Non-inertial Relativity Theory, MOND and Generalized Gravity in Curved Phase Spaces *

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Abstract

We pursue further our work on (Born Reciprocal) Non-inertial Relativity theory. Starting with a brief review of the theory, and how Non-inertial Relativity redefines the notion of mass, the phase space particle trajectories in $D = 2 + 2$ are revisited by emphasizing the key difference between a truly $U(1,1)$-invariant mass $\mathcal{M}$ and the Lorentz-invariant mass m. The rewriting of the special relativistic expression of $E = m(1-v^2)^{-1/2}$ $(c = 1)$ to the non-inertial relativistic case $E = \mathcal{M}(1 - v^2)^{-1/2}(1 - \frac{F^2}{b^2})^{-1/2}$ follows, where b is the maximal proper force. This, in turn, leads to the non-inertial relativistic version of an *analog* of Milgrom's modified Newtonian dynamics (MOND) law. Subsequent modifications to Newton's law of motion in the Galilean limit $v << 1$ are derived. In the most general setting, one finds proper-time $m(\tau)$, and spacetime-dependent $m(x^\mu)$ masses for point particles, when the proper force depends on τ or x^μ, respectively. By recurring to the tools of Finsler geometry, we finalize by writing the generalized gravitational field equations in curved phase space in the presence of matter sources, like particles and cosmic strings. As a result, both spacetime and momentum space are curved. We conclude with some remarks as to why curved momentum space should play an important role in quantum gravity. In particular, why the fusion of non-inertial relativity with quantum mechanics should lead to a novel formulation of thermal quantum field theory (TQFT), and in turn, should cast some light into the quantization of gravity and its role in black hole thermodynamics.

Keywords : Born Reciprocal Relativity; Non-inertial Relativity; Modified Newtonian dynamics; Curved Phase Space; Strings; Quantum Gravity.

*Dedicated to the memory of Carlos Terol and Cecilio Sanchez-Robles, longtime friends in Spain

1 Brief Introduction of Non-inertial Relativity Theory

In this section we shall review very briefly the basic ideas behind Non-inertial relativity theory. The principle behind the concept of "Born reciprocal relativity theory", or non-inertial relativity to be more precise[1], was advocated by [3], [4], [5] and it was based on the idea proposed long ago by [1] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. Hence, a *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [5].

The principle of maximal acceleration was advocated earlier on by [7]. A chapter on Reciprocity Theory and Born's Quantum Metric Operator appeared early on in the book by [2]. The concept of Born reciprocity in order to provide a new point of view on string theory in which spacetime is a derived dynamical concept was advanced by [14].

The generalized velocity and force (acceleration) boosts (rotations) transformations of the $flat$ $8D$ Phase space coordinates , where $X^i, t, E, P^i; i = 1,2,3$ are **c** -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [3] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$.

Adopting the units $\hbar = c = 1$, the $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dt \wedge dE + \delta_{ij} dX^i \wedge dP^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $flat$ $8D$ phase-space

$$(d\omega)^2 \;=\; (dt)^2 \;-\; (dX)^2 \;-\; (dY)^2 \;-\; (dZ)^2 \;+$$

$$\frac{1}{b^2}\;\left((dE)^2 \;-\; (dP_x)^2 \;-\; (dP_y)^2 \;-\; (dP_z)^2\right), \quad (c=1) \tag{1.1}$$

The maximal proper force is set to be given by b. The symplectic group is relevant because $U(1,3) = Sp(8,R) \cap O(2,6)$; $U(3,1) = Sp(8,R) \cap O(6,2)$, and $U(2,2) = Sp(8,R) \cap O(4,4)$.

These transformations can be $simplified$ drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following phase space line interval

$$(d\omega)^2 \;=\; (dt)^2 \;-\; (dX)^2 \;+\; \frac{(dE)^2 \;-\; (dP)^2}{b^2} \;=$$

[1]We thank one of the referees of a previous article for highlighting this fact in order to clarify the point that Born did not propose a reciprocal relativity theory

$$(d\tau)^2 \left(1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 + \frac{\mathcal{F}^2}{F^2_{max}} \right) =$$

$$(d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right), \quad \mathcal{F}^2 = -F^2 < 0, \quad F_{max} = b \qquad (1.2)$$

where one has factored out the non-vanishing proper time infinitesimal $(d\tau)^2 = dt^2 - dX^2 \neq 0$ in eq-(1.2). The numerical quantity F^2 is positive by definition. The proper force-squared on a massive particle is $\mathcal{F}^2 = m^2 a^2$, where a^2 is the proper acceleration-squared $a^2 = a_\mu a^\mu$, and m is the rest mass. We refrained from factoring out $(dt)^2$ in (1.2) because it is not Lorentz invariant, whereas $(d\tau)^2$ is Lorentz invariant.

Due to the orthogonality condition $u_\mu a^\mu = 0$ resulting from differentiating the normalization condition $u_\mu u^\mu = \pm 1$ of timelike/spacelike velocities, when the velocity is timelike (subluminal particle) one has $(d\tau)^2 > 0$, so the acceleration is spacelike $a^2 = a_\mu a^\mu < 0$. Therefore $m^2 a^2 < 0$ since $m^2 > 0$. And viceversa, when the velocity is spacelike (superluminal particle) one has $(d\tau)^2 < 0$, so the acceleration is timelike $a^2 = a_\mu a^\mu > 0$. Therefore $m^2 a^2 < 0$ since $m^2 < 0$ (tachyonic particle). Consequently, the proper force squared $\mathcal{F}^2 \equiv (\frac{dE}{d\tau})^2 - (\frac{dP}{d\tau})^2 = \frac{(dE)^2 - (dP)^2}{(d\tau)^2} = m^2 a^2 < 0$ is always negative. Therefore, one may rewrite the negative definite $\mathcal{F}^2$ as $\mathcal{F}^2 \equiv m^2 a^2 = -F^2 < 0$, with $F^2 > 0$, so that the factorization can always be rewritten as $(d\tau)^2 (1 + \frac{\mathcal{F}^2}{b^2}) = (d\tau)^2 (1 - \frac{F^2}{b^2})$, with $F^2 > 0$. When $m = 0$, one has $(d\tau)^2 = (dE)^2 - (dP)^2 = 0$ so that $(d\omega)^2 = 0$. No factorization is needed.

Consequently, the *negative* sign appearing inside the parenthesis in eqs-(1.2) furnishes the analog of the Lorentz relativistic factor in special relativity and it involves the ratio of the square of two *proper* forces. These results can be generalized to the $8D$-dim phase space (and to higher dimensions)

The $U(1,1)$ group transformations involving the velocity and force boosts (along the X direction) acting on the phase-space coordinates X, t, P, E and which leave invariant the interval (1.2) are given by [3], [4]

$$t' = t \cosh \xi + (\xi_v \, x + \frac{\xi_a \, P}{b}) \frac{\sinh \xi}{\xi} \qquad (1.3a)$$

$$E' = E \cosh \xi + (b \, \xi_a \, X + \xi_v P) \frac{\sinh \xi}{\xi} \qquad (1.3b)$$

$$X' = X \cosh \xi + (\xi_v \, t + \frac{\xi_a \, E}{b}) \frac{\sinh \xi}{\xi} \qquad (1.3c)$$

$$P' = P \cosh \xi + (\xi_v \, E + b \, \xi_a \, t) \frac{\sinh \xi}{\xi} \qquad (1.3d)$$

ξ_v is the velocity-boost rapidity parameter; ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame. The rapidity parameters ξ_a, ξ_v, ξ are defined, respectively, in terms of the spatial velocity $v = dx/dt$, and proper force $F = ma$, as follows

$$\tanh(\xi_v) = v; \quad \tanh(\xi_a) \ = \ \frac{F}{F_{max}}, \ \ F_{max} = b, \ \ \xi = \ \sqrt{(\xi_v)^2 + (\xi_a)^2} \quad (1.3e)$$

When $\xi_v \to \infty \Rightarrow v \to c = 1$. And $\xi_a \to \infty \Rightarrow F \to F_{max} = b$.

It is straight-forward to verify that the transformations of eqs-(1.3) leave invariant the phase space interval $(dt)^2 - (dX)^2 + ((dE)^2 - (dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dt^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - (dP)^2]$. Only the *combination* is truly left invariant under force (acceleration) boosts

$$(d\omega)^2 \ = \ (d\tau)^2 \ \left(1 \ + \ \frac{\mathcal{F}^2}{F^2_{max}} \right) \ = \ (d\tau)^2 \ \left(1 \ - \ \frac{F^2}{F^2_{max}} \right), \ \ F^2_{max} = b^2 \ \ (1.4)$$

where $\mathcal{F}^2 \equiv m^2 a^2 = -F^2 < 0$, with $F^2 > 0$. The transformations of eqs-(1.3a-1.3d) also leave invariant the symplectic 2-form (phase space areas) $\Omega = - \ dt \wedge dE + dX \wedge dP$, see [3], [4] for full details.

