx_gamma)
print("True Value =", mp.euler)
```

Listing 2: Python code to compute the multinomial Taylor expansion of the Euler-Mascheroni constant γ

We get the output values as,

`Approximated Value = 0.57721566490153234793601501168190412216145089809756`

```
True Value = 0.57721566490153286060651209008240243104215933593992
```

It can be observed that the decimal value of the approximation done by the Python code is correct upto the first 15 decimal points. Thus for the first 50 terms, we get upto 15 correct decimal points.

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