Analytical Computation of the Area of a Spherical Triangle

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1. Introduction

A spherical triangle is a triangle drawn on the spherical surface with a finite radius such that each of its three sides is a great circle arc and the sum of its interior angles is more 180^o [1,2]. In this paper, analytical formulae are derived to compute the area enclosed by a spherical triangle using previously established relations [3,4] for two practical cases: (i) when the aperture angles, i.e., the angles subtended by the sides of the triangle at the centre of the sphere, are known, and (ii) when all three sides of the triangle, expressed as great-circle arcs, are known. For both cases, the interior angles of the spherical triangle are evaluated using the inverse cosine formula [5] and subsequently, the numerical examples are illustrated to compute the area and solid angle of spherical triangle. The derived formulae are simply applicable for computing important geometric quantities, including the solid angle subtended at the centre of the sphere, the enclosed spherical surface area, and the interior angles of the triangle. In addition, the derived equations are applied to the triangular pyramid formed by joining the vertices of the spherical triangle to the centre of the sphere, enabling analytical determination of parameters such as the normal height, the angles between consecutive lateral edges, and the area of the triangular base.

2. Area of spherical triangle

The area of the spherical triangle having each side in form of a great circle arc on the spherical surface will be computed for two cases when (i) aperture angle subtended by each of three sides at the centre of sphere are known, (ii) arc length of each of three sides is known. The analytic formula will be derived to compute area of spherical triangle in both these cases as given below.

2.1. Area of spherical triangle when aperture angles subtended by the sides at the centre of sphere are known

Consider a spherical triangle ABC having each side in form a great circle arc on a spherical surface of radius R such that $\alpha, \beta \;\&\; \gamma$ are aperture angles subtended at the centre O by the sides (each as a great circle-arc) BC, AC & AB respectively. Join the vertices A, B & C with each other & to the centre O of sphere by the dotted straight lines to get a tetrahedron OABC (as shown by plane-projection in the Figure 1).

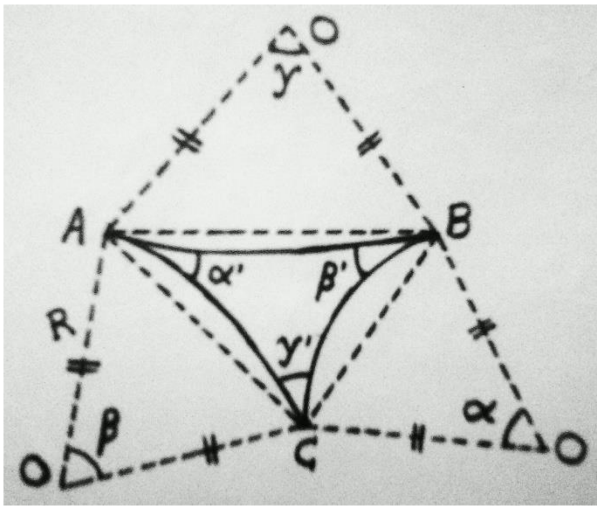

Figure 1: Plane projection of vertices A, B, C & centre O. $\boldsymbol{\alpha}, \boldsymbol{\beta} \;\&\; \boldsymbol{\gamma}$ are aperture angles subtended by the sides (each as a great circle arc) BC, AC & AB respectively of spherical ΔABC at the centre O of sphere of radius R. OABC is a tetrahedron obtained by joining vertices A, B & C & centre O by dotted straight lines.

Now, if $\alpha', \beta' \;\&\; \gamma'$ are the interior angles of spherical ΔABC opposite to the aperture angles $\alpha, \beta \;\&\; \gamma$ respectively, the interior angle α' of spherical triangle opposite to aperture angle α is given by inverse cosine formula for tetrahedron OABC [5] as follows

$$\boldsymbol{\alpha}' = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) \quad \ldots\ldots\ldots(1)$$

Similarly, angle β' opposite to the angle β is given as

$$\boldsymbol{\beta}' = \cos^{-1}\left(\frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}\right) \quad \ldots\ldots\ldots(2)$$

Similarly, angle γ' opposite to the angle γ is given as

$$\boldsymbol{\gamma}' = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) \qquad \dots\dots\dots(3)$$