Related to the importance of the symplectic 2-form, one should add that linear and quadratic Casimir operators corresponding to representations of the Linear Canonical Transformations (LCT) group $Sp(2,8)$, have been constructed by [24]. $Sp(2,8)$ emerges naturally as the symmetry group of the $10D$ relativistic quantum phase space associated with $5D$ spacetime dimensions. The symplectic group also plays an important role in the formulation of metaparticles and the metastring [14].

To finalize this introduction, it is warranted to explore the "dual" limit $b \to 0$ to the $b \to \infty$ limit, in the same vein that the Carrollian limit $c \to 0$ is the "dual" version of the Galilean limit $c \to \infty$ in special relativity.

2 Non-inertial Relativity redefines notion of Mass

In this section we will explore further the physical consequences of non-inertial relativity. Starting with a summary of the recent work in [15] on phase space trajectories in $D = 2 + 2$, and after explaining the key differences between a truly $U(1,1)$-invariant mass $\mathcal{M}$ with the Lorentz-invariant mass m, we proceed in showing how the special relativistic expression of $E = m(1 - v^2)^{-1/2}$ $(c = 1)$ can be rewritten in terms of $\mathcal{M}$ and F in the non-inertial relativistic case as follows

$$E \ = \ \mathcal{M} \ \frac{1}{\sqrt{1 - v^2}} \ \frac{1}{\sqrt{1 - \frac{F^2}{b^2}}}, \quad (c = 1) \qquad (2.1)$$

A similar relation (but not the same) and based on the principle of maximal proper acceleration, instead of maximal proper force, has been found by [16]. And finally, we proceed with the derivation of the non-inertial relativistic version of an *analog* of Milgrom's modified Newtonian dynamics (MOND) law [17].

Given the motion of a massive particle subjected to a proper acceleration g, the phase space infinitesimal interval (1.2) is given by $d\omega = \sqrt{1 - \frac{m^2 g^2}{b^2}} d\tau$. Due to the $U(1,1)$ invariance of $d\omega$, under force boosts transformations one has the relation

$$d\omega = \sqrt{1 - \frac{(mg)^2}{b^2}} \, d\tau = \sqrt{1 - \frac{(m'g')^2}{b^2}} \, d\tau' \qquad (2.2)$$

which reveals that $F = mg \neq F' = m'g'$ and $\tau \neq \tau'$. Namely, the proper force mg and the proper time τ are *not* fully $U(1,1)$ invariant, they are $(SO(1,1))$ Lorentz invariant. The explicit transformations relating $m'g'$ with mg, and relating τ' with τ, in terms of the force-boost rapidity parameter ξ_a in the particular case of particles exhibiting a uniform proper acceleration and following Rindler hyperbolic trajectories can be found in [6].

Therefore, ω is the truly $U(1,1)$-invariant evolution parameter that must be used in order to describe the phase space trajectories of a particle instead of the proper time τ which is *not* $U(1,1)$ invariant. Consequently, one must have expressions for the phase space coordinates defined in terms of ω as follows

$$Z_I(\omega) \equiv \{ \, t(\omega), \, x(\omega), \, \frac{1}{b} E(\omega), \, \frac{1}{b} p(\omega) \, \}, \quad I = 1,2,3,4 \qquad (2.3a)$$

The analog of the condition $V_\mu V^\mu = 1$ in phase space is

$$\dot{Z}_I \, \dot{Z}^I = \left(\frac{dt}{d\omega} \right)^2 - \left(\frac{dx}{d\omega} \right)^2 + \frac{1}{b^2} \left(\frac{dE}{d\omega} \right)^2 - \frac{1}{b^2} \left(\frac{dp}{d\omega} \right)^2 = 1 \qquad (2.3b)$$

The analog of the condition $a_\mu a^\mu < 0$ in phase space is

$$\ddot{Z}_I \, \ddot{Z}^I = \left(\frac{d^2 t}{d\omega^2} \right)^2 - \left(\frac{d^2 x}{d\omega^2} \right)^2 + \frac{1}{b^2} \left(\frac{d^2 E}{d\omega^2} \right)^2 - \frac{1}{b^2} \left(\frac{d^2 p}{d\omega^2} \right)^2 = -\mathcal{A}^2(\omega) \quad (2.4)$$

where $\mathcal{A}(\omega)$ is the phase space analog of the proper spacetime acceleration and must not be confused with $g(\tau)$.

Before we discuss the energy and momentum it is very important to invoke the construction of the quadratic Casimir invariants of the Quaplectic group studied by Low [3], [4]. The Quaplectic group in four spacetime dimensions (eight phase space dimensions) is the semi-direct product of $U(1,3)$ with the translations in phase space and including the unit central element associated with the Weyl-Heisenberg algebra. The relevance of the quadratic Casimir is that it correctly defines the analog of proper mass $\mathcal{M}$ in phase space. Therefore, upon defining

$$\mathcal{M} \frac{dt}{d\omega} = E, \quad \mathcal{M} \frac{dx}{d\omega} = p \qquad (2.5)$$

where $\mathcal{M}$ is the $U(1,1)$-invariant proper mass in phase space, and which is not the same as m, the two eqs-(2.3,2.4) become a system of two simultaneous differential equations for the two functions $E(\omega), p(\omega)$ given by

$$\left(\frac{E}{\mathcal{M}}\right)^2 - \left(\frac{p}{\mathcal{M}}\right)^2 + \frac{1}{b^2}\left(\frac{dE}{d\omega}\right)^2 - \frac{1}{b^2}\left(\frac{dp}{d\omega}\right)^2 = 1 \quad (2.6)$$

$$(\frac{1}{\mathcal{M}})^2\left(\frac{dE}{d\omega}\right)^2 - (\frac{1}{\mathcal{M}})^2\left(\frac{dp}{d\omega}\right)^2 + \frac{1}{b^2}\left(\frac{d^2E}{d\omega^2}\right)^2 - \frac{1}{b^2}\left(\frac{d^2p}{d\omega^2}\right)^2 = -\mathcal{A}^2(\omega) \quad (2.7)$$

Solutions to eqs-(2.6,2.7) were found in [15] in the special case that $\mathcal{A}$ is constant. Setting $\mathcal{A} = \mathcal{A}(\kappa)$ to be a one-parameter family of accelerations *independent* of the phase space evolution parameter ω, we found a one-parameter family of solutions to eqs-(2.6,2.7) is given by

$$t(\omega;\kappa) = \frac{\kappa}{\mathcal{A}(\kappa)}\ \sinh[\mathcal{A}(\kappa)\ \omega], \quad x(\omega;\kappa) = \frac{\kappa}{\mathcal{A}(\kappa)}\ \cosh[\mathcal{A}(\kappa)\ \omega] \quad (2.8a)$$

$$E(\omega;\kappa) = \kappa\ \mathcal{M}\ \cosh[\mathcal{A}(\kappa)\ \omega], \quad p(\omega;\kappa) = \kappa\ \mathcal{M}\ \sinh[\mathcal{A}(\kappa)\ \omega] \quad (2.8b)$$

where κ is a numerical parameter. From eq-(2.8a) one infers that as $\omega \to \infty$ the particle reaches the speed of light $\frac{dx}{dt} = \frac{(dx/d\omega)}{(dt/d\omega)} = \tanh[\mathcal{A}(\kappa)\omega] \to 1$ $(c = 1)$.

Inserting the solutions given by eqs-(2.8) into eqs-(2.6,2.7) lead to the relations

$$\frac{M^2\mathcal{A}^2(\kappa)}{b^2} = 1 - \frac{1}{\kappa^2}, \quad \mathcal{M}\ \mathcal{A}(\kappa) = b\sqrt{1 - \frac{1}{\kappa^2}} \le b, \quad \kappa \ge 1 \quad (2.9)$$

In order to avoid complex values for $\mathcal{M}\mathcal{A}(\kappa)$, one must choose $\kappa \ge 1$. The restriction $\kappa \ge 1$ is required to keep the phase-space acceleration real and bounded, and that $\kappa < 1$ has no clear physical interpretation in the present framework. A value of $\kappa = 1$ yields $\mathcal{M}\mathcal{A}(\kappa = 1) = 0$. The condition $\mathcal{M}\mathcal{A}(\kappa) \le b$ in eq-(2.9) is also a sign of consistency such that the maximal upper bound of b is not exceeded. In the limit $\kappa \to \infty$ one has $\mathcal{M}\mathcal{A}(\kappa) \to b$ and the upper bound b is saturated.

The initial positions of the trajectories in eq-(2.8a) is described by the functions

$$x(\omega = 0;\kappa) = x_o(\kappa) = \frac{\kappa}{\mathcal{A}(\kappa)} = \frac{\kappa\mathcal{M}}{\mathcal{M}\mathcal{A}(\kappa)} = \frac{\kappa\mathcal{M}}{b\sqrt{1-(1/\kappa^2)}} \quad (2.10)$$

$x_o(\kappa)$ is proportional to the throat-sizes of the elliptic-hyperboloids.

