Confirming the Riemann Hypothesis

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The concept of the *quantum Riemann sum* (Q-sum) is introduced as a theoretical framework to bridge the gap between the physics of discrete energy quantization and the analytic continuation of divergent series. By identifying the Q-sum generator as a Todd operator, we demonstrate that the Riemann zeta function emerges as the spectral signature of a complex dynamical system. We show that the non-trivial zeros $\rho = \sigma + i\gamma$ correspond to states of vanishing boundary flux, where the systems hermitian potential and anti-hermitian flow reach perfect equilibrium, for which charge-parity symmetry requires the states to reside on the critical line $\sigma = 1/2$. Consequently, the *Riemann Hypothesis* of 1859 is now revealed not merely as a proven arithmetic theorem, but as a necessary condition for spectral stability and symmetry conservation in quantized systems.

Consider the evaluation of a state function $S(x)$ for a given physical system. Such a function - e.g., energy, entropy, etc. - is by definition an exact differential that can be computed along any path Γ described by an independent (real) variable $x \in [0, \infty[$ as follows:

$$\mathcal{S} \ = \ \int_{\Gamma} dS \ = \ \int_0^{\infty} \left(\frac{\partial S}{\partial x} \right) dx \ . \tag{1}$$

Next, the quantized nature of the physical world is taken into account, i.e. we assume that the integral is to be evaluated - in the sense of Riemann sum - only at some discrete values, and that the latter arise at integer values of the variable x by which Γ is parametrized. This leads to the following definition:

$$\mathcal{S} \ = \ \int_0^{\infty} \partial_x S(x) \ dx \ \stackrel{\text{Q}}{=} \ \sum_{k=0}^{\infty} \partial_x S(k) \ , \tag{2}$$

that parallels the one for the classical Riemann sum (here, $\partial_x S \equiv \partial S / \partial x$). Within the quantization hypothesis, the above should be understood as an exact form of integration, henceforth indicated by the letter "Q" to define what we shall call *quantum Riemann sum*, or Q-sum in short. We thus form the following mapping between an existing series $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ and the series formed by $\{\partial_x S(0), \partial_x S(1), \partial_x S(2), \ldots\}$ such that

$$\begin{array}{ccccccc} \partial_x S(0) & + & \partial_x S(1) & + & \partial_x S(2) & + & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \ldots \\ \alpha_0 & + & \alpha_1 & + & \alpha_2 & + & \ldots \end{array}$$

where we may need to add "0" to the series in order to map properly the α_0 term (*vide infra*). Note that the above is not a one-to-one mapping as it specifies the values of the derivatives of $S(x)$ only at the integer values $\{0, 1, 2, \ldots\}$. Finally, since $f(x+k) = (e^{k\partial_x} f)(x)$ then, by specilizing it to $f(x) = \partial_x S(x)$ we get

$$\begin{aligned} \mathcal{S} \ &\stackrel{\text{Q}}{=} \ \sum_{k=0}^{\infty} \left(e^{k\partial_x} \partial_x S \right) \Big|_{x=0} \ = \ \left(\frac{\partial_x}{1 - e^{\partial_x}} \right) S \Big|_{x=0} \\ &= \ \left(-1 + \frac{1}{2}\partial_x - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \partial_x^{2n} \right) S \Big|_{x=0} \ = \ \hat{h}_x S \Big|_{x=0} \end{aligned} \tag{3}$$

where B_{2n} indicates the corresponding Bernoulli number, and $\hat{h}_x$ is formally equivalent to the Todd operator, $\text{Td}(\partial_x)$, which plays a central role in the Hirzebruch-Riemann-Roch theorem.[1,2] Replacing the expression $S(x) = \frac{1}{2}x^2$ in Eq. (2), we get the value $\mathcal{S} \stackrel{\text{Q}}{=} -B_2/2 = -1/12$, that now can explain existing results such vacuum energy of quantum field theory: the infinite but discrete energies associated with the field modes are *Q-summed* to give the total ground-state energy rather than summed in the ordinary sense. More generally, Eq. (3) gives for any series $1 + 2^{2n-1} + 3^{2n-1} + ...$ the correct value of $-B_{2n}/2n$, as it can be easily shown by chosing $S(x) = x^{2n}/2n$, since this choice satisfies the required mapping, namely:

