

Structural Link Between Sophie Germain Numbers and Triangular Numbers

Jean-Yves BOULAY

Jean-Yves BOULAY independent researcher – FRANCE – jean-yvesboulay@orange.fr ORCID: 0000-0001-5636-2375
<https://www.researchgate.net/profile/Jean-Yves-Boulay>

Abstract: This work highlights a simple yet remarkably overlooked connection between the arithmetic structure underlying Sophie Germain numbers and the classical theory of triangular numbers. Although these two notions arise in distinct contexts, one in the study of prime constellations, the other in figurate number theory, they share a common algebraic backbone that becomes explicit once one examines the product $x(2x + 1)$ arising from the transformation mapping x to $(2x + 1)$.

Keywords: Sophie Germain primes, Safe primes, Triangular numbers, Figurate numbers, Integer sequences, Quadratic parametrization, Number genetics, Numerical stability.

1 Introduction

In connections to previous paper* (Boulay, 2025) to further explore the mechanics of fusion between value and rank, we examine here the product of the entity x and its transformation which we call* " $G(x)$ ". This operation is not merely a simple algebraic multiplication; it reveals a structural bridge between Germain's descent and the geometry of figurate numbers.

* See Appendix for explicit details.

Indeed, the product $xG(x)$ allows the concept of "safe prime" within a precise arithmetic reality. We observe that this product systematically generates triangular numbers, highlighting an inherent algebraic stability within the number series.

2 The Product of x and $G(x)$

The following tables (Figure 1 and Figure 2) illustrate this convergence depending on whether x belongs to the set of natural integers or negative integers.

$x \in \mathbb{N}_0 : x(2x + 1) = T_{2x}$											
x	0	1	2	3	4	5	6	7	8	9	10
$G(x) = 2x + 1$	1	3	5	7	9	11	13	15	17	19	21
$xG(x) = x(2x + 1)$	0	3	10	21	36	55	78	105	136	171	210
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
T_{2x}	T_0	T_2	T_4	T_6	T_8	T_{10}	T_{12}	T_{14}	T_{16}	T_{18}	T_{20}

Figure 1: Correlation between the product $xG(x)$ and even triangular numbers T_{2x} for $x \in \mathbb{N}_0$.

It is observed Figure 1 that for any non-negative x ($x \geq 0$), the product of x by its Germain transformation $G(x) = 2x + 1$ corresponds exactly to the triangular number of index $2x$. This relationship, $x(2x + 1) = T_{2x}$, demonstrates that the Germain structure is intrinsically linked to triangular geometric progression, acting as the "keystone" of numerical stability.

$x \in \mathbb{Z}_{(<0)} : x(2x + 1) = T_{(-2x-1)}$										
x	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$G(x) = 2x + 1$	-19	-17	-15	-13	-11	-9	-7	-5	-3	-1
$xG(x) = x(2x + 1)$	190	153	120	91	66	45	28	15	6	1
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$T_{(-2x-1)}$	T_{19}	T_{17}	T_{15}	T_{13}	T_{11}	T_9	T_7	T_5	T_3	T_1

Figure 2: Correlation between the product $xG(x)$ and odd triangular numbers $T_{(-2x-1)}$ for $x \in \mathbb{Z}_{(<0)}$.

In the domain of negative integers ($x \leq -1$), Figure 2, the symmetry of the model is maintained. The product $x(2x + 1)$ here aligns with the series of triangular numbers with odd indices: $x(2x + 1) = T_{(-2x-1)}$. This persistence of the triangular form, despite the change in sign, confirms the universality of the Germain transformation as a structural pivot for the model.

3 Background: Sophie Germain's Transformation

A *Sophie Germain prime* is a prime number p such that $2p + 1$ is also prime. The map $p \mapsto 2p + 1$ is therefore the defining transformation of the concept. Independently of primality, this transformation is simply an affine function with slope 2 and intercept 1.

The associated product $p(2p + 1)$ appears naturally in several analytic and combinatorial treatments of Sophie Germain primes, but its intrinsic algebraic structure is rarely emphasized.

