Disproof of the Riemann Hypothesis 2

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Abstract

This paper disproves the Riemann Hypothesis by a different method.

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Contents

1 Introduction

Let $s = \sigma + it$ be complex variable. Let $\zeta(s)$ be the Riemann Zeta Function. Let "$\sim$" stand for "is asymptotic to".

Start with [1, eq.2.1.3], valid in the half-plane $\sigma > 1$,

$$\sum_{n=a+1}^{b} \frac{1}{n^s} = \frac{b^{1-s} - a^{1-s}}{1-s} - s \int\limits_a^b \frac{\{x\} - \frac{1}{2}}{x^{s+1}} dx + \frac{b^{-s} - a^{-s}}{2} \tag{1}$$

If one takes $\sigma > 1$, $a = 1$, $b \to \infty$ and adds 1 to both sides, one obtains

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} - s \int_1^{\infty} \frac{\{x\} - \frac{1}{2}}{x^{s+1}} dx + \frac{1}{2} \tag{2}$$

The left hand side is $\zeta(s)$ in the half-plane $\sigma > 1$, and hence in the same half-plane

$$\zeta(s) = \frac{1}{s-1} - s \int_1^{\infty} \frac{\{x\} - \frac{1}{2}}{x^{s+1}} dx + \frac{1}{2} \tag{3}$$

$\zeta(s)$ being complex analytic, this expression is correct on the half-plane $\sigma > 0$ by analytic continuation [1, p.14].

2 Series parts

For $\sigma > 0$ with $b \to \infty$ and by adding 1 to both sides of the equation, Eq. (1) reads

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = -s \int_1^{\infty} \frac{\{x\} - \frac{1}{2}}{x^{s+1}} dx - \frac{1}{1-s} + \frac{1}{2} + \lim_{b \to \infty} \frac{b^{1-s}}{1-s} \tag{4}$$

By Eq. (3), the first three terms on the right are $\zeta(s)$. Hence, in the Critical Strip $0 < \sigma < 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) + \lim_{b \to \infty} \frac{b^{1-s}}{1-s} \tag{5}$$

3 The Approximate Functional Equation

The Approximate Functional Equation [1, pp.78,79,95] in the Critical Strip $0 < \sigma < 1$ is

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} + \chi(s) \sum_{n \le y} \frac{1}{n^{1-s}} + O\left(x^{-\sigma}\right) + O\left(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1}\right) \tag{6}$$

$$2\pi x y = |t| \tag{7}$$

$$\left|\chi(s)\right| \sim \left\{\frac{t}{2\pi}\right\}^{\frac{1}{2}-\sigma} \tag{8}$$

Let $h(s) = \exp\left\{\mathrm{i} \arg\left(\chi(s)\right)\right\}$ and $f(s) = \exp\left\{\mathrm{i}\, t \log\left(\frac{t}{2\pi}\right)/2\right\}$.

Set $x = y = \sqrt{\frac{t}{2\pi}}$ and let $t \to \infty$ and apply Eq. (5) to the Approximate Functional Equation above,

$$\zeta(s) \sim \zeta(s) + \frac{\left\{\frac{t}{2\pi}\right\}^{\frac{1-s}{2}}}{1-s} + h(s)\left\{\frac{t}{2\pi}\right\}^{\frac{1}{2}-\sigma}\zeta\left(1-s\right) + h(s)\frac{\left\{\frac{t}{2\pi}\right\}^{\frac{1}{2}-\sigma+\frac{s}{2}}}{s} \tag{9}$$

Consequently,

$$\zeta(1-s) \sim \frac{1}{h(s)\left\{\frac{t}{2\pi}\right\}^{\sigma/2}}\left\{\frac{1}{f(s)\left[1-s\right]} - \frac{f(s)}{s}\right\} \tag{10}$$

Hence, due to the continuity of $\zeta(s)$, in the Critical Strip $0 < \sigma < 1$, as t increases,

$$\left|\zeta(s)\right| = O\left(1\right) \tag{11}$$

4 Conclusion

This result is the same as in [2]. Hence, all the conclusions are the same as well. Most notably, the Riemann Hypothesis is false: not all non-trivial zeros of $\zeta(s)$ are on the Critical Line $\sigma = \frac{1}{2}$.

References

[1] Titchmarsh, E.C., Heath-Brown, D.R.: The Theory of the Riemann Zeta-function, second revised edn. Clarendon Press, Oxford [Oxfordshire] (1986)

[2] Hrnčić, I.: Disproof of the Riemann Hypothesis. https://www.vixra.org/abs/2508.0055