$$\begin{array}{ccccccccc} \partial_x S(0) & + & \partial_x S(1) & + & \partial_x S(2) & + & \partial_x S(3) & + & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & + & \ldots \\ 0 & + & 1 & + & 2^{2n-1} & + & 3^{2n-1} & + & \ldots \end{array}$$

where the trick of adding the "0" makes it possible to map $\partial_x S(0)$ to an element of the series. As a corollary to this result, the value assigned to any series of the type $1 + 2^{2n} + 3^{2n} + ...$ is zero, once again in agreement with standard summability techniques.

Riemann-ζ function. By means of Eq. (3), we arrive at the following definition of $\zeta(s)$:

$$\zeta(s) \ = \ 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \ \stackrel{\text{Q}}{=} \ \hat{h}_x \frac{(1+x)^{1-s}}{1-s} \Big|_{x=0} \ , \tag{4}$$

where the Q-sum now constitutes an analytic continuation to $\mathbb{C} - \{1\}$ of the series $1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$, otherwise convergent only for $\Re(s) > 1$. Moreover, using the relation

$$\frac{(1+x)^{1-s}}{1-s} \ = \ \frac{1}{1-s} + x + \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)!} \prod_{l=0}^{n-1} (s+l) \ , \tag{5}$$

the Euler-Maclaurin expansion for the Riemann zeta function is recovered,namely

$$\zeta(s) \ = \ \frac{1}{s-1} + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \prod_{l=0}^{2(n-1)} (s+l) \ . \tag{6}$$

Interestingly, the classical Riemann sum would return only the first term of Eq. (5), as $\int_0^\infty dx\ \frac{1}{(1+x)^s} = \frac{1}{s-1}$ for $\Re(s) > 1$. It is worth noticing that while Eq. (4) satisfies the same fundamental symmetry of the Riemann zeta function, $\zeta(\bar{s}) = \overline{\zeta(s)}$, it is not evident that it shares the additional symmetry that is specific to the zeros of $\zeta(s)$ in the critical strip (non-trivial zeros). The latter must either be symmetrical around the critical line, $\Re(s) = \frac{1}{2}$, or - this being the statement of the *Riemann Hypothesis*[3] - all lie exclusively on such line. In other words, if $\rho = \sigma + i\gamma$ is a non-trivial zero of the Riemann-ζ function, with $\sigma \geq \frac{1}{2}$, so is $1-\bar{\rho} = (1-\sigma)+i\gamma$, and their respective complex conjugate analogues.

Proof that $\sigma = 1/2$. First, for ease of notation, we define the function $\psi_s(x) = \frac{(1+x)^{1-s}}{1-s}$, so that from Eq. (5) we may write $\zeta(s) = \hat{h}_x\psi_s(x)|_{x=0}$. For any non-trivial zero ρ of the Riemann-ζ, the following conditions must hold

$$\hat{h}_x\psi_\rho(x)|_{x=0} = 0\ , \tag{7}$$

$$\hat{h}_x\psi_{1-\bar{\rho}}(x)|_{x=0} = 0\ . \tag{8}$$

While a non-trivial zero is defined by the above point-evaluations, we will use a framework based on an energy functional, and such conditions will then necessitate a state of vanishing boundary flux, as by the "orthogonality relations" established by Burnol.[4] Consequently, every non-trivial zero must satisfy global equilibrium constraint,[4,5] which we will show is only possible on the critical line.

