

Invariant Averaging Over the Kottler Family of Solutions of the Einstein Vacuum Equation with a $\pm\Lambda$ -term

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September 24, 2026

ABSTRACT

This study proposes a unified approach to the Kottler family of metric solutions of the Einstein vacuum equation with a cosmological constant. To combine ten metrics with signatures $(+ - - -)$ and $(- + + +)$ within a single system, we employed the method of invariant averaging of the associated scalars and curvature invariants. As the main results, averaged values of the squared Ricci tensor, squared Weyl tensor, and scalar curvature are obtained ($\langle C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \rangle = 12r_b^2/r^6$, $\langle R_{\mu\nu}R^{\mu\nu} \rangle = 36/r_a^4$, $\langle g_{00} \rangle = 1$, $\langle R \rangle = 0$, $\langle R_{00} \rangle = 3c^2r^2/r_a^4$, $\langle K \rangle = 12r_b^2/r^6 + 24/r_a^4$, $\langle l_r^{(+)} \rangle = \langle \frac{\Delta r}{r} \rangle = \langle \sqrt{g_{11}} - 1 \rangle$ and others), which are independent of the choice of coordinate system. Based on these results, mathematical models of two opposite stable spherical vacuum formations, called the Kottler–Schwarzschild–de Sitter “core” and “anti-core,” are constructed. The limitations of this approach include the impossibility of describing the interaction between these formations and their evolution. To overcome these difficulties, an extended Einstein vacuum equation is proposed, which includes a finite difference between a set of positive and negative Λ -terms: It is shown that this equation reduces to the classical Einstein equation with an effective cosmological constant and can serve as a basis for a hierarchical cosmological model within the program of geometrization of physics. This work is of interest to researchers in gravitation, cosmology, and mathematical physics, as it offers a new perspective on the averaging problem in general relativity and its applications in modeling the structure of vacuum.

Keywords: averaging problem, Kottler metric, Einstein manifold, de Sitter / anti-de Sitter space, metric signature, vacuum, metric solutions, vacuum equation, cosmological model, relative elongation of the vacuum

1 Introduction

The Einstein vacuum equation with a Λ -term (or Einstein manifold equation) has the form:

$$R_{ik} = \Lambda g_{ik}, \quad (1)$$

where g_{ik} are the components of the metric tensor of a smooth 4-dimensional space with the pseudo-Riemannian metric $ds^2 = g_{ik} dx^i dx^k$,

$$R_{ik} = \frac{\partial \Gamma_{ik}^l}{\partial x^l} - \frac{\partial \Gamma_{il}^l}{\partial x^k} + \Gamma_{ik}^l \Gamma_{lm}^m - \Gamma_{il}^m \Gamma_{mk}^l - \text{Ricci tensor}, \quad (2)$$

$$\Gamma_{ik}^\lambda = \frac{1}{2} g^{\lambda\mu} \left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu} \right) - \text{Christoffel symbols}, \quad (3)$$

$$T_{ik} = 0 - \text{zero energy-momentum tensor of an external material source (no source is present)}, \quad (4)$$

$$\Lambda = 3/r_a^2 - \text{a constant (where } r_a \text{ is the radius of the sphere or the de Sitter radius)}. \quad (5)$$

The vacuum equation (1), As a result of solving the Einstein–Hilbert variational problem, rests on the following fundamental principles and axioms underlying modern physics [1–14]: general covariance, coordinate invariance, equivalence, independence of the speed of light from the reference frame, causality, extremal action, relativity, quasi-linearity, diffeomorphism invariance, scale invariance, and self-consistency.

In addition, in the 3+1 ADM decomposition, the vacuum equation (1) is represented as a set of four constraint equations and six evolution equations, which resemble the conservation laws:

$$\nabla_j [R_{ik} - \Lambda g_{ik}] = 0. \quad (6)$$

The reason is that the tensor under consideration is identically zero ($R_{ik} - \Lambda g_{ik} = 0$): $\nabla_j 0 = \frac{\partial 0}{\partial x^j} - \Gamma_{ij}^l 0 - \Gamma_{kj}^l 0 = 0$.