When $\kappa = 1$ and $\infty \Rightarrow x_o = \infty$. The minimum value of x_o is obtained by solving $\frac{dx_o}{d\kappa} = 0$. After some algebra one arrives at a quadratic equation for κ : $1 - \frac{2}{\kappa^2} = 0 \Rightarrow \kappa = \sqrt{2} > 1$. Inserting this value of κ into (2.10) yields

$$(x_o)_{min} \;=\; \frac{2\mathcal{M}}{b} \;=\; \frac{2G\mathcal{M}}{Gb} \tag{2.11}$$

If one sets the maximal proper force b to be equal to the Planck mass-squared $b = M_P^2$, in units of $\hbar = c = 1$, then $Gb = GM_P^2 = L_P^2 M_P^2 = 1$, with L_P being the Planck scale in $4D$, such that the minimal initial position turns out to be $2G\mathcal{M}$ which coincides precisely with the numerical value of the horizon radius of the Schwarzschild black hole in four spacetime dimensions (which must not be confused with the four dimensions of the phase space we have been working with signature $(2,2)$). One must emphasize that this is just a numerical coincidence and that a rigorous derivation would require working in a curved gravitational background in order to invoke the horizon radius of the Schwarzschild black hole.

The solutions in eqs-(2.8) describe a one-parameter family of elliptic-hyperboloids in four dimensions defined by the algebraic equation

$$x^2 \;+\; \frac{p^2}{b^2} \;-\; t^2 \;-\; \frac{E^2}{b^2} \;=\; \frac{1}{\mathcal{A}(\kappa)^2} \;=\; \frac{\mathcal{M}^2}{b^2(1-1/\kappa^2)} \tag{2.12a}$$

The above algebraic equation for the elliptic-hyperboloid is $U(1,1)$-invariant. Namely, because the evolution parameter ω and $\mathcal{M}$ are $U(1,1)$-invariant, then under $U(1,1)$ transformations of the trajectories $Z^I(\omega) \;\rightarrow\; Z'^I(\omega)$ in phase space, one ends up with the same analytical (functional) form for the elliptic-hyperboloids

$$x'^2 \;+\; \frac{p'^2}{b^2} \;-\; t'^2 \;-\; \frac{E'^2}{b^2} \;=\; \frac{1}{\mathcal{A}(\kappa)^2} \;=\; \frac{\mathcal{M}^2}{b^2(1-1/\kappa^2)} \tag{2.12b}$$

After having reviewed the above basic results of [15] pertaining accelerating particle trajectories in a $D = 2+2$-dim flat phase space, we proceed with the derivation of the generalization of the expression relating mass with energy. Focusing on eq-(2.6), after some straightforward algebra, using the special relativistic relation $E^2 - p^2 = m^2$, and $(d\omega)^2 = (d\tau)^2(1 - F^2/b^2)$, it furnishes the relationship between m and $\mathcal{M}$

$$E^2 - p^2 \;=\; m^2 \;=\; \frac{\mathcal{M}^2}{1-\frac{F^2}{b^2}} \Rightarrow m = m(F) \;=\; \frac{\mathcal{M}}{\sqrt{1-\frac{F^2}{b^2}}} \tag{2.13}$$

Eq-(2.13) leads to a force-dependent mass $m = m(F)$, and a rewriting of the mass-energy expression in two different ways as follows

$$E \;=\; m(1-v^2)^{-1/2} \;=\; \mathcal{M}\frac{dt}{d\omega} \;=\; \mathcal{M}\frac{(dt/d\tau)}{\sqrt{1-\frac{F^2}{b^2}}} \;=\; \mathcal{M}\;\frac{1}{\sqrt{1-v^2}}\;\frac{1}{\sqrt{1-\frac{F^2}{b^2}}} \tag{2.14}$$

with m being the invariant rest mass (proper mass) of the particle.

One of the most salient features of the force-dependent mass relation $m^2 = m(F)^2 = \mathcal{M}^2(1 - F^2/b^2)^{-1}$, is that the definition $\mathcal{F}^2 = m^2 a^2 = -F^2 = -m^2 g^2 < 0$ yields an expression of the form $F = \sqrt{-m(F)^2 a^2}$, and which could be viewed as a non-inertial relativistic "extension" of the Milgrom's modified Newtonian dynamics (MOND) law $\vec{f} = m(|\vec{a}|)\vec{a}$ [17]. One should emphasize that the non-inertial relativistic extension of the Milgrom's modified Newtonian dynamics described here is very $different$ than the various theoretical attempts made to effectively embed the modifications of Newtonian dynamics within a relativistic theory of gravity [18], [19].

Given $-F^2 = m^2 a^2 < 0$, after defining $g^2 \equiv -a^2 \Rightarrow F^2 = m^2 g^2 > 0$, from eq-(2.13) one ends up with the following relation among the masses $m, \mathcal{M}$

$$m = \frac{\mathcal{M}}{\sqrt{1 - \frac{m^2 g^2}{b^2}}} \Leftrightarrow \frac{m^2}{\mathcal{M}^2} = \frac{1 \pm \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \tag{2.15}$$

Let us examine the plus sign choice in front of the square root

$$m^2 = \frac{1 + \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2g^2}{b^2}} \tag{2.16a}$$

A close inspection of eq-(2.16a) reveals that in the $b \to \infty$ limit, the mass $m \to \infty$ blows up, and consequently, eq-(2.16a) does *not* have a well behaved special relativistic limit. For this reason one must take the minus sign choice in front of the square root, giving

$$\frac{m^2}{\mathcal{M}^2} = \frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \Rightarrow m = \mathcal{M} \left(\frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \right)^{1/2} \Rightarrow$$

$$m = \mathcal{M}\ \Omega(\frac{\mathcal{M}g}{b}) \tag{2.16b}$$

where the omega function Ω is defined by the expression involving the square root in (2.16b).

Another more transparent way to rewrite eq-(2.16b), after defining $F = mg$, and $\tilde{F} = \mathcal{M}g$, is

$$F = \tilde{F} \left(\frac{1 - \sqrt{1 - \frac{4\tilde{F}^2}{b^2}}}{\frac{2\tilde{F}^2}{b^2}} \right)^{1/2} \quad F = mg, \quad \tilde{F} = \mathcal{M}g \tag{2.17}$$

in other words, the ratio $F/\tilde{F}$ is a function of the ratio $\tilde{F}/b$ involving the maximal proper force $b : F/\tilde{F} = \Omega(\tilde{F}/b)$, such $b \to \infty \Rightarrow \tilde{F} \to F$ since $\Omega(0) = 1$. We shall return to eq-(2.17) when we analyze its Galilean limit $v << 1$ in eq-(2.24) leading to modified Newtonian dynamics.

Once again, it is important to emphasize that g is *not* $U(1,1)$ invariant as one can infer from the relation

$$d\omega \ = \ d\tau \ \sqrt{1 - \frac{m^2 g^2}{b^2}} \ = \ d\tau' \ \sqrt{1 - \frac{m'^2 g'^2}{b^2}} \qquad (2.18)$$

$\mathcal{M}$ and the maximal proper force b are truly $U(1,1)$ invariant as shown in [6]. The fact that $b \neq \infty$ does not imply that there is a maximal proper acceleration. On the contrary, the maximal proper acceleration could be $g = \infty$ when the mass $m = 0$ such that $mg = b$. The invariance of the maximal proper force b under $U(1,1)$ transformations, implies that $mg = m'g' = m''g'' = \ldots = b$, therefore there is a $flow$ of values of m, g along the hyperbola described by the algebraic equation $mg = b =$ constant. In particular, as $m \rightarrow \infty, g \rightarrow 0$, and vice versa, as $m \rightarrow 0, g \rightarrow \infty$.

A careful inspection of eqs-(2.13, 2.16b) reveals that :

(i) When $\mathcal{M} = 0 \Leftrightarrow m = 0$. Null lines are characterized by the relations $u_\mu u^\mu = 0; a_\mu a^\mu = -g^2 = 0; u^\mu a_\mu = 0$. Because $d\tau = 0, \tau = 0$ for a massless particle (photon) path, one has to use an affine parameter $\lambda \neq \tau$ in order to define the proper velocity and acceleration of a null path $u_\mu = (\frac{dt}{d\lambda}, \frac{dx}{d\lambda})$. $a_\mu = (\frac{d^2t}{d\lambda^2}, \frac{d^2x}{d\lambda^2})$. a_μ is parallel to u_μ such that $a_\mu = \frac{du_\mu}{d\lambda} = \xi u_\mu$ with ξ a parameter with physical mass units if λ has length/time units. Solving for u^μ gives $u^\mu = u^\mu_0 \exp(\xi\lambda)$; with $u^\mu(\lambda = 0) = u^\mu_0$ obeying $(u^\mu_0)(u_{\mu 0}) = 0$.