Now, the area of spherical triangle ABC having interior angles α', β' & γ' from (1), (2), and (3) on the spherical surface of radius R is given by the Spherical Pythagoras theorem [4] as follows

$$\text{Area of spherical } \Delta ABC = (\alpha' + \beta' + \gamma' - \pi)R^2 \qquad \dots\dots\dots\dots\dots(4)$$

2.2. Area of spherical triangle when arc lengths of all three sides are known

Consider a spherical triangle ABC having each of sides BC, AC & AB as a great circle arc of lengths a, b & c respectively on a spherical surface of radius R. Now, aperture angles α, β & γ subtended at the centre O by the sides (each as a great circle-arc) $BC = a$, $AC = b$ & $AB = c$ respectively (see the above Figure 1) are given as

$$\alpha = \frac{\text{great arc length BC}}{\text{radius of sphere}} = \frac{a}{R}$$

Similarly,

$$\beta = \frac{\text{great arc AC}}{R} = \frac{b}{R}, \quad \gamma = \frac{\text{great arc AB}}{R} = \frac{c}{R}$$

Now, setting the values of $\alpha, \beta, \& \gamma$ in the formula of case-1, the interior angles α', β' & γ' opposite to the sides $BC = a, AC = b$ & $AB = c$ respectively of spherical ΔABC on a sphere of radius R (see Fig. 1) are given as

$$\boldsymbol{\alpha}' = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos\frac{a}{R} - \cos\frac{b}{R}\cos\frac{c}{R}}{\sin\frac{b}{R}\sin\frac{c}{R}}\right) \qquad \dots\dots\dots(5)$$

$$\boldsymbol{\beta}' = \cos^{-1}\left(\frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos\frac{b}{R} - \cos\frac{a}{R}\cos\frac{c}{R}}{\sin\frac{a}{R}\sin\frac{c}{R}}\right) \qquad \dots\dots\dots(6)$$

$$\boldsymbol{\gamma}' = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) = \cos^{-1}\left(\frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}}\right) \qquad \dots\dots\dots(7)$$

Now, the area of spherical triangle ABC having interior angles α', β' & γ' from (5), (6), and (7) on the spherical surface of radius R is given as

$$\text{Area of spherical } \Delta ABC = (\alpha' + \beta' + \gamma' - \pi)R^2 \qquad \dots\dots\dots(8)$$

3. Solid angle subtended by spherical triangle at the centre of sphere

The solid angle subtended by the spherical triangle is obtained from the fundamental definition of solid angle [6] as follows

$$\omega_{\Delta} = \int \frac{dA}{r^2} = \frac{\int dA}{R^2} = \frac{\text{Area covered by the spherical triangle}}{(\text{Radius of sphere})^2} = \frac{(\alpha' + \beta' + \gamma' - \pi)R^2}{R^2} = \alpha' + \beta' + \gamma' - \pi$$

$$\omega_{\Delta} = \alpha' + \beta' + \gamma' - \pi \qquad \dots\dots\dots..(9)$$

The above equation (9) is a generalized formula to compute the solid angle subtended by a spherical triangle at the centre of a sphere which can also be derived and validated with the values obtained from the Theory of Polygon [7,8].

4. Illustrative numerical examples

In this section, the analytical formulae derived above (Eqs. (4) and (8)) are applied to compute the area enclosed by spherical triangles through illustrative numerical examples.

Q1. Compute the area of spherical triangle on a sphere of radius 15 cm such that $29^o, 50^o$ & 65^o are the aperture angles subtended by the sides (each as a great circle arc) at the centre of sphere.

Solution. The interior angles α', β' & γ' of a spherical triangle opposite to the aperture angles $\alpha = 29^o, \beta = 50^o$ & $\gamma = 65^o$respectively subtended by the sides at the centre of a sphere are computed by using general formula (above case-1) as follows

$$\alpha' = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos 29^o - \cos 50^o \cos 65^o}{\sin 50^o \sin 65^o}\right) = 29^o 43' 0.34'' = 0.51865532 \text{ rad.}$$

$$\beta' = \cos^{-1}\left(\frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos 50^o - \cos 29^o \cos 65^o}{\sin 29^o \sin 65^o}\right) = 51^o 33' 40.24'' = 0.89991232 \text{ rad.}$$

$$\gamma' = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) = \cos^{-1}\left(\frac{\cos 65^o - \cos 29^o \cos 50^o}{\sin 29^o \sin 50^o}\right) = 112^o 4.52' 31.37''$$
$$= 1.956084404 \text{ rad.}$$

