Why was it so difficult to prove the twin prime conjecture?

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Abstract:

The twin prime conjecture, asserting there are infinitely many pairs of primes differing by 2, was popularized by French mathematician Alphonse de Polignac in 1849 [1] [2] We are pleased to present an astounding and overwhelming proof with a clasic *reductio ad absurdum* flavour revealing, by the way, a perhaps not so amazing relationship with the Goldbach conjecture and testing, since the core of reasoning is the same, that both statements are strongly connected [3].

Definitions:

From now on, *m* and *n* are positive integer numbers, *p, q* are prime numbers and p_i (i = 1,2,3..., k) is the prime number sequence begining with p_1=5.

Twin prime conjecture states that there are infinitely many pairs of primes that differ by 2: 11-13; 17-19; 29-31, and so on.

Let´s suppose for the sake of contradiction that there is one last pair of twin primes p_k and p_{k-1} so $p_k - p_{k-1} = 2$. Note with respect to the next prime number p_{k+1} that $p_{k+1} - p_k \geq 4$. Now, our goal is to prove that it cannot happen that there is not at least one new pair of twin primes inside the interval $p_k^2 < m < p_{k+1}^2$ where $p_{k+1}^2 \geq (p_k + 4)^2 = p_k^2 + 8(p_k + 2) -$

If q and q+2 are twin prime numbers greater than 3 they are of the form 6n ± 1 so let´s see the conditions that 6n ± 1 ($p_k^2 < 6n < p_{k+1}^2$) must fullfill to became twin primes: Obviusly 6n ± 1 must not be multiple of any prime number less than or equal to p_k

Twin prime conditions
for 6n

$6n \pm 1 \not\equiv 0 \bmod 5$	or	$6n \not\equiv \pm 1 \bmod 5$
$6n \pm 1 \not\equiv 0 \bmod 7$	or	$6n \not\equiv \pm 1 \bmod 7$
$6n \pm 1 \not\equiv 0 \bmod 11$	or	$6n \not\equiv \pm 1 \bmod 11$

$6n \pm 1 \not\equiv 0 \bmod 13$ or $6n \not\equiv \pm 1 \bmod 13$

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$6n \pm 1 \not\equiv 0 \bmod p_k$ or $6n \not\equiv \pm 1 \bmod p_k$

Hence for each p_i there are p_i-2 remainders moduli p_i that fullfill the conditions. That amounts up to $(p_1-2)(p_2-2)(p_3-2)\ldots(p_k-2)$, id est, $3.5.9.11\ldots.(p_k-2)$ different systems of linear congruences with prime moduli. The chinese remainder theorem ensures that each one of them has a different and unique solution moduli $5.7.11.13\ldots p_k$.

It´s necessary then to prove that exists at least a multiple of 6 that fullfills the preceding conditions inside the aformentioned interval:

Let be M the greatest number of consecutive occurrences of 6m that do not fullfill the conditions. It is not easy to figure out the value of M, given the unpredictable nature of prime number distribution[1]. But we can prove that exists an upper bound L for M such that, for sufficient large n, L is less than the interval amplitude A:

$$A = 8(p_k+2)/6 = 4(p_k+2)/3$$

k	**p_k**	**M**	**A**
2	7	4	12
4	13	8	20
6	19	15	28

Given a series of S consecutive 6n´s, each residue class mod p appears about S/p times, so prime p covers roughly 2S/p of these S multiples of 6 and hence the density ***d*** of multiples not covered by any prime is about:

$$d = \prod_{5 \le p \le p_k} \left(1 - \frac{2}{p}\right) \quad (1)$$

Taking adventage of the simplification $\log(1 - x) \approx -x$ for small values of x [4]:

[1] For all those who, like myself, enjoy practical questions that sometimes shed light on some more abstract matter of discussion, the problem to determine an accurate value for **M** is the same as the following: Suppose you may not work on 2 predetermined days in five, 2 predetermined days in seven, 2 days in 11, 2 in 13 and so on until 2 days in p_k days. What is the maximum number, as a function of p_k, of consecutive days off?

$$\prod_{5\le p\le p_k}\left(1-\frac{2}{p}\right)\approx\ exp\left(-2\sum_{5\le p\le p_k}\frac{1}{p}\right)$$

Since the series between brackets is the well known partial summation of the reciprocal of the primes[5]:

$$\sum_{p\le x}\frac{1}{p}\approx \mathrm{loglog(x)}$$

Then:

$$\prod_{5\le p\le p_k}\left(1-\frac{2}{p}\right)\approx \exp(-2\log\log \mathrm{p_k})=\frac{1}{\log^2\mathrm{p_k}}$$

And the tipical gap between uncovered numbers is $\log^2 p_k$ so the longest run of consecutive multiples of 6 that do not generate a twin prime within the aforementioned interval grows as most on the order of the square of the logarithm of p_k.

Now, let´s find a **suitable upper bound L for M**: The inequality (valid for $0 < x < \frac{1}{2}$):

$$\log(1-x)\ge -x-x^2 \quad \text{with } x=2/p$$

gives:

$$\log\left(1-\frac{2}{p}\right)\ge -\frac{2}{p}-\frac{4}{p^2}$$

Applying in (1):

$$\log d\ge -2\sum_{5\le p\le p_k}\frac{1}{p}-4\sum_{5\le p\le p_k}\frac{1}{p^2}$$

In turn[2]:

$$\sum_{5\le p\le x}\frac{1}{p}\le \mathrm{loglogx}+1$$

And[2]

[2] Of course tighter bounds are available but not necessary at all for the sake of roudness and simplicity.

$$\sum_{p\geq 5}\frac{1}{p^2} \leq 0.1$$

Hence

$$\log d \geq -2\log\log p_k - 2 - 0.4 = -2\log\log p_k - 2.4$$

$$d \geq e^{-2.4}(\log p_k)^{-2}$$

$$d \geq \frac{0{,}09}{(\log p_k)^2}$$

Finally

$$L \leq \frac{1}{d} \leq \frac{1}{0{,}09}(\log p_k)^2$$

Arriving to the explicit bound:

$$L \leq 11(\log p_k)^2$$

So for sufficient large k, (let´s say for $p_k > 400$) M is smaller than p_k while A is always greater.

Now, given twin primes p_{k-1} and p_k , there is always at least a pair of twin primes between p_k^2 and p_{k+1}^2 . Hence it is inmediate to conclude that there are infinitely many twin primes.

That completes the demonstration.

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References:

[1] Alphonse de Polignac: *Recherches nouvelles sur les nombres premiers*. Comptes Rendus des Sciences de l'Academie des Sciences, 1849.

[2] S Nazardoyavi: *Some History about Twin Prime Conjecture*. Arxiv.2012.

[3] Óscar E. Chamizo Sánchez: *Why was that difficult to prove the Goldbach conjecture*. In https://rxiverse.org/abs/2602.0044.

[4] Apostol, Tom M: *Calculus.Volume I*: 434-437.John Wiley&Sons.1967.

[5] Pollack, Paul: *Euler and the partial sums of the prime harmonic series*. University of Georgia. Athens. Georgia.