However, in curved spacetime, obtaining integrals of motion from a divergence requires integration over a volume, but the covariant derivative contains Christoffel symbols, which prevents the application of the Gauss–Ostrogradsky theorem to tensors of rank two or higher (in particular, $\Gamma_{ik}^l \Gamma_{lm}^m - \Gamma_{il}^m \Gamma_{mk}^l$). Nevertheless, vacuum equation (1) is suitable for identifying stable vacuum formations.

This combination of fundamental principles in a single vacuum equation (1) gives its solutions special significance.

The vacuum equation (1) admits a family of spherically symmetric Kottler solutions, encompassing the Schwarzschild, de Sitter, and anti-de Sitter metrics, as well as their combinations. Typically, these solutions are treated individually, with metrics possessing $(+ - - -)$ and $(- + + +)$ signatures regarded merely as different representations of the same geometry.

However, the question of combining them into a single system remains open: due to the nonlinearity of equation (1), direct summation or component-wise averaging of its metric-solutions leads to unphysical results, while the averaging of tensor quantities in the theory of relativity depends on the choice of coordinates.

This paper proposes an approach based on the invariant averaging of scalars and curvature invariants associated with the ten metrics of the Kottler family. Instead of averaging the metric tensor components, the study considers coordinate-independent quantities: the square of the Ricci tensor, the square of the Weyl tensor, scalar curvature, the norms of the Killing vector, and the relative vacuum expansion. This method relies on established results by Zalaletdinov, Buchert, Green, and Wald regarding the validity of averaging scalars and curvature invariants.

2. Methods for combining solutions to the vacuum equation

2.1. The Einstein Vacuum Equation and the Family of Kottler Metrics

The generalized Birkhoff theorem * states that only the spherically symmetric solutions of equation (1)

$$R_{ik} - \Lambda g_{ik} = 0 \quad (7)$$

are given by the Kottler family of metrics [1,2,5,13]

$$ds_1^2 = \pm f(r) c^2 dt^2 \mp \frac{dr^2}{f(r)} \mp r^2 d\theta^2 \mp r^2 \sin^2 \theta d\phi^2. \quad (8)$$

Of these, five metric solutions have the signature $(+ - - -)$ in a single coordinate chart (t, r, θ, ϕ) :

$$\text{I} \quad ds_1^{(+2)} = \left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (9)$$

$$\text{H} \quad ds_2^{(+2)} = \left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right) c^2 dt^2 - \frac{dr^2}{\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (10)$$

$$\text{V} \quad ds_3^{(+2)} = \left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (11)$$

$$\text{H}' \quad ds_4^{(+2)} = \left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right) c^2 dt^2 - \frac{dr^2}{\left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (12)$$

$$i \quad ds_5^{(+2)} = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2; \quad (13)$$

and five analogous metric solutions had opposite signatures $(- + + +)$ in an analogous chart.

$$\text{H}' \quad ds_1^{(-2)} = -\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (14)$$

$$\text{V} \quad ds_2^{(-2)} = -\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right) c^2 dt^2 + \frac{dr^2}{\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (15)$$

$$\text{H} \quad ds_3^{(-2)} = -\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (16)$$

$$\text{I} \quad ds_4^{(-2)} = -\left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right) c^2 dt^2 + \frac{dr^2}{\left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (17)$$

$$i \quad ds_5^{(-2)} = -c^2 dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (18)$$

where r_b is the Schwarzschild radius.

*The generalized Birkhoff theorem states that any spherically symmetric solution of equation (7) with a non-constant areal radius is locally isometric to one of the solutions of the Kottler family (9) – (18); the exceptional case of a constant areal radius leads to the Nariai spacetime [15], which is not considered here.