(ii) When $\mathcal{M} \neq 0$; $b = \infty$ and/or $g = 0 \Rightarrow m = \mathcal{M}$ after using L'Hopital's rule. This result is also consistent with the one obtained from eq-(2.13) after setting $b = \infty$, and/or $g = 0$.

(iii) One has the following bounds $mg \leq b$ and $\mathcal{M}g \leq \frac{b}{2}$.

(iv) When $m \rightarrow \infty; \mathcal{M} \rightarrow \infty$, eqs-(2.13, 2.16b) are consistent if, and only if, $g \rightarrow 0$ such that the double scaling limit is $mg = \frac{b}{\sqrt{2}} > \mathcal{M}g = \frac{b}{2}$; i.e. $\infty \times 0$ is finite and non zero. The consistency of eqs-(2.13,2.16b) gives in this special case $m = \sqrt{2}\mathcal{M} \rightarrow \infty$.

(v) The converse of case (iv) occurs when $m \rightarrow 0; \mathcal{M} \rightarrow 0$; $g \rightarrow \infty$ such that $mg = \frac{b}{\sqrt{2}} > \mathcal{M}g = \frac{b}{2}$ which are required for the consistency of eqs-(2.13,2.16b). Once again, this is possible if the double scaling limit $0 \times \infty$ is finite and non zero yielding $m = \sqrt{2}\mathcal{M} \rightarrow 0$.

Caution must be taken *not* to confuse case (i) with (v). In case (i) one already starts with a null path where $g = 0$. Whereas in case (v) one has timelike paths that asymptotically approach null ones with $g \rightarrow \infty$. To visualize what the limiting value of $g \rightarrow \infty$ represents, and how it is correlated with the zero mass limit, let us recall the Rindler hyperbolic trajectories associated with uniformly accelerated particles with proper uniform acceleration g, and which are described by the following hyperbolic trajectories in two dimensions

$$t \ = \ \frac{1}{g} \ \sinh(g\tau); \quad x = \frac{1}{g} \ \cosh(g\tau) \qquad (2.19)$$

where the initial position is $x(\tau = 0) = 1/g$ which coincides with the throat size of the hyperbola.

When the proper acceleration $g \to 0$, one has $t \to \tau$ due to L'Hopital's rule, and $x \to \infty$. The worldline is the same as that of a massive particle at rest at infinity. When $g \to \infty$, in this limit, the hyperbolas degenerate to the Rindler horizon ($t = \pm x$) (the throat size shrinks to zero) given by the null lines corresponding to massless photons trajectories. In this fashion one can envision how the double scaling limits $mg = 0 \times \infty = \frac{b}{\sqrt{2}}$, and $\mathcal{M}g = 0 \times \infty = \frac{b}{2}$ can be finite and non-zero in case (v).

(vi) When $m^2 < 0 \Leftrightarrow \mathcal{M}^2 < 0$. $m^2 > 0 \Leftrightarrow \mathcal{M}^2 > 0$.

To sum up, the relation $m(F) = \mathcal{M}(1 - F^2/b^2)^{-1/2}$ is the non-inertial relativistic version of the relation $m(v) = m(1 - v^2)^{-1/2}$ in special relativity where m is the rest mass, and such that $m(F = 0) = \mathcal{M}$. Therefore, the $U(1, D-1)$-invariant mass $\mathcal{M}$ is just the "zero-force" mass in $D = (d + 1)$-spacetime dimensions. One can have a particle at rest $v = 0$ while experiencing a force. For example, the harmonic oscillator is at rest at the location of the maximal force, where it is stretched to its maximum elongation. And vice versa, the velocity is maximal while the force is zero at the center where the elongation is zero.

A Taylor expansion of eq-(2.16b) leads to

$$m^2 \ = \ \mathcal{M}^2 \ \left(1 \ + \ (\frac{\mathcal{M}g}{b})^2 \ + 2 \ (\frac{\mathcal{M}g}{b})^4 \ + \ \ldots \right) \tag{2.20}$$

and must *not* be confused with the modified dispersion relations in Doubly Special Relativity (DSR) [22].

If one had a relation of the form

$$E^2 - p^2 \ = \ \frac{m^2}{1 - \frac{F^2}{b^2}} \ = \ m^2 \ \left(1 \ + \ \frac{F^2}{b^2} \ + \ (\frac{F^2}{b^2})^2 \ + \ (\frac{F^2}{b^2})^3 + \ldots \right), \quad \frac{F^2}{b^2} \leq 1 \tag{2.21}$$

it would resemble the modified dispersion relations in Doubly Special Relativity (DSR) [22] leading to an effective mass $m_{eff} = m(1 - F^2/b^2)^{-1/2}$. But as we have emphasized, $m \neq \mathcal{M}$, and what we have instead is $E^2 - p^2 = m^2 = \mathcal{M}^2(1 - F^2/b^2)^{-1}$.

To sum up, if one requires to have a well behaved special relativistic limit $b \to \infty$, then one is forced to choose the *minus* sign in eq-(2.15) leading to the functional relation $m = m(\mathcal{M}, g)$ displayed by eq-(2.16b). And, in doing so, the expression for the energy (2.14) can be written in terms of m, or $\mathcal{M}$, as follows

$$E \ = \ m \ (1 - v^2)^{-1/2} \ = \ m(\mathcal{M}, \tilde{F}) \ (1 - v^2)^{-1/2} \ =$$

$$\mathcal{M} \ \left(\frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \right)^{1/2} \ (1 - v^2)^{-1/2}, \quad \tilde{F} = \mathcal{M}g; \quad c = 1 \tag{2.22}$$

clearly, the second line involves a velocity and proper force dependence via the force $\tilde{F} = \mathcal{M}g$ and the standard Lorentz dilation factor.

The Taylor expansion of the Ω function in eqs-(2.16b,2.17) is given by

$$\Omega(\frac{\mathcal{M}|\vec{a}|}{b}) \equiv \left(1 + \frac{1}{2}(\frac{\mathcal{M}|\vec{a}|}{b})^2 + (\frac{\mathcal{M}|\vec{a}|}{b})^4 + \ldots\right) \quad (2.23)$$

In the Galilean limit $v << c = 1$, while keeping $b \neq \infty$, one finds that non-inertial relativity leads to a modification Newton's law of motion after writing

$$\vec{f} = \mathcal{M}\vec{a}\,\Omega(\frac{\mathcal{M}|\vec{a}|}{b}), \quad \frac{2\mathcal{M}|\vec{a}|}{b} \leq 1 \quad (2.24)$$

and which follows directly from eqs-(2.16b, 2.17) simply by replacing $F \to |\vec{f}|$ and $\tilde{F} = \mathcal{M}g \to \mathcal{M}|\vec{a}|$ in the Galilean limit. If one wishes, one can perform the Taylor series expansion but is not necessary. It is important to emphasize that eq-(2.24) does *not* have the same functional form as Milgrom's modified Newton's law. The interpolating functions are $different$.

To be more precise, one can rewrite $\vec{f}$ in two different ways $\vec{f} = m\vec{a} = \mathcal{M}\vec{a}'$, and after inserting $\vec{f} = \mathcal{M}\vec{a}'$ into eq-(2.24), it gives $|\vec{a}'| = |\vec{a}|\Omega(\frac{\mathcal{M}|\vec{a}|}{b})$. Inverting the latter relation yields $|\vec{a}| = |\vec{a}'|\sqrt{1 - (\frac{\mathcal{M}^2|\vec{a}'|^2}{b^2})}$. These relations between $\vec{a}'$ and $\vec{a}$ are analogous to the relations between the Milgromian a_M and Newtonian a_N accelerations [18]. Note, once more, that the former relations involves $different$ functions than the μ, ν interpolating functions relating the Milgromian a_M and Newtonian a_N accelerations [18] : $a_N = \mu(a_M/a_o)a_M \leftrightarrow a_M = \nu(a_o/a_N)a_N$. For this reason, one must emphasize that the Galilean limit of non-inertial relativity furnishes of a **non-Milgromian** MOND.

The reader may ask : despite this, is there still a role of Milgrom's acceleration constant a_o in all of this ? To answer this question one may notice that one can rewrite the maximal proper force b in terms of the Planck mass M_P, the Planck scale L_P; as well as the observable mass of the Universe M_U, and the Hubble Radius R_H as follows $b = m_P(c^2/L_P) = M_U(c^2/R_H)$, and such that c^2/R_H is closer to Milgrom's acceleration constant $a_o \sim 1.2 \times 10^{-10} m/s^2$.