Hence, the area of spherical triangle ABC having interior angles α', β' & γ' on the spherical surface of radius $R = 15\ cm$ is

$$= (\alpha' + \beta' + \gamma' - \pi)R^2 = (0.51865532 + 0.89991232 + 1.956084404 - \pi)15^2 = \mathbf{52.43836295\ cm^2}$$

$$\Rightarrow \boldsymbol{\omega_\Delta} = 0.51865532 + 0.89991232 + 1.956084404 - \pi \approx \mathbf{0.23305939\ sr}$$

Q2. Compute the area of spherical triangle having each side as a great circle arc of lengths $6\ cm, 8\ cm, 10\ cm$ on a sphere of radius 24 cm.

Solution. The interior angles α', β' & γ' of a spherical triangle opposite to the sides (each as a great circle arc) $a = 6\ cm, b = 8\ cm$ & $c = 10\ cm$ respectively on a sphere of radius $R = 24\ cm$ are computed by using general formula (above case-2) as follows

$$\alpha' = \cos^{-1}\left(\frac{\cos\frac{a}{R} - \cos\frac{b}{R}\cos\frac{c}{R}}{\sin\frac{b}{R}\sin\frac{c}{R}}\right) = \cos^{-1}\left(\frac{\cos\frac{6}{24} - \cos\frac{8}{24}\cos\frac{10}{24}}{\sin\frac{8}{24}\sin\frac{10}{24}}\right) = 0.65763203 \text{ rad.}$$

$$\beta' = \cos^{-1}\left(\frac{\cos\frac{b}{R} - \cos\frac{a}{R}\cos\frac{c}{R}}{\sin\frac{a}{R}\sin\frac{c}{R}}\right) = \cos^{-1}\left(\frac{\cos\frac{8}{24} - \cos\frac{6}{24}\cos\frac{10}{24}}{\sin\frac{6}{24}\sin\frac{10}{24}}\right) = 0.941391672 \text{ rad.}$$

$$\gamma' = \cos^{-1}\left(\frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}}\right) = \cos^{-1}\left(\frac{\cos\frac{10}{24} - \cos\frac{6}{24}\cos\frac{8}{24}}{\sin\frac{6}{24}\sin\frac{8}{24}}\right) = 1.584848266 \text{ rad.}$$

Hence, the area of spherical triangle ABC having interior angles α', β' & γ' on the spherical surface of radius $R = 24\ cm$ is

$$= (\alpha' + \beta' + \gamma' - \pi)R^2 = (0.65763203 + 0.941391672 + 1.584848266 - \pi)24^2 = \mathbf{24.35288494\ cm^2}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta}} = 0.65763203 + 0.941391672 + 1.584848266 - \pi \approx \mathbf{0.0422793144\ sr}$$

Important note: In any spherical ΔABC drawn on a sphere of radius R, the interior angles A, B & C opposite to the sides $BC = a, AC = b, AB = c$ (each as a great circle arc) respectively are given by the general formula (as derived in above case-2) as follows

$$A = \cos^{-1}\left(\frac{\cos\frac{a}{R} - \cos\frac{b}{R}\cos\frac{c}{R}}{\sin\frac{b}{R}\sin\frac{c}{R}}\right),\ B = \cos^{-1}\left(\frac{\cos\frac{b}{R} - \cos\frac{a}{R}\cos\frac{c}{R}}{\sin\frac{a}{R}\sin\frac{c}{R}}\right)\ \&\ C = \cos^{-1}\left(\frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}}\right)$$

Conclusion: The analytical formulae derived in this work provide a systematic method for computing the principal geometric parameters of a spherical triangle, including the enclosed surface area, the solid angle subtended at the centre, and the interior angles. These results are directly applicable to the corresponding plane triangle obtained by joining the vertices with straight line segments. Furthermore, the developed relations enable analytical evaluation of the key geometric parameters of the triangular pyramid formed by connecting the vertices of a spherical triangle to the centre of the sphere.

Note: Above articles had been developed *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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