ON THE CAUCHY PROBLEM FOR THE HARMONIC OSCILLATOR EQUATION

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The article proves that the solution to the Cauchy problem for the harmonic oscillator equation is not unique,and can have the most unusual properties.

Key words: Cauchy problem; harmonic oscillator; non-uniqueness of solution; Schrödinger equation.

Non-stationary equation of a harmonic oscillator

$$ih\frac{\partial u}{\partial t} = -h^2\Delta u + |\mathbf{x}|^2 u, \quad \mathbf{x} = (x_1, x_2, ..., x_n),$$

is one of the few equations of quantum mechanics for which exact solution is known ([1],[2],[3]).

Obviously, any solution of his equation, defined for all t and for all x, is a solution of the corresponding Cauchy problem.

For simplicity of calculations, we restrict ourselves to the case of a one-dimensional oscillator.

Assuming $\lambda = h^{-1}$ we will look for a solution to the harmonic oscillator equation

$$ih\frac{\partial u}{\partial t} = -h^2\frac{\partial^2 u}{\partial x^2} + x^2 u, \tag{1}$$

in the form a wave packet

$$u(t,x) = \int\limits_{p_1(t)}^{p_2(t)} e^{i\lambda(px^2/2)}\varphi(t,p)dp. \tag{2}$$

Substituting (2) into (1), we obtain

$$-ihe^{i\lambda(p_2(t)x^2/2))}\varphi(t,p_2(t))p_2'(t) + ihe^{i\lambda(p_1(t)x^2/2)}\varphi(t,p_1(t))p_1'(t)+$$

$$+\int\limits_{p_1(t)}^{p_2(t)} e^{i\lambda(px^2/2)}[(x^2(p^2+1) - ihp)\varphi(t,p) - ih\varphi_t(t,p)]dp = 0. \tag{3}$$

Integrating by parts, we transform the integral included in (3)

$$\int\limits_{p_1(t)}^{p_2(t)} e^{i\lambda(px^2/2)}x^2(p^2+1)\varphi(t,p)dp = -2ihe^{i\lambda(px^2/2)}(p^2+1)\varphi(t,p)|_{p_1(t)}^{p_2(t)} +$$

$$+\int\limits_{p_1(t)}^{p_2(t)} e^{i\lambda(px^2/2}2ih(2p\varphi(t,p)+(1+p^2)\varphi_p(t,p))dp. \tag{4}$$

Substituting (4) into (3), we eliminate the terms outside the integral,requiring that

$p_1'(t)=-2(p_1^2(t)+1),\ p_2'(t)=-2(p_2^2(t)+1),$

then,

$p_i(t)=\operatorname{tg}2(t_i-t), i=1,2,$

where t_1,t_2 are arbitrary constants. As a result, we obtain the equality

$$\int\limits_{p_1(t)}^{p_2(t)} e^{i\lambda(px^2/2)}(-ih\varphi_t(t,p)+2ih(p^2+1)\varphi_p(t,p)+3ihp\varphi(t,p))dp=0. \tag{5}$$

We will satisfy this equality by assuming

$$\varphi_t-2(p^2+1)\varphi_p-3p\varphi=0. \tag{6}$$

The solution to equation (6) will be

$$\varphi(t,p)=\Phi(2t+\operatorname{arctg}p)(p^2+1)^{-3/4}, \tag{7}$$

where $\Phi(\xi)$ - is an arbitrary continuously differentiable function defined for any ξ.

Let $t_1=n\pi/2$ and $t_2=(n+1/4)\pi/2$, where n is some fixed integer. In this case, both integration limits cannot be simultaneously infinity.

The lower limit of integration becomes infinite if $n\pi-2t=k\pi+\pi/2$, where k- is an arbitrary integer, hence $2t=m\pi-\pi/2$, where m - is an arbitrary integer. Since $p\in(p_1(t),p_2(t))$, the infinite lower limit corresponds to $\operatorname{arctg}p=\pm\pi/2$ and hence $\xi=2t+\operatorname{arctg}p=l\pi$, where l is an arbitrary integer.