It is generally accepted that metrics with signatures $(+---)$ and $(-+++)$, other things being equal, describe the same spacetime using different notations. Therefore, modern scientific studies usually employ only one of them.

Moreover, metric solutions such as (9) – (13) are mostly considered separately for various cosmological problems (see Table 1).

Table 1. Metric solutions of the vacuum equation (7) for the signature $(+---)$

No.	$f(r)$	Metric, $\Lambda^\dagger$
1	$1 - r_b/r + r^2/r_a^2$	$\Lambda > 0$ Schwarzschild–anti-de Sitter (Sch-AdS)
2	$1 + r_b/r - r^2/r_a^2$	$\Lambda < 0$ anti-Schwarzschild–de Sitter (ASch-dS)
3	$1 - r_b/r - r^2/r_a^2$	$\Lambda < 0$ Kottler–Schwarzschild–de Sitter ** (standard Kott-Sch-dS metric)
4	$1 + r_b/r + r^2/r_a^2$	$\Lambda > 0$ anti-Schwarzschild–anti-de Sitter (ASch-AdS) (repulsive central solution)
5	1	$\Lambda = 0$ Minkowski (flat Minkowski spacetime)

† The sign of Λ refers to Equation (1) with the Ricci tensor (2) and signature $(+---)$; in the convention most common in the literature, de Sitter-type solutions correspond to $\Lambda > 0$.

** Kottler first wrote down a metric of the form (11) in [3], published in March 1918. In the limiting case $r_a \neq \infty$ and $r_b = 0$, the Kottler metric (11) becomes the de Sitter metric. In the opposite limit, $r_a = \infty$ and $r_b \neq 0$, it becomes the Schwarzschild metric. When both $r_a = \infty$ and $r_b = 0$, metrics (9) – (12) reduce to the Minkowski metric of flat spacetime. For this reason, the metric solutions (9) – (12) are usually called Schwarzschild–de Sitter and Schwarzschild–anti-de Sitter metrics (abbreviated Sch-dS and Sch-AdS).

This study attempts to unify the metric solutions of the Kottler families (9) – (18) within a single system. To this end, we considered several possible ways to achieve this unification.

2.2. The Problem of Unifying Metric Solutions

Summing all 10 metric solutions (9) – (18) in a single coordinate chart.

$$ds_0^{(\pm)2} = \sum_{i=1}^5 ds_i^{(+)2} + \sum_{j=1}^5 ds_j^{(-)2} = 0 \quad (19)$$

is not an admissible operation because of the nonlinearity of equation (7), but it leads to two zero metrics with the total signature $(+---) + (-+++)$ = (0 0 0 0):

$$ds_0^{(\pm)2} = \pm 0 \cdot c^2 dt^2 \mp 0 \cdot dr^2 \mp 0 \cdot r^2 d\theta^2 \mp 0 \cdot r^2 \sin^2 \theta d\phi^2 = 0. \quad (20)$$

It is well established that the zero metrics (20) are not solutions to equation (7) because the Christoffel symbols (3) are undefined in this case.

$$\lim_{g_{fl} \rightarrow 0} \Gamma_{ik}^\lambda = \lim_{g_{fl} \rightarrow 0} \left[\frac{1}{2} g^{\lambda\mu} \left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu} \right) \right] \rightarrow \frac{0}{0} = 0 \cdot \infty, \text{ where the indices } f \text{ and } l \text{ run over all values } \lambda, \mu, i, k,$$

because the expression $g^{\lambda\mu} = 1/g_{i\mu} = 1/0$ was incorrect. Moreover, the inverse metric $g^{\lambda\mu}$ is the inverse of the matrix $g_{i\mu}$, and not an element-wise division, followed by the limit “ $1/0 \rightarrow \infty$ ”.