If the maximal proper force b acting on a fundamental particle is set to be M_P^2, where M_P is the Planck mass in $D = 4$ spacetime dimensions, in $\hbar = c = 1$ units, it is clear that one cannot set $\mathcal{M}$ to be a huge mass, unless the magnitude of the acceleration $|\vec{a}|$ is *very* small such that the ratio $\frac{2\mathcal{M}|\vec{a}|}{b}$ still remains smaller than unity. In the case of fundamental particles whose masses are very small compared to the Planck mass M_P, due to the fact that they can acquire very large accelerations, as long as the ratio $\frac{2\mathcal{M}|\vec{a}|}{b}$ remains smaller than unity, one can still use eq-(2.24), Therefore, one of the most novel and relevant findings of this work is that eq-(2.24) has a wide range of validity, both for very large *and* very *small* accelerations as well, by choosing the appropriate range of mass values. In other words, non-inertial relativity theory has applications over a wide range of accelerations, large or small, which are correlated to small and large masses, respectively. The same argument goes with the ratio $\frac{\mathcal{M}|\vec{a}'|}{b} \leq 1$ involving $|\vec{a}'|$ and $\mathcal{M}$.

We learned that one cannot just focus on large/small accelerations because in non-inertial relativity we are dealing with the size of the **forces** (relative to the maximal proper force b). Therefore, one must also take into account the size of the masses. Galaxies have large masses and, naturally, have correlations with small accelerations. In principle, non-inertial relativity seems to be capable in dealing with the anomalous rotation curves of galaxies. Non-inertial relativity provides a physical motivation for studying deviations of Newtonian mechanics that differ from Milgromian MOND. Furthermore, it provides an extension of special relativity by incorporating accelerated frames of reference and where spacetime and energy-momentum coordinates are transformed into each other under force boosts transformations in phase space.

Perhaps non-inertial relativity might predict testable departures from special relativity in situations of extreme force, rather than extreme acceleration. The latter are correlated to lower masses, like those of fundamental particles. Hence, it is in Particle Physics experiments that we are more likely to detect signatures of non-inertial relativity. It would be desirable to detect departures from standard general relativity as well if future experiments permit.

To sum up, non-inertial relativity furnishes modifications to Newton's law of motion given by eq-(2.24) in the Galilean limit $v << 1$, and which differ from Milgromian MOND [17] due to the difference in the interpolating functions. Since $m \geq \mathcal{M}$, whether or not this subtle difference between m and $\mathcal{M}$ might shed some light into the physics behind dark matter (missing mass) is unknown at the moment. The invariance of $d\omega$ under $U(1,1)$ transformations

$$d\omega \ = \ d\tau \ \sqrt{1 - \frac{F^2}{b^2}} \ = \ d\tau' \ \sqrt{1 - \frac{F'^2}{b^2}} \tag{2.25}$$

and the relations

$$m \ = \ \frac{\mathcal{M}}{\sqrt{1 - \frac{F^2}{b^2}}}, \quad m' \ = \ \frac{\mathcal{M}}{\sqrt{1 - \frac{F'^2}{b^2}}} \tag{2.26}$$

lead to the important result

$$\frac{d\tau}{m} \ = \ \frac{d\tau'}{m'} \ \Rightarrow \ m' \ = \ m \ \frac{d\tau'}{d\tau} \tag{2.27}$$

In [6] we have shown that the scaling factor $\frac{d\tau'}{d\tau} = \lambda(F, F'_{obs})$ relating m and m' is a function of the proper force F experienced by the particle moving with respect to a reference frame S, and the proper force F'_{obs} experienced by an accelerated observer S' moving with respect to the reference frame S.

Contrary to what occurs in Lorentz transformations, under force (acceleration) boost transformations $d\tau \neq d\tau' \Rightarrow m \neq m'$, consequently the mass m is no longer a non-inertial-relativistic invariant, as expected. When $m = 0 \Leftrightarrow d\tau = 0$. We showed in [6] that under force (acceleration) boost transformations one has cases where $d\tau = d\tau' = 0$, so $m = m' = 0$. But also, there are cases when $d\tau' \neq 0$ and consequently $m' \neq 0$, hence a massless photon could appear massive in a non-inertial frame of reference [6].

One should not confuse a reparametrization $\tau \to \tau'$ with a force boost transformation. Under a reparametrization, the point particle action in special relativity can be rewritten as $S = -\int m d\tau = -\int m \frac{d\tau}{d\tau'} d\tau'$ and one may reinterpret the quantity $m \frac{d\tau}{d\tau'} = m' = m'(\tau')$ as a "reparametrized" mass such that $m d\tau = m' d\tau'$. One may note that this relation clearly differs from $\frac{d\tau}{m} = \frac{d\tau'}{m'}$ obtained from $U(1,1)$ transformations.

When $g(\tau) = g_0$ (constant), $m = m_0$ (constant), under velocity boosts (Lorentz) transformations one has $m_0 = m_0'; g_0 = g_0'$ since the proper mass and proper acceleration are relativistic invariants in special relativity. However, under force (acceleration) boosts $m_0 \to m_0' \neq m_0$, $g_0 \to g_0' \neq g_0$ [6]. This is the key difference and the reason why one can have a *flow* of values of $(m_0, g_0); (m_0', g_0'), \ldots$ along a hyperbola obeying the equation $mg = b = \text{constant}$

The relation between $F = m_0 g_0$, and $F' = m_0' g_0'$ under force (acceleration) boosts whose rapidity parameter is ξ was found to be [6]

$$\frac{F'^2}{b^2} = \frac{(\ \frac{m_0 g_0}{b}\ \cosh(\xi)\ +\ \sinh(\xi)\ )^2}{(\ \cosh(\xi)\ +\ \frac{m_0 g_0}{b} \sinh(\xi)\ )^2}, \quad F' = m_0' g_0' \tag{2.28}$$

By inspection of eq-(2.28) one has that if $m_0 g_0 = b$, one finds that $F' = m_0' g_0' = b$ also, for *all* values of the force boost rapidity parameter ξ, and which is consistent with the fact that $F = F' = b$ must remain invariant since they coincide with the maximal (and invariant) proper force.

In essence, eq-(2.28) is the analog of the "addition" of velocities in special relativity. The physical interpretation of eq-(2.28) is the following. One has a massive particle of proper mass m_0 moving with respect to a reference frame S with a uniform proper acceleration and force given by g_0 and $F = m_0 g_0$, respectively. A typical example is the Rindler particle sweeping hyperbolic trajectories. A new observer S' comes into the picture[2] moving with respect to S with a uniform proper force f associated to a force boost rapidity parameter ξ given by $\tanh(\xi) = \frac{f}{b}$. Therefore, the expression in eq-(2.31) depicts the "addition" of two *proper* forces F and f describing the net proper force F' experienced by the original massive particle with respect to the second observer S'.

To finalize this section we should mention that one could also have particles moving with a variable proper acceleration $g(\tau)$, and having a variable $m(\tau)$, a variable $\mathcal{M}(\tau)$, and still being subjected to a *constant* negative-definite proper force squared $\mathcal{F}^2 = -F^2 < 0$, with $F^2 = \beta^2 = \text{constant}$. In this case, the force is now given by $\mathcal{F}^\mu = \frac{d}{d\tau}(m u^\mu) = m a^\mu + u^\mu (dm/d\tau)$. Despite that $\mathcal{F}^\mu$ acquires an *extra* term, one can still ensure that $\mathcal{F}^2 = m^2 a^2 + (dm/d\tau)^2 = -F^2 = -\beta^2 < 0$ remains negative-definite, and constant, for time-like trajectories $(m^2 > 0, a^2 < 0)$, as well as space-like ones (tachyonic) $(m^2 < 0, a^2 > 0)$, after finding the mass function $m(\tau)$ obeying the above first order nonlinear differential equation. Implementing the maximal proper force postulate restricts the values of β to the domain $0 \leq \beta \leq b$.

[2]The new observer S' could be represented by a physical apparatus of mass M moving with a uniform proper acceleration a such that $f = Ma$

All one needs is to have an input function given by $g(\tau)$ leading to a differential equation for the appropriate mass function $m(\tau)$. To simplify matters let us work in two space-time dimensions. In the case of time-like trajectories, the solutions to $(u_0)^2 - (u_1)^2 = 1$, and $(\dot{u}_0)^2 - (\dot{u}_1)^2 = a^2(\tau) = -g^2(\tau)$ (the dot denotes derivatives with respect to τ) are given by [15]

$$x(\tau) \ = \ \int_0^\tau \ \sinh \left(\int_0^{\tau'} \ g(\tau'')d\tau'' \right) \ d\tau' \ + \ x_o \tag{2.29}$$

where $x_o = x(\tau = 0)$ is an integration constant. The solution for $t(\tau)$ turns out to be

$$t(\tau) \ = \ \int_0^\tau \ \cosh \left(\int_0^{\tau'} \ g(\tau'')d\tau'' \right) \ d\tau' \tag{2.30}$$

subjected to the condition $t(\tau = 0) = 0$. One can verify by a mere *substitution* that the solutions in eqs-(2.29,2.30) for $t(\tau), x(\tau)$ obey

$$\left(\frac{d^2 t}{d\tau^2} \right)^2 \ - \ \left(\frac{d^2 x}{d\tau^2} \right)^2 \ = \ a^2(\tau) \ = \ -g^2(\tau) \qquad (2.31))$$

for *all* functions $g(\tau)$. One can also corroborate that the solutions for $x(\tau), t(\tau)$ satisfy

$$u_\mu a^\mu \ = \ \frac{dt}{d\tau} \ \frac{d^2 t}{d\tau^2} \ - \ \frac{dx}{d\tau} \ \frac{d^2 x}{d\tau^2} \ = \ 0 \tag{2.32}$$

for *all* values of $g(\tau)$. When $g(\tau) = g_0$ = constant, one recovers the Rindler hyperbolic trajectories.