If we require the condition $\Phi(\xi)=0$, if $\xi\in(l\pi-\varepsilon,l\pi+\varepsilon)$, then for t close to $l\pi+\pi/2$ and for large $|p|$, the integrand is zero. This eliminates the possibility of improper integrals with an infinite lower limit.

Similarly, cases of an improper integral with infinite upper limit are excluded if $\Phi(\xi)=0$ for $\xi\in(l\pi+\pi/4-\varepsilon,l\pi+\pi/4+\varepsilon)$.

We further require that $\Phi(\xi)=0$ if $\xi<4n\pi$ or $\xi>(4n+2)\pi$, i.e., if $2t+\operatorname{arctg}p<4n\pi$ or $2t+\operatorname{arctg}p>(4n+2)\pi$ and hence the integrand, and with it $u(t,x)$ can be nonzero only for $t\in(2n\pi-\pi/4,(2n+1)\pi+\pi/4)$.

Note that most of the previous conditions follow from this condition.

Thus, the following is proved

Theorem. For any integer n, the solution to equation (1) is the function

$$u_n(t,x) = \int\limits_{p_1(t)}^{p_2(t)} e^{i\lambda(x^2 p/2)} \Phi(2t + \operatorname{arctg} p)(p^2+1)^{-3/4} dp, \tag{8}$$

which is defined for any t and x, where

$p_1(t) = \operatorname{tg}(n\pi - 2t)$, $p_2(t) = \operatorname{tg}(n\pi + \pi/4 - 2t)$,

and $\Phi(\xi)$ is an arbitrary continuously differentiable function satisfying the following conditions:

1) $\Phi(\xi) = 0$ if $\xi \in (l\pi - \varepsilon, l\pi + \varepsilon)$ or $\xi \in (l\pi + \pi/4 - \varepsilon, l\pi + \pi/4 + \varepsilon)$, where l is an arbitrary integer,

2) $\Phi(\xi) = 0$, if $\xi < 4n\pi$ or $\xi > (4n+2)\pi$.

This solution can be nonzero only for $t \in (2n\pi - \pi/4, (2n+1)\pi + \pi/4)$.

Corollary 1. The solution to the Cauchy problem for the equation (1) is not unique. Indeed, if $u(t,x)$ is a solution to some Cauchy problem with the initial condition $u(0,x) = u_0(x)$, then for arbitrary $n > 0$, $u(t,x) + u_n(t,x)$ - is a different solution satisfying the same initial condition.

Corollary 2. Suppose $U(t,x) = \sum\limits_{n=1}^{\infty} u_n(t,x)$, since for any t this sum has no more than one nonzero term and the question of the convergence of this series does not arise, we obtain a solution that satisfies the zero initial condition, equal to zero on some time intervals and nonzero on others.

Here, if we assume that the solution describes the behavior of some physical object, we can say that it appears, disappears, reappears and so on.

Corollary 3. If $u(t,x)$ - is a solution of equation (1), then $u(t,x) + U(t,x)$, where $U(t,x)$ is the solution from the previous corollary, is a solution of equation (1)that corresponds to the same initial condition as $u(t,x)$, coinciding with the solution $u(t,x)$ on some time intervals and differing from it on other time interval.

A natural question arises: what physical meaning can an equation have whose solutions exhibit such strange properties? More precisely, how can use in theoretical constructions an equation for which different solutions can be chosen at will in the same initial situation? Are these unusual properties of the solution explained by imperfections in the adopted model, or to they reflect some properties of physical reality that this equation is supposed to describe?

If the latter is true, then the behavior of physical objects associated with this equation is determined not by their state at the initial moment of time, but by certain hidden parameters (in particular by the function $\Phi(\xi)$).

One can also assume that the solution to the Cauchy problem of the Schr$\ddot{o}$dinger equation is not unique for certain cases of other potentials.

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