

# EVOLUTION OF THE CONCEPT OF ANGULAR MOMENTUM

YU. E. ZEVATSKYI

*New types of angular momentum of elementary particles arising from the special theory of relativity are examined. A total of 14 types have been identified, including the classical angular momentum. The calculated values of relativistic angular momenta can be applied both to the determination of fundamental properties of elementary particles and to the quantitative description of various types of interactions between them.*

**Keywords:** angular momentum, special relativity, local time, relativistic velocity.

Angular momentum (also referred to as moment of momentum) is one of the key concepts of classical mechanics. For a moving material point of mass  $m$ , the angular momentum vector with respect to the origin of Euclidean space is given by:

$$\mathbf{L} = m \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{pmatrix}, \quad (1)$$

where the dot over a coordinate symbol denotes differentiation with respect to absolute time. This angular momentum is used in Kepler's second law (the law of areas) in the form:

$$\mathbf{l} = \frac{\mathbf{L}}{m} \quad (2)$$

With the advent of the theory of relativity, the classical velocity in the laws of mechanics had to be replaced by the relativistic velocity [1]. This resulted from the introduction of the concept of local time  $\tau$ , the interval  $\Delta\tau$  of which is invariant under inhomogeneous Lorentz transformations [2]:

$$(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2. \quad (3)$$

$\Delta t$  is the corresponding interval of absolute time measured by a clock at rest. In a number of reference sources, the change in the relativistic angular momentum is attributed to a change in the relativistic mass [3, 4]:

$$m = \frac{m_0}{\sqrt{1 - \mathbf{v}^2/c^2}}, \quad (4)$$

where  $m_0$  is the rest mass of the particle, and  $\mathbf{v}$  is the classical velocity vector:

$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \quad (5)$$

L. B. Okun', in his work [5], criticized the concept of relativistic mass. He convincingly demonstrated the existence of only the invariant mass of particles; thus, the relativistic angular momentum takes the following form:

$$\mathbf{L}' = m_0 \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ x' & y' & z' \end{pmatrix}, \quad (6)$$

where the prime over a coordinate symbol denotes differentiation with respect to local time  $\tau$ .

## SPACE-LOCAL TIME

One of the avenues for advancing the application of the special theory of relativity (STR) to various physical problems is the Euclidean space-Lorentz local time framework, which follows from equation (3). Passing to differentials and defining  $t$  as a natural parameter in four-dimensional Euclidean space with coordinates  $(c\tau, x, y, z)$ , *the phenomenon of motion of any particle with a constant limiting speed of light  $c$  is obtained:*

$$\left(c \frac{d\tau}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = c^2. \quad (7)$$

The main postulates and results of this STR model are described in detail in preprint [6]. Previously, these ideas were employed to derive the Born–Jordan quantum conditions [7], to solve the gravitational two-body problem and to explain the phenomenon of accelerated recession of galaxies [8], to establish the difference in the nature of virtual and real particles [9], and to compare the motion parameters of a linear oscillator and a rigid rotor in their lowest quantum states [10, 11]. Among the new invariants of motion, there figure moments of three types:

$$\mathbf{l}_1 = \begin{pmatrix} \mathbf{h} & \mathbf{j} & \mathbf{k} \\ c\tau & y & z \\ c\dot{\tau} & \dot{y} & \dot{z} \end{pmatrix}, \quad \mathbf{l}_2 = \begin{pmatrix} \mathbf{h} & \mathbf{i} & \mathbf{k} \\ c\tau & x & r_3 \\ c\dot{\tau} & \dot{x} & \dot{r}_3 \end{pmatrix}, \quad \mathbf{l}_3 = \begin{pmatrix} \mathbf{h} & \mathbf{i} & \mathbf{j} \\ c\tau & x & y \\ c\dot{\tau} & \dot{x} & \dot{y} \end{pmatrix}. \quad (8)$$

where  $\mathbf{h}$  is the unit vector along the local time axis, orthogonal to all spatial unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ . The conservation of these moments (2) and (8) as a consequence of the invariance under rotations of any three of the four unit vectors, yields solutions that involve motions not along geodesics in the intrinsic harmonic potential field of an elementary particle. Furthermore, it was shown in [7] that, for the two-body problem to be solvable, their local times must be complex. In [8] it was demonstrated that for real particles to exist, as opposed to virtual ones, their local time also contains an imaginary component and is complex:

$$c\tau = w_R + iw_I. \quad (9)$$

Thus, there exists another triplet of moments of the form (8), but with complex-valued coefficients at the unit vectors..