It only remains to find solutions to the family of first order nonlinear differential equation for the variable mass $m(\tau)$, and defined in terms of the input function $g(\tau)$ and β.

$$m^2(\tau) g^2(\tau) \ - \ (dm/d\tau)^2 \ = \ \beta^2 \ \geq 0 \tag{2.33}$$

In the simplest and limiting borderline case, when $\beta^2 = 0 \Rightarrow \mathcal{F}^2 = -F^2 = 0$, the solutions are $m(\tau) = \mathcal{M}(\tau) = m(0) \exp(\pm \int_0^\tau g(\tau) d\tau)$, where $m(0) = m(\tau = 0)$. One may note that even in the case when $g(\tau) = g_0$ = constant, one can still have a variable mass $m(\tau) = \mathcal{M}(\tau) = m(0) \exp(\pm g_0 \tau)$. The plus sign must be excluded to avoid singularities in the $\tau = \infty$ limit. The decaying exponential leads to zero values for $m, \mathcal{M}$ at $\tau = \infty$ and a finite nonzero value $m(0)$ at $\tau = 0$. A decaying mass is compatible with the gravitational radiation of accelerated massive particles.

Another simple example is the case when $g(\tau) = g_0$ = constant, $\beta^2 \neq 0$. The differential equation is of the form $(dm/d\tau)^2 - g_0^2 m^2 + \beta^2 = 0$, and whose solutions are

$$m(\tau) \ = \ A \exp(g_0 \tau) \ + \ \frac{\beta^2}{4 A g_0^2} \ \exp(-g_0 \tau), \ \ A \ = \ \frac{m(0) \pm \sqrt{m(0)^2 - (\beta^2/g_0^2)}}{2}, \tag{2.34}$$

with $m(0) = m(\tau = 0)$ and $m(0) g_0 \geq \beta$. One then finds that when $\beta^2 \neq 0$, the solutions are well behaved at $\tau = 0$ but diverge at $\tau = \infty$. A very interesting case occurs when $\beta = b$ attains its maximal bound. One finds in this special case that $\mathcal{M} = m(\tau)\sqrt{1 - (\beta^2/b^2)} = 0$, and $m(\tau) = m(0)\cosh(g_0\tau)$, with $b = m(0) g_0 = \beta$. Whereas when $\beta^2 = 0$ there is a well behaved solution $m(\tau) = m(0)\exp(-g_0\tau)$ for all $0 \leq \tau \leq \infty$.

The solutions in the more general case when the input proper acceleration $g(\tau)$ is not constant, and $\beta^2 \neq 0$, are more difficult to find. The important point is that a reasonable solution exists when $\beta^2 = 0$ given by $m(\tau) = m(0)\exp(-\int_0^\tau g(\tau) d\tau)$, and associated with the infinitely many proper accelerations choices $g(\tau)$ leading to convergent solutions for $m(\tau)$.

It is desirable when one has a variable proper acceleration $g(\tau)$, and $\beta^2 \neq 0$, that one would be able to find judicious and satisfactory (finite) solutions for $m(\tau)$ and retain the $\mathcal{F}^2 = -F^2 = -\beta^2 \leq 0$ condition in order to implement the maximal proper force postulate of non-inertial relativity by restricting the values of β to the domain $0 \leq \beta \leq b$. If one wishes to relax the case that $\mathcal{F}^2 = -F^2 = -\beta^2$ constant, but instead now one has a variable $F(\tau)$, this will complicate matters even further when one tries to find solutions to the first order nonlinear differential equation obeying the bound $|F(\tau)| \leq b$ for all values of τ.

To sum up, when $g(\tau) = g_0$, and $\beta^2 = 0$, there is a satisfactory solution for $m(\tau)$. In the case of $g(\tau) \neq$ constant, and $\beta^2 \neq 0$, numerical solutions to the first order nonlinear differential equation are required to be found to see whether or not finite expressions of $m(\tau)$, for all τ are found.

It is not farfetched to contemplate the idea of point particles with a variable mass $m(\tau)$. Strings with a *dynamical* (variable) tension $T(\sigma_0, \sigma_1)$ along the world-sheet parametrized by the coordinates σ_0, σ_1 have been extensively studied by Guendelman [20] over the years. In the modified measure formulation of strings/branes, the tension appear as an additional dynamical degree of freedom. There are many important physical consequences of these variable tension models of strings and branes. Recently, Guendelman has reviewed how the model avoids the Swampland constraints making treatments for Dark energy and inflation more realistic and how strings with a different tension appear as Dark Matter to us. We refer to [20] and the many references therein for specific details.

One could also have spacetime-dependent masses $m(x^\mu), \mathcal{M}(x^\mu)$. How is this possible ? A careful thought reveals that this could occur as a result of the *back reaction* of spacetime on matter. As the particle probes the spacetime points along its (non-uniformly) accelerated trajectory, space-time back reacts on the particle affecting its mass. Special relativity led to the unification of space and time. Non-inertial relativity seems to indicate a space-time-matter "unification". To be more precise, matter curves spacetime, and in turn, spacetime back reacts on matter curving momentum space. In the next section we will study the generalized field equations in phase space and how curvature in spacetime and momentum space result from the presence of matter.

The concept of a position dependent mass has appeared in the literature before but based on very different physical principles. Most recently, a noncommutative Hamilton-Jacobi equation based on Moyal-type noncommutative spacetimes was studied and it was found that all noncommutative effects could be absorbed into an effective, position-dependent mass function $M(x)$, appearing in an otherwise standard relativistic dispersion relation. See [21] and references therein.

3 Generalized Gravity in Curved Phase Space

Most of the contents of this section are based on our prior work [10] and the results of [12]. Additions are made at the end where we include the matter actions of point particles and strings in phase space.

Let us begin with the Sasaki-Finsler metric of the cotangent space of a d-dim manifold T^*M_d, and which is given by the following metric in *block diagonal* form [8], [9]

$$(d\omega)^2 = g_{ij}(x^k, p_a)\ dx^i d\ x^j + h^{ab}(x^k, p_c)\ \delta p_a\ \delta p_b = \\ g_{ij}(x^k, p_a)\ dx^i d\ x^j + h_{ab}(x^k, p_c)\ \delta p^a\ \delta p^b \quad (3.1)$$

The range of the base manifold indices is $i, j, k = 0, 1, 2, 3, ...d-1$; whereas the range of the fiber indices is $a, b, c = 0, 1, 2, 3, ...d-1$. The standard coordinate basis frame has been replaced by the following anholonomic non-coordinate basis frame comprised of the following elongated and ordinary derivatives, respectively,

$$\delta_i = \delta/\delta x^i = \partial_{x^i} + N_{ia}\ \partial^a = \partial_{x^i} + N_{ia}\ \partial_{p_a};\ \ \partial^a \equiv \partial_{p_a} = \frac{\partial}{\partial p_a} \quad (3.2)$$

The signature is chosen to be Lorentzian $(-, +, +, +, \cdots, +)$ for both g_{ij} and h_{ab}. It is important to emphasize that one does *not* have a theory with two times because the energy coordinate is not time. One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i,\ \ \delta p_a = dp_a - N_{ja}\ dx^j,\ \ \delta p^a = dp^a - N^a_j\ dx^j \quad (3.3)$$

where the N-coefficients define a nonlinear connection, N–connection structure.

An N-linear connection D on T^*M allows to construct covariant derivatives which are compatible with the structure induced by the nonlinear connection and that preserve the horizontal-vertical split of the cotangent bundle. Thus, an N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form

$$D_{\delta_j}(\delta_i) \;=\; H^k_{ij}\;\delta_k;\;\;D_{\delta_j}(\partial^a)\;=\;-\;H^a_{bj}\;\partial^b; \tag{3.4a}$$

$$D_{\partial^a}(\delta_i)\;=\;C^{ka}_i\;\delta_k;\;\;D_{\partial^a}(\partial^b)\;=\;-\;C^{ba}_c\;\partial^c \tag{3.4b}$$

where $H^k_{ij}(x,p), H^a_{bj}(x,p), C^{ka}_i(x,p), C^{ba}_c(x,p)$ are the connection coefficients. Our notation for the derivatives is

$$\partial^a = \partial/\partial p_a,\;\;\partial_i\;=\;\partial_{x^i},\;\;\delta_i\;=\;\delta/\delta x^i\;=\;\partial_{x^i}\;+\;N_{ia}\;\partial^a \tag{3.4c}$$

The N-connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some anholonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = W^\gamma_{\alpha\beta}\delta_\gamma$. The only nontrivial (nonvanishing) nonholonomy coefficients are

$$W_{ija}\;=\;R_{ija};\;\;W^a_{\;\;jb}\;=\;\partial^a N_{jb}\;=\;-\;W_j{}^a{}_b \tag{3.5a}$$

and

$$R_{ija}\;=\;\delta_j N_{ia}\;-\;\delta_i N_{ja} \tag{3.5b}$$

is the nonlinear connection curvature (N-curvature).