4 The Quadratic Identity

For any integer $x \in \mathbb{Z}$, consider the expression $x(2x + 1) = 2x^2 + x$. This quadratic form is precisely the general expression of a triangular number. Indeed, recalling that $T_n = \frac{n(n+1)}{2}$, one verifies immediately that $2x^2 + x = T_{2x} = T_{(2x-1)}$

Thus we obtain the following identity.

5 Theorem

For every integer $x \in \mathbb{Z}$, the product $x(2x + 1)$ is a triangular number.

This identity is elementary, but its conceptual significance is substantial: the affine map $x \mapsto 2x + 1$ that defines Sophie Germain primes is exactly the linear component of the quadratic form generating triangular numbers.

6 Consequence for Sophie Germain Numbers

As an immediate corollary, every Sophie Germain prime p determines a triangular number via $T = (2p + 1)p$.

No primality assumption is required for the identity itself; the structure is purely algebraic. What is noteworthy is that the *Sophie Germain transformation* sits naturally inside the *triangular number structure*.

This observation provides a conceptual bridge between two classical objects of number theory:

- the **linear** transformation $x \mapsto 2x + 1$, central to the definition of Sophie Germain primes,
- the **quadratic** transformation $x \mapsto (2x^2) + x$, which parametrizes triangular numbers.

The former is, in a precise sense, the “first layer” of the latter.

7 General theorem

This theorem establishes a direct and elegant bridge between the set of relative integers (\mathbb{Z}) and the sequence of triangular numbers.

7.1 A Quadratic Parametrization of Triangular Numbers

The theorem demonstrates that the expression $2n^2 + n$ systematically generates a triangular number, whether n is positive, negative, or zero. Unlike classical methods, this approach proves that all triangular numbers without exception can be obtained using this single formula. For each value of n , a precise rank (k) is found that corresponds to the number's position in the pyramid of triangular numbers (for example, $n = 2$ gives the 4th triangular number, which is 10).

We prove that every triangular number admits a quadratic parametrization over the integers. More precisely, the set of triangular numbers coincides with the image of the map $n \mapsto 2n^2 + n$ for $n \in \mathbb{Z}$. This provides a symmetric description of triangular numbers indexed by all integers.

7.2 Theorem

For every integer $n \in \mathbb{Z}$, the quadratic expression $2n^2 + n$ is a triangular number.

Proof. Recall that the triangular numbers are given by $T_k = k(k + 1)/2$ for $k \in \mathbb{N}$.

We seek k such that $2n^2 + n = T_k$.

Multiplying by 2 yields $4n^2 + 2n = k^2 + k$.

Viewed as a quadratic equation in k , this gives $k^2 + k - (4n^2 + 2n) = 0$, whose discriminant is $\Delta = 1 + 16n^2 + 8n = (4n + 1)^2$.

Thus $k = (-1 \pm (4n + 1)) / 2$, leading to the two integer solutions $k = 2n$ and $k = -2n - 1$.

For $n \geq 0$, the index $k = 2n$ is non-negative; for $n \leq -1$, the index $k = -2n - 1$ is positive.

In both cases, $k \in \mathbb{N}$, and therefore $2n^2 + n = T_k$ is triangular.

7.3 Proposition

The set of triangular numbers is exactly the image of \mathbb{Z} under the mapping $n \mapsto 2n^2 + n$

7.4 Example

$$n = 2 \rightarrow 2 \times 2^2 + 2 = 10 \rightarrow \text{triangular number } T_4$$

$$n = -2 \rightarrow 2 \times (-2)^2 + (-2) = 6 \rightarrow \text{triangular number } T_3$$

$$n = 3 \rightarrow 2 \times 3^2 + 3 = 21 \rightarrow \text{triangular number } T_6$$

$$n = -3 \rightarrow 2 \times (-3)^2 + (-3) = 15 \rightarrow \text{triangular number } T_5$$

$$n = 4 \rightarrow 2 \times 4^2 + 4 = 36 \rightarrow \text{triangular number } T_8$$

$$n = -4 \rightarrow 2 \times (-4)^2 + (-4) = 28 \rightarrow \text{triangular number } T_7$$

8 Discussion and Conclusion

The results presented in this work reveal that the arithmetic structure underlying Sophie Germain's transformation is far more universal than traditionally acknowledged. By examining the product $x(2x+1)$ not merely as an auxiliary expression but as a structural object in its own right, we uncover a direct and symmetric parametrization of the triangular numbers across the entire set of integers. This observation places the Germain transformation at the heart of a broader figurate geometry, where linear and quadratic forms interact in a remarkably stable manner.