By analogy with the relativistic angular momentum (6) and (2), three additional types of moments with real-valued coefficients can be written:

$$\mathbf{l}'_1 = \begin{pmatrix} \mathbf{h} & \mathbf{j} & \mathbf{k} \\ w & y & z \\ 1 & y' & z' \end{pmatrix}, \quad \mathbf{l}'_2 = \begin{pmatrix} \mathbf{h} & \mathbf{i} & \mathbf{k} \\ w & x & z \\ 1 & x' & z' \end{pmatrix}, \quad \mathbf{l}'_3 = \begin{pmatrix} \mathbf{h} & \mathbf{i} & \mathbf{j} \\ w & x & y \\ 1 & x' & y' \end{pmatrix}. \quad (10)$$

For complex values of local time (9), the final triplet of moments of the form (10) is obtained. The total number of types according to formulas (2), (6), (8), and (10) amounts to 14.

## CALCULATION OF MOMENTS FOR ABSOLUTELY STATIONARY PARTICLES

As an example for the calculation of relativistic angular momenta, hypothetical absolutely stationary particles, the existence of which was proposed in a lecture presentation [12], will be used. According to the Big Bang theory, these particles could exist at its epicenter. In accordance with the terminology employed, these particles possess no spatial displacements associated with the recession of galaxies, thermal motions, or other collisions.

According to the results of calculations reported in [6, 8], there exists a trivial solution for which the local time is real and  $d\tau = dt$ . The second solution assumes complex values of  $w$ . In this case, local time acquires the properties of a cyclic coordinate. In the simplest case, the equations of motion in the complex plane of local time–Euclidean space are as follows:

$$w = R(\cos \omega t + i \sin \omega t) \quad x, y, z = \text{const} \quad (11)$$

$$\dot{w} = \omega R(-\sin \omega t + i \cos \omega t) \quad \dot{x}, \dot{y}, \dot{z} = 0 \quad (12)$$

$$\omega R = c \quad (13)$$

When relativistic velocities are used:

$$w' = 1 \quad \omega R = 1 \quad x', y', z' = 0 \quad (14)$$

Substitution of the equations of motion (11)–(13) into the expressions for moments (2) and (8) yields:

$$\mathbf{l} = 0 \quad (15)$$

$$\mathbf{l}_1 = \omega R(\sin \omega t - i \cos \omega t)(y\mathbf{k} - z\mathbf{j}) \quad (16)$$

$$\mathbf{l}_2 = \omega R(\sin \omega t - i \cos \omega t)(x\mathbf{k} - z\mathbf{i}) \quad (17)$$

$$\mathbf{l}_3 = \omega R(\sin \omega t - i \cos \omega t)(x\mathbf{j} - y\mathbf{i}) \quad (18)$$

Substitution of the equations of motion (11), (14) into the expressions for moments (6) and (10) yields:

$$\mathbf{L}' = 0 \quad (19)$$

$$\mathbf{l}_1 = \omega R(y\mathbf{k} - z\mathbf{j}) \quad (20)$$

$$\mathbf{l}_2 = \omega R(x\mathbf{k} - z\mathbf{i}) \quad (21)$$

$$\mathbf{l}_3 = \omega R(x\mathbf{j} - y\mathbf{i}) \quad (22)$$

## DISCUSSION OF THE RESULTS

The interpretation of the physical content of moments (16)-(18) and (20)-(22) currently presents certain difficulties. Upon a change of the origin of coordinates, the vectors of the moments retain their location in a single plane. The sum of moments (19)-(22) constitutes a matrix

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 1 & -1 & 1 \end{pmatrix}, \quad (23)$$

whose physical meaning remains unclear.

Some assistance may be derived from the application of the concept of moment of inertia, for which purpose the concept of mass must be introduced. For stationary circular motion in a harmonic potential field, the equality of centrifugal and centripetal forces holds:

$$m\omega^2 R = \kappa R, \quad (24)$$

in the classical notation of velocities, and

$$m\omega = \kappa R, \quad (25)$$

when relativistic velocities are employed, where  $\kappa$  is the proportionality coefficient of the harmonic field. The only distinction between equations (24) and (25) is the presence of the factor  $c$ ; accordingly, relativistic velocities were employed in the subsequent analysis. From (25), it follows that

$$m = \frac{\kappa}{\omega^2} = \kappa R^2. \quad (26)$$

for a point moving in a circle in the complex plane of local time, its moment of inertia about an axis orthogonal to this plane is

$$I = mR^2 = \frac{\kappa}{\omega^4} = \kappa R^4. \quad (27)$$

From this, the value of the angular momentum is obtained

$$I\omega = \frac{\kappa}{\omega^3} = \kappa R^3 \quad (28)$$

Comparison of this expression with the matrix (23) constitutes the direction for further investigation of the problem outlined above.

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