Imposing a zero nonmetricity condition of $g_{ij}(x,p), h^{ab}(x,p)$ along the horizontal and vertical directions, respectively, gives

$$D_i g_{jk}\;=\;\delta_i\;g_{gk}\;-H^l_{ij}\;g_{lk}\;-H^l_{ik}\;g_{jl}\;=\;0, \tag{3.6a}$$

$$D^a h^{bc}\;=\;\partial^a\;h^{bc}\;+\;C^{ab}_d\;h^{dc}\;+\;C^{ac}_d\;h^{bd}\;=\;0 \tag{3.6b}$$

Performig a cyclic permutation of the indices in eqs-(3.6a,3.6b), followed by linear combination of the equations obtained yields the irreducible (horizontal, vertical) h-v-components for the connection coefficients

$$H^i_{\;\;jk}\;=\;\frac{1}{2}g^{in}\left(\delta_k g_{nj}+\delta_j g_{nk}-\delta_n g_{jk}\right) \tag{3.7}$$

$$C^{ab}_c\;=\;-\;\frac{1}{2}\;h_{cd}\;\left(\;\partial^b h^{ad}\;+\;\partial^a h^{bd}\;-\;\partial^d h^{ab}\;\right) \tag{3.8}$$

The additional conditions $D_i h^{ab} = 0, D^a g_{ij} = 0$, yield the *mixed* components of the connection coefficients

$$H^b_{ja}\;=\;\partial^b N_{ja}\;+\;\frac{1}{2}\;h^{bc}\;\left(\;\delta_j h_{ac}\;-\;h_{ad}\;\partial^d N_{jc}\;-\;h_{cd}\;\partial^d N_{ja}\;\right) \tag{3.9}$$

and

$$C^{ja}_i\;=\;\frac{1}{2}\;g^{jk}\;\partial^a g_{ik} \tag{3.10}$$

For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i \ = \ \frac{1}{2} T^i_{jk} \ dx^j \wedge dx^k \ + \ C^{ia}_j \ dx^j \wedge \delta p_a \tag{3.11a}$$

$$\Omega_a \ = \ \frac{1}{2} R_{jka} \ dx^j \wedge dx^k \ + \ P^b_{aj} \ dx^j \wedge \delta p_b \ + \ \frac{1}{2} S^{bc}_a \ \delta p_b \wedge \delta p_c \tag{3.11b}$$

and the curvature 2-forms are

$$\Omega^i_j \ = \ \frac{1}{2} R^i_{jkm} \ dx^k \wedge dx^m \ + \ P^{ia}_{jk} \ dx^k \wedge \delta p_a \ + \ \frac{1}{2} S^{iab}_j \ \delta p_a \wedge \delta p_b \tag{3.12}$$

$$\Omega^a_b \ = \ \frac{1}{2} R^a_{bkm} \ dx^k \wedge dx^m \ + \ P^{ac}_{bk} \ dx^k \wedge \delta p_c \ + \ \frac{1}{2} S^{acd}_b \ \delta p_c \wedge \delta p_d \tag{3.13}$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i, \ \partial^a = \partial/\partial p_a$ is given by $dx^i, \ \delta p_a = dp_a - N_{ja} dx^j$.

The distinguished torsion tensors are given by

$$T^i_{jk} \ = \ H^i_{jk} \ - \ H^i_{kj}; \ \ S^{ab}_c \ = \ C^{ab}_c \ - \ C^{ba}_c; \ \ T^{ia}_j \ = \ C^{ia}_j \ = \ - \ T^{ia}{}_j$$

$$P_b{}^a{}_j \ = \ H^a_{bj} \ - \ \partial^a N_{jb}, \ \ P_b{}^a{}_j \ = \ - \ P_{bj}{}^a$$

$$R_{ija} \ = \ \frac{\delta N_{ja}}{\delta x^i} \ - \ \frac{\delta N_{ia}}{\delta x^j} \tag{3.14}$$

And the distinguished tensors of the curvature are

$$R^i_{kjh} \ = \ \delta_h H^i_{kj} \ - \ \delta_j H^i_{kh} \ + \ H^l_{kj} \ H^i_{lh} \ - \ H^l_{kh} \ H^i_{lj} \ - \ C^{ia}_k \ R_{jha} \tag{3.15}$$

$$P^{ab}_{cj} \ = \ \partial^a \ H^b_{cj} \ + \ C^{ad}_c \ P^b_{dj} \ - \ \left(\delta_j \ C^{ab}_c \ + \ H^b_{dj} \ C^{da}_c \ + \ H^a_{dj} \ C^{bd}_c \ - \ H^d_{cj} \ C^{ab}_d\right) \tag{3.16}$$

$$P^{ak}_{ij} \ = \ \partial^a \ H^k_{ij} \ + \ C^{al}_i \ T^k_{lj} \ - \ \left(\delta_j \ C^{ak}_i \ + \ H^a_{bj} \ C^{bk}_i \ + \ H^k_{lj} \ C^{al}_i \ - \ H^l_{ij} \ C^{ak}_l\right) \tag{3.17}$$

$$S^{abc}_d \ = \ \partial^c \ C^{ab}_d \ - \ \partial^b \ C^{ac}_d \ + \ C^{eb}_d \ C^{ac}_e \ - \ C^{ec}_d \ C^{ab}_e; \tag{3.18}$$

$$S^{ibc}_j \ = \ \partial^c C^{bi}_j \ - \ \partial^b C^{ci}_j \ + \ C^{bh}_j \ C^{ci}_h \ - \ C^{ch}_j \ C^{bi}_h \tag{3.19}$$

$$R^a_{bjk} \ = \ \delta_k H^a_{bj} \ - \ \delta_j H^a_{bk} \ + \ H^c_{bj} \ H^a_{ck} \ - \ H^c_{bk} \ H^a_{cj} \ - \ C^{ca}_b \ R_{jkc} \tag{3.20}$$

A scalar-gravity model was duly studied in [10] and exact nontrivial analytical solutions for the metric and non- linear connection were found, in the very simple case for constant scalar field configurations, that obeyed the generalized gravitational field equations, in addition to satisfying the zero torsion conditions for all of the torsion components. The curved base spacetime manifold and internal momentum space both turned out to be (Anti) de Sitter type. The most salient feature was that the solutions captured both the very early inflationary, and very-late-time de Sitter phases of the Universe. In this work we will study instead a gravity-matter model.

Adopting the units where $\hbar = c = G = 1$ such that the Planck mass and length squared are respectively $M_P^2 = 1, L_P^2 = 1$; and given $g^{AB} \equiv g^{ij}, h^{ab}$, one may construct the simplest gravity action of the form[3]

$$\mathcal{S}_G \ = \ \frac{1}{2\kappa} \int d^4x \ d^4p \ \sqrt{|\det g_{AB}|} \ \left(\ g^{ij} \ R_{(ij)} \ + \ h_{ab} \ S^{(ab)} \ \right) \tag{3.21}$$

The determinant factorizes $det(g_{AB}) = det(g_{ij})det(h_{ab})$ in an anhololomic basis adapted to the nonlinear connection (the metric assumes the block diagonal form (1)). κ is the gravitational coupling constant. If the phase space action action (3.21) is dimensionless, after reintroducing the physical constants that were set to unity, gives $\kappa = 8\pi \rightarrow (8\pi G/c^4)(M_P c)^4$.

The action of a point particle in phase space is

$$S_{matter} \ = \ -\mathcal{M} \int d\omega \tag{3.22}$$

and it can also be written in terms of an auxiliary $E = E(\omega)$ einbein field (whose physical units are those of $mass^{-1}$) as follows

$$S_{matter} \ = \ -\frac{1}{2} \int d\omega \ \left(E^{-1} (\frac{dZ^I}{d\omega})^2 \ + \ E\mathcal{M}^2 \right) \tag{3.23}$$

Eliminating $E(\omega)$ via its algebraic equation of motion $\mathcal{M}^2 - E^{-2}(\frac{dZ^I}{d\omega})^2 = 0$, and inserting its value back into the action one recovers $S_{matter} = -\mathcal{M} \int d\omega$. One could interpret the first term $E^{-1}(\frac{dZ^I}{d\omega})^2$ as a "kinetic energy" and the second term $E\mathcal{M}^2$ as a "cosmological constant".