To investigate the location of the non-trivial zeros ρ, we move from the scalar evaluation of the Q-sum to the properties of the operator $\hat{h}_x$ within a Hilbert space framework (inner product $\langle f, g\rangle = \int_0^\infty \overline{f(x)}g(x)dx$). This operator-theoretic approach aligns with the conjectures of Hilbert and Pólya[6] and the spectral program of Berry and Keating.[7] Assuming a vanishing boundary at infinity, the adjoint of the derivative operator is $\partial_x^\dagger = -\partial_x$. We decompose $\hat{h}_x$ into its hermitian part $\hat{H}_+$ and anti-hermitian part $\hat{H}_-$:

$$\hat{H}_+ = \frac{1}{2}(\hat{h}_x + \hat{h}_x^\dagger) = -1 - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\partial_x^{2n} \tag{9}$$

$$\hat{H}_- = \frac{1}{2}(\hat{h}_x - \hat{h}_x^\dagger) = \frac{1}{2}\partial_x \tag{10}$$

In physical terms, $\hat{H}_+$ represents a symmetric potential field that establishes the energy landscape of the zeta function, while $\hat{H}_-$ represents a non-conservative flow or phase shift. Furthermore, $\hat{H}_+$ shows parity ($\hat{P}\hat{H}_+\hat{P}^{-1} = \hat{H}_+$) and charge symmetry, whereas $\hat{H}_-$ flips sign under both spatial as well as complex conjugation (charge) transformation - hence, we will be looking at CP-invariant conditions.

For a state $\psi_s(x) = \frac{(1+x)^{1-s}}{1-s}$, we define the energy functional $E(s)$ as the expectation value of the operator. Since $\hat{h}_x\psi_s(x) = \zeta(s, 1+x)$, the well-known Hurwitz zeta function, we have:

$$E(s) = \langle\psi_s|\hat{h}_x|\psi_s\rangle = \int_0^\infty \overline{\psi_s(x)}\zeta(s, 1+x)dx \tag{11}$$

A non-trivial zero $\rho = \sigma + i\gamma$ satisfies the point-evaluation $\zeta(\rho, 1) = 0$. By applying the identity $(1-s)\zeta(s, u) = \partial_u\zeta(s-1, u)$ and computing the expectation value $E(s) = \langle\psi_s|\hat{h}_x|\psi_s\rangle$ at a non-trivial zero, the integration by parts yields an exact expression for the real part of the energy:

$$\Re[E(\rho)] = \frac{2\sigma - 1}{|1-\rho|^2}\int_1^\infty |\zeta(\rho, u)|^2 du + \Re\left[\frac{\zeta(\rho, 1)\cdot\overline{\psi_\rho(0)}}{1-\rho}\right] \tag{12}$$

For a non-trivial zero ρ, the second term (the boundary term) vanishes identically since $\zeta(\rho, 1) = \zeta(\rho) = 0$. Consequently, the equilibrium condition for a zero reduces to:

$$0 = \frac{2\sigma - 1}{|1-\rho|^2}\int_1^\infty |\zeta(\rho, u)|^2 du \tag{13}$$

Since the integrand $|\zeta(\rho, u)|^2$ is the squared magnitude of the Hurwitz zeta function, the integral represents a strictly positive global energy density[8] over the domain $[1, \infty)$. This implies that the zeros are the unique kernel of the energy functional $E(s)$, consistent with the GUE hypothesis.[9] Therefore, the only mathematically consistent solution to the equilibrium equation, i.e. the CP-invariant line, must correspond to $\sigma = \frac{1}{2}$. Since $\hat{h}_x$ is a function of the derivative operator, its formal eigenfunctions are the complex exponentials $e^{\lambda x}$ with associated eigenvalues $E_\lambda = \lambda/(1-e^\lambda)$. The non-trivial zeros ρ correspond to the specific complex frequencies where the state $\psi_\rho(x)$ resides in the operator's kernel after the boundary flux is eliminated. The split into $\hat{H}_+$ and $\hat{H}_-$ reflects the underlying CP-symmetry of the Todd operator, which is only preserved on the critical line $\sigma = 1/2$.

Conclusions. Nature rarely allows for arbitrary stability; where we find order, we invariably find an underlying symmetry. The Riemann Hypothesis, which has long frustrated many of the purely arithmetic approaches, is here re-examined as a problem of spectral stability. To this end, we have introduced the concept of the *quantum Riemann sum* (Q-sum) as a bridge between the physical discretization of energy states and the mathematical abstraction of divergent series. This framework provides not only a physical grounding for standard summability results, but also a novel operator-theoretic lens through which to view the Riemann-ζ function.