A central outcome of this study is the demonstration that the affine map $x \mapsto 2x + 1$, which defines Sophie Germain primes, is precisely the linear component of the quadratic form generating triangular numbers.

The identity $x(2x+1) = T_{2x} = T_{-2x-1}$ shows that every integer—positive, negative, or zero—naturally produces a triangular number through this mechanism. This dual indexing, one even and one odd, highlights a deep symmetry that is usually absent from classical presentations of figurate numbers, which restrict themselves to non-negative indices.

From a conceptual standpoint, this structural bridge enriches the interpretation of Sophie Germain numbers. While primality conditions remain external to the identity itself, the fact that every Sophie Germain prime p automatically generates a triangular number T_{2p} or T_{-2p-1} suggests that the Germain descent is embedded within a stable geometric framework. The transformation is not an isolated arithmetic curiosity but a natural component of a larger generative system.

This perspective aligns with the broader program of “number genetics” developed in earlier work (Boulay, 2025). The decomposition of a whole number into its intrinsic emanations—value $V(x)$, rank $R(x)$, and descent $G(x)$ —finds here a concrete algebraic manifestation. The triangular identity confirms that the fusion of these emanations produces not only the classical Germain structure but also a canonical figurate form. In this sense, triangular numbers appear as the geometric trace of the reproductive mechanism underlying the enriched arithmetic of whole numbers.

More broadly, the results invite a reconsideration of how linear and quadratic transformations interact in number theory. The fact that a single quadratic expression, $(2n^2 + n)$, parametrizes the entire sequence of triangular numbers over \mathbb{Z} suggests that figurate structures may admit more symmetric and unified descriptions than traditionally assumed. This opens the door to further investigations into other figurate families—square, pentagonal, or hexagonal numbers—and their potential connections with affine transformations of arithmetic origin.

In conclusion, the structural link established in this paper demonstrates that the Germain transformation is not merely a tool for defining a special class of primes but a fundamental arithmetic mechanism with geometric consequences. By revealing its intrinsic connection to triangular numbers, we provide a unified framework that bridges prime constellations, quadratic parametrizations, and figurate geometry. This synthesis strengthens the conceptual foundations of number genetics and suggests several promising directions for future research, particularly in exploring how other classical number sequences may emerge from similar generative principles.

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Appendix

Here is an excerpt (using the original numbering chapter) from the article: *Boulay, J.-Y. (2025). Expanding Sophie Germain primes concept to ultimate numbers. Number Genetics and the 3 to 2 ratio.*

9 The numerical genetics of the whole numbers

We will now finalize in great detail the introduction of the concept of number genetics by describing it as a pure system of biology applying to whole numbers.

9.1 Whole number emanations

The algebraic mechanics of Sophie Germain making it possible to respectively qualify the entities " x " and " $2x + 1$ " of Sophie Germain prime and pure prime—and whose concept we have expanded to fertile number and pure number—compares two mathematics entities one of which ($2x + 1$) consists of a part (1 and more precisely $+ 1$) which does not identify with x .

We are now going to propose that this second entity which is " $2x + 1$ " is in fact totally derived from the first entity " x ".

In mathematical literature, in number theory, there is always the debate whether or not to integrate the exotic number zero (0) into the set \mathbb{N}_0 . As we specified in the paper introduction, we have decided to include this number in the set of whole numbers.

This integration has an important consequence regarding the attributes of the different whole numbers. Indeed, as the first number of the set \mathbb{N}_0 has 0 for value, this value is different from the value of its rank which is equal to 1. Thus, for any x , x being any whole number, the value of its rank is equal to $x + 1$. We can therefore already say that any x has two attributes: its value, which we call $V(x)$ and the value of its rank which we call $R(x)$. We propose to consider these two different attributes (these two values) as intrinsic emanations of x .