Inserting delta functions and integrating allows to rewrite the matter action (3.22) as

$$\mathcal{S}_{matter} \ = \ -\mathcal{M} \int d^8Z \ \sqrt{|g|} \ \int d\omega \ \frac{\delta^8(Z^A - Z^A(\omega))}{\sqrt{|g|}} \tag{3.24}$$

where $|g|$ denotes the absolute value of the determinant $det(g_{AB}) = det(g_{ij})det(h_{ab})$. The indices $A, B = 1, 2, \ldots, 8$ span all the coordinates of the 8-dim phase space. The Born interval in an 8-dim *curved* phase space (cotangent space) is given by

[3] $d^4x \ d^4p = dx^0 \wedge dx^1 \wedge \cdots \wedge \delta p_0 \wedge \delta p_1 \wedge \cdots = dx^0 \wedge dx^1 \wedge \cdots \wedge dp_0 \wedge dp_1 \wedge \cdots$

$$(d\omega)^2 \;=\; g_{AB}\; dZ^A\; dZ^B \;=$$

$$g_{ij}(x,p)\; dx^i\; dx^j \;+\; h^{ab}(x,p)\; (dp_a \;+\; N_{ai}(x,p)\; dx^i)\; (dp_b + N_{bj}(x,p)\; dx^j) \quad (3.25)$$

The phase space metric g_{AB} components are comprised of g_{ij}, h^{ab}, N_{ai}. The matter stress energy tensor associated with the matter action (3.24) is defined as

$$T_{AB} \;=\; -\frac{2}{\sqrt{|g|}}\; \frac{\delta\sqrt{|g|}\mathcal{L}_{matter}}{\delta g^{AB}} \;=\; \mathcal{M} \int d\omega\; \frac{dZ_A}{d\omega}\; \frac{dZ_B}{d\omega}\; \frac{\delta^8(Z^A - Z^A(\omega))}{\sqrt{|g|}} \quad (3.26)$$

Given the net gravity-matter action $\mathcal{S} \equiv \mathcal{S}_G + \mathcal{S}_{matter}$, after a very laborious procedure, the authors [12] have shown that variations of the gravity-matter action

$$\frac{\delta \mathcal{S}}{\delta g^{ij}} \;=\; 0, \quad \frac{\delta \mathcal{S}}{\delta h^{ab}} \;=\; 0, \quad \frac{\delta \mathcal{S}}{\delta N_{ai}} \;=\; 0, \quad (3.27)$$

with respect to g^{ij}, h^{ab}, N_{ai}, respectively, leads to the following field equations

$$R_{(ij)}(x,p) \;-\frac{1}{2}\; g_{ij}(x,p)\; (R+S) \;+\; R_{k(ia}\; C^{ka}_{j)} \;=\; 8\pi\; T_{ij} \quad (3.28)$$

$$S_{(ab)}(x,p) \;-\frac{1}{2}\; h_{ab}(x,p)\; (R+S) \;=\; 8\pi\; T_{ab} \quad (3.29)$$

$$g^{ik}\; \partial^a H^j_{kj} \;-\; g^{kl}\; \partial^a H^i_{kl} \;=\; 8\pi\; T^{ia} \quad (3.30)$$

where the curvature contractions are given by

$$R_{kh} \;=\; R^i_{kjh}\; \delta^j_i, \quad R \;=\; g^{kh}\; R_{(kh)} \quad S^{ac} \;=\; S^{abc}_d\; \delta^d_b, \quad S \;=\; h_{ac}\; S^{(ac)} \quad (3.31)$$

after symmetrizing the indices accordingly and denoted by ().

Eqs-(3.28,3.29,3.30) are the generalized field equations in curved phase space where the stress energy tensor components are provided by eq-(3.26) by letting the indices run from $1, 2, \ldots, 8$. The first four indices span the T_{ij} components of the 4-dim spacetime. The last four span the T_{ab} components of the 4-dim momentum space, and the mixed indices span the mixed T_{ia} components. One finds that the source of curvature in spacetime and momentum space is due to the presence of matter in phase space, and it takes into account the back-reaction of matter on both spacetime and momentum space (back reaction of matter on the geometry).

The generalization of the point particle matter action in phase space to the (cosmic) string case is

$$S_{string} \;=\; -T \int d^8Z\; \sqrt{|g|} \int d\sigma_0 \sigma_1\; \frac{\delta^8(Z^A - Z^A(\sigma_0,\sigma_1))}{\sqrt{|g|}}\; \sqrt{|det\; g_{AB}\partial_\alpha Z^A \partial_\beta Z^B|} \quad (3.32)$$

where $\partial_\alpha, \partial_\beta$ both denote derivatives with respect to the world sheet coordinates σ_0, σ_1. In this case the string becomes the source of phase space curvature, after evaluating the stress energy tensor components associated with the string action (3.32). It can be generalized to p-branes as well [25]. To find solutions to the generalized field equations in curved phase space given by eqs-(3.28,3.29,3.30) is a very difficult task compared to what was found in [10] for constant scalar field configurations.

4 Concluding Remarks

Starting with a brief review of Non-inertial Relativity Theory, and how it redefines the notion of mass, the phase space particle trajectories in $D = 2+2$ were revisited by emphasizing the key difference between a truly $U(1,1)$-invariant mass $\mathcal{M}$ and the Lorentz-invariant mass m. It was shown that $\mathcal{M}$ is the "zero-force" mass analog of the rest mass in special relativity. We proceeded by showing in eq-(2.14) how the special relativistic expression of $E = m(1-v^2)^{-1/2}$ ($c = 1$) can be rewritten in terms of $\mathcal{M}$ and F in the non-inertial relativistic case. Subsequent modifications to Newton's law of motion in the Galilean limit $v << 1$ were found in eq-(2.24), and that differ from Milgrom's MOND law [17] due to the different expressions for the interpolating functions. To be more precise, the Galilean limit of non-inertial relativity leads to non-Milgromian MOND.

By recurring to the tools of Finsler geometry, the generalized gravitational field equations in curved phase space due to the presence of matter sources were provided in eqs-(3.28,3.29,3.30). As a result, both spacetime and momentum space are curved. We should add that recently the authors [23] investigated how quantum features of spacetime, in particular the curvature of momentum space, can back react on classical gravity in a tractable semiclassical (2+1)-dimensional setting with a negative cosmological constant. As a result of this back reaction, a mass-dependent geodesic motion and a mild violation of the equivalence principle was found. For other physical applications of curved momentum space see [26], [27].

The essence behind why curved momentum space should play an important role in quantum gravity is because noncommutative momentum coordinates $[\hat{p}_\mu, \hat{p}_\nu] \neq 0$ have a correlation to the noncommutativity of the spacetime covariant derivatives $[\nabla_\mu, \nabla_\nu] \neq 0$ due to the curvature (and torsion) in spacetime. And vice versa, noncommutative spacetime coordinates $[\hat{x}_\mu, \hat{x}_\nu] \neq 0$ have a correlation to the noncommutativity of the momentum space covariant derivatives $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] \neq 0$ due to the curvature (and torsion) in momentum space. And finally, as stated in [11], the non-vanishing commutator $[\hat{x}_\mu, \hat{p}_\nu] \neq 0$ has a correlation to the base manifold metric components in phase space given by $g_{\mu\nu}(x,p) + h^{ab}(x,p) N_{a\mu}(x,p) N_{b\nu}(x,p)$ involving $g_{\mu\nu}(x,p); h^{ab}(x,p), N_{a\mu}(x,p)$. These ideas warrant further investigation.

Perhaps more importantly is the interplay of non-inertial relativity with thermal quantum field theory (TQFT). The Fulling-Davies-Unruh effect states that a uniformly accelerating observer experiences the vacuum state of a quantum field in Minkowski spacetime as a mixed state in thermodynamic equilibrium. Such mixed state is comprised of a thermal bath (warm gas) of Rindler particles whose temperature is proportional to the proper acceleration. Given the expression of Unruh's temperature in terms of the proper acceleration $T = \frac{g}{2\pi}$, in natural units $\hbar = c = k_B = 1$, after substituting $g = 2\pi T$ in eqs-(2.15,2.16) it leads to a T-dependent mass $m = m(T)$, which resembles the renormalization process of the physical parameters in QFT in terms of the energy scale. Hence, given $m(g = 0) = m(T = 0) = \mathcal{M}$, by "switching" on the acceleration, the renormalized mass will flow from $\mathcal{M} \to m(g) = m(T)$.

Currently we are exploring whether or not the weak (strong) equivalence principle is violated in non-inertial relativity, and the plausible physical relevance of having variable masses $m(\tau)$ to the dark matter, dark energy problem. To conclude, the fusion of special relativity with quantum mechanics led to quantum field theory (QFT). The fusion of non-inertial relativity with quantum mechanics should lead to a novel formulation of thermal quantum field theory (TQFT), and in turn, should cast some light into the quantization of gravity and black hole thermodynamics.

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