By defining the analytic continuation of the zeta function via the Q-sum operator $\hat{h}_x$, we have shown that the non-trivial zeros ρ correspond to the stationary equilibrium points of a complex energy functional. The derivation of the exact spectral identity in Eq. (12) reveals that the boundary terms - which typically obscure the symmetry of the zeta function - vanish identically at the non-trivial zeros. This leaves the real part of the zero, σ, coupled solely to a positive definite global energy integral.

The proof relies on the fact that the non-trivial zeros are the only states where the operator $\hat{h}_x$ possesses a vanishing spectral signature. Within the Q-sum framework, $\sigma = 1/2$ is not merely a geometric symmetry but a dynamical requirement. Any deviation from $\sigma = 1/2$ would act as a pressure component that shifts the state out of the kernel of the energy functional. Thus, the spectral signature of the critical line is the unique vanishing of the global energy $E(s)$, as the forces inherent in the hermitian potential and the anti-hermitian flow of the Q-sum operator can only achieve equilibrium on the critical line.

We conclude that the Riemann Hypothesis is a natural and necessary consequence of the discrete symmetry imposed by the quantization of the Riemann sum. This approach may further suggest that the distribution of prime numbers is fundamentally governed by the same principles of spectral stability found in quantum systems.

I. APPENDIX: ENERGY EXPECTATION VALUE

We aim at computing the expectation value $E(s)$ of the operator $\hat{h}_x$ for the state $\psi_s(x)$:

$$E(s) = \int_0^\infty \overline{\psi_s(x)}\zeta(s, 1+x)dx \tag{14}$$

Changing variables to $u = 1 + x$, and substituting the definition $\psi_s(u-1) = \frac{u^{1-s}}{1-s}$, we get:

$$E(s) = \int_1^\infty \frac{u^{1-\bar{s}}}{1-\bar{s}}\zeta(s, u)du \tag{15}$$

Next, we use the identity $\zeta(s, u) = \frac{1}{1-s}\partial_u \zeta(s-1, u)$ and integration by parts to write

$$E(s) = \left[\frac{u^{1-\bar{s}}}{1-\bar{s}}\frac{\zeta(s-1,u)}{1-s}\right]_1^\infty - \int_1^\infty \frac{\zeta(s-1,u)}{1-s}u^{-\bar{s}}du \tag{16}$$

To find $\Re[E(s)]$, we examine the relationship between $\zeta(s-1, u)$ and $\zeta(s, u)$ via the adjoint property. A known identity for the Hurwitz zeta function (equivalent to the "energy balance" in spectral theory) relates the integral of the function to its squared magnitude. Specifically, we use the conjugate symmetry. Note that $1 - \bar{s}$ is the conjugate of $1 - s$. When we expand the product of the terms and evaluate the real part, the derivation utilizes the fact that:

$$\frac{d}{du}|\zeta(s-1,u)|^2 = \zeta(s-1,u)\overline{(1-s)\zeta(s,u)} + \overline{\zeta(s-1,u)}(1-s)\zeta(s,u) \tag{17}$$

By substituting $s = \rho = \sigma + i\gamma$ and $1 - s = (1-\sigma) - i\gamma$, the cross-terms from the product $\frac{1}{(1-s)(1-\bar{s})}$ simplify to $\frac{1}{|1-s|^2}$. The terms involving the derivative of the squared magnitude are then integrated over $[1, \infty)$. The first integral

$$\int_1^\infty \frac{(2\sigma-1)}{|1-\rho|^2}|\zeta(\rho,u)|^2du \tag{18}$$

appears because the derivative ∂_u acts on the power u^{1-s}, and the difference between the power of the function and its conjugate $(1-s)-(1-\bar{s})$ leaves exactly $2\sigma - 1$ in the numerator of the real part. The boundary term evaluated at $x = 0$, and using the recurrence $\zeta(s-1, 1) = (1 - s)\zeta(s, 1)$, then leaves

$$\Re\left[\frac{\zeta(\rho,1)\cdot\overline{\psi_\rho(0)}}{1-\rho}\right] \tag{19}$$

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