Thus, we suggest that the classical algebraic notation " $2x + 1$ " used to define the Germain's descent of x be replaced by $V(x) + R(x)$. We also propose to call $G(x)$ the result of this *fusion* of the two entities $V(x)$ and $R(x)$. We therefore have the following mechanism of numerical genetics: the descent of x is equal to the sum of the value of x and the value of the rank of x . We thus write " $G(x) = V(x) + R(x)$ " and consider that these three values are only emanations of x . We consider these values as real and intrinsic emanations of number x since no other mathematical entity not depending on x composes them. This, in contrast to the algebraic operation " $f(x) = 2x + 1$ " where the numeric entity " 1 " is foreign to it.

9.2 $V(x)$, $R(x)$ and $G(x)$ emanations

We have logically proposed the names $V(x)$ and $R(x)$ in reference to the Value of x and the Rank of x , two emanations of x . For this third emanation, we propose the letter G in reference to "generation of x " but also in reference (and homage) to Sophie Germain, initiator of the concept of Germain prime and safe prime; concept that we expand in this paper. Table in Figure 49 visually explains these concepts of emanations and their different novel conventional writing.

entity (emanation)	→	<i>Value of x</i>	<i>Rank of x</i>	<i>Descent of x</i>
notation	→	$V(x)$	$R(x)$	$G(x)$
formula	→	$V(x) = x$	$R(x) = V(x) + 1$	$G(x) = V(x) + R(x)$
final value (standard value)	→	x	$x + 1$	$2x + 1$

Figure 49: Numeric emanations of entity x as integrate parts of x (x being any whole number). See also Figure 50.

9.3 Enriched number genetics mechanism

We have thus demonstrated that the entity generated from x using the conventional formula " $2x + 1$ " is, in fact, a pure emanation of x . The illustrations in Figure 50 clearly describe how x first generates two primary entities (pure attributes of x), which then result in the secondary entity $G(x)$, the final emanation of x . The whole number 3 is used here as an example to enhance the understanding of this new arithmetic concept.

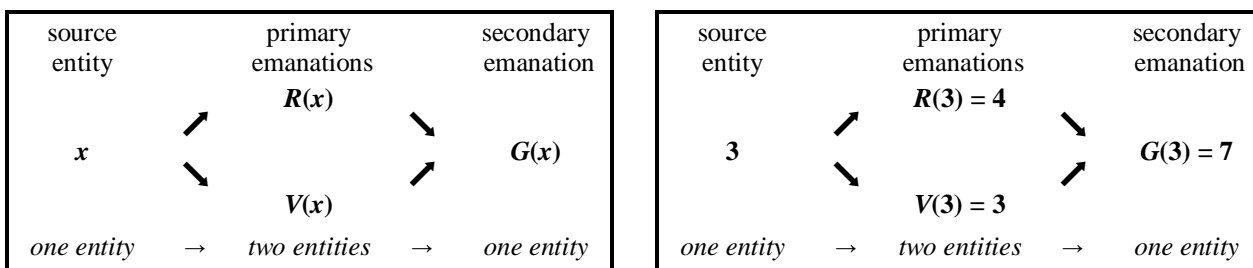


Figure 50: Numeric emanations of entity x as integrate parts of x . (x being any whole number). Whole number 3 as illustrated example (See also Figure 49).

9.3.1 Enriched number writing

Building upon this enriched number genetics mechanism, we logically propose an enriched number writing system. Therefore, any whole number can be written in this expanded form (5 as example):

$$R(x) \chi G(x) \rightarrow {}^6 5_{11}$$

Table in Figure 51 lists this wider writing of the first ten numbers.

Rank of $x \rightarrow R(x) = x + 1$	1	2	3	4	5	6	7	8	9	10
Value of $x \rightarrow V(x) = x$	0	1	2	3	4	5	6	7	8	9
Descent of $x \rightarrow G(x) = V(x) + R(x) = 2x + 1$	1	3	5	7	9	11	13	15	17	19
wider writing of x	${}^1 0_1$	${}^2 1_3$	${}^3 2_5$	${}^4 3_7$	${}^5 4_9$	${}^6 5_{11}$	${}^7 6_{13}$	${}^8 7_{15}$	${}^9 8_{17}$	${}^{10} 9_{19}$

Figure 51: Wider genetics writing of the first ten whole numbers.

9.4 Numeric genetics identifying with a biological process

Given the extensive discussion in this article regarding number genetics, particularly in the development of the famous Sophie Germain concept, we now introduce a close analogy with the field of biology.

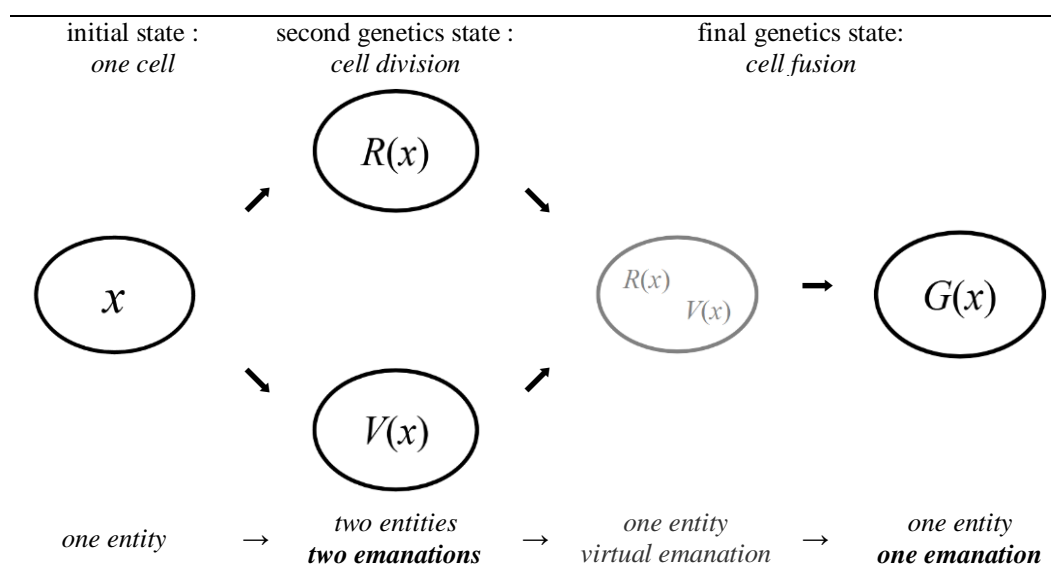


Figure 52: Arithmetic Sophie Germain's concept as a biological process of cellular reproduction (See also Figures 50 and 53).

We therefore introduce the idea that the arithmetic mechanism producing $G(x)$ from x is analogous to a biological process of cellular reproduction. Thus, x (any whole number) effectively divides like a biological cell into two daughter cells to generate $V(x)$ and $R(x)$. Subsequently, these two cells biologically merge to form a new cell, which is none other than $G(x)$.

Figure 52 illustrates this number biology through the creation of an intermediate virtual cell that forms immediately prior to $G(x)$. Since mathematical mechanics is timeless, this entity should therefore only be considered a ghost in the formation of $G(x)$. Figure 53 completes this illustration using the whole number 3 as an example: numbers 4 and 7 are pure emanations of number 3.

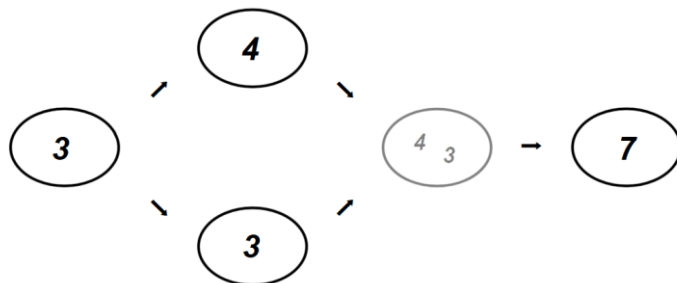


Figure 53: Arithmetic Sophie Germain's concept as a biological process of cellular reproduction. Whole number 3 as illustration example (See also Figures 50 and 52).