However, it is not ruled out that in some cases, the Ricci tensor (2) may turn out to be zero if, under certain conditions, the indeterminacies cancel each other through their differences:

$$\lim_{g_{fl} \rightarrow 0} R_{ik} = \lim_{g_{fl} \rightarrow 0} \left[\frac{\partial \Gamma_{ik}^l}{\partial x^l} - \frac{\partial \Gamma_{il}^l}{\partial x^k} + \Gamma_{ik}^l \Gamma_{lm}^m - \Gamma_{il}^m \Gamma_{mk}^l \right] \rightarrow \left[\frac{\partial(0 \cdot \infty)}{\partial x^l} - \frac{\partial(0 \cdot \infty)}{\partial x^k} + (0 \cdot \infty)^2 - (0 \cdot \infty)^2 \right] \rightarrow [0 + 0] = 0.$$

We propose returning to this important limiting case by considering the degenerate limit of the one-parameter family $g_{i\mu}(\varepsilon)$ with a controlled limit $\varepsilon \rightarrow 0$, for which $R_{i\mu}(\varepsilon)$ is computed explicitly. This is beyond the scope of this article; however, resolving this problem would make it possible to introduce a zero signature (i.e., a neutral element) into the group of Algebra of Signatures under addition [16–20].

Equation (7) determines the metric-dynamic state of a certain region of spacetime (i.e., of the Einstein vacuum), and it cannot be ruled out that all Lorentzian metric solutions (9) – (13) and (14) – (18) in a single coordinate chart are interrelated. The following question arises: Is it possible to unify these solutions in a single system?

Because of the nonlinearity of equation (7), the sums and differences of the metric solutions (9) – (18) are not solutions of this equation, except for the zero operations of Type (19) (which require proof).

Simultaneously, one may assume that each metric solution of equation (7) belongs to a common system with a certain probability. In this case, one may attempt to apply an averaging operation.

As shown above, averaging all 10 metrics (9) – (18) leads to the zero result (20), while metrics (13) and (18) characterize not a curved state but the initial state of the vacuum (i.e., *flat Minkowski spacetime*). Therefore, we propose averaging metrics (9) – (12) and (14) – (17) separately, because they are defined in the same chart and have the same areal radius r .

However, averaging metrics is one of the most difficult problems in general relativity, known as the Isaacson–Buchert “averaging problem,” on which an extensive literature exists, in particular [21–25]. The main question is how can tensors be averaged while preserving their transformation law under a change in coordinates? However, there is no general answer to this question.

Reviews on this topic point out that there is currently no simple procedure for averaging vector and tensor quantities in general relativity. The point is that the average metric components are tied to selected coordinates. If the coordinates are changed, the transformed components are averaged, and the result is compared with the previous result of averaging the components of the original metric; the results do not coincide. Averaging and coordinating transformations do not commute.

Zalaletdinov, Buchert, Green, and Wald showed in [21–23] that averaging the components of the metric tensor over a region of spacetime is a non-invariant operation, whereas averaging the curvature invariants and scalars associated with these metrics is permissible.

Here, we face a related problem: if, for example, we perform component-wise averaging of the ensembles of the four metrics (9) – (12) or (14) – (17), we obtain the averaged metrics

$$\langle ds_{\Sigma}^{(\pm)2} \rangle = \frac{1}{4} \sum_{i=1}^4 ds_i^{(\pm)2} = \pm c^2 dt^2 \mp \langle g_{rr}^{(\pm)} \rangle dr^2 \mp r^2 d\theta^2 \mp r^2 \sin^2 \theta d\phi^2, \quad (21)$$

$$\text{where } \langle g_{rr}^{(\pm)} \rangle = \frac{1}{4} \left[\frac{1}{\left(1 - \frac{r_b}{r} + \frac{r_a^2}{r^2}\right)} + \frac{1}{\left(1 + \frac{r_b}{r} - \frac{r_a^2}{r^2}\right)} + \frac{1}{\left(1 - \frac{r_b}{r} - \frac{r_a^2}{r^2}\right)} + \frac{1}{\left(1 + \frac{r_b}{r} + \frac{r_a^2}{r^2}\right)} \right]. \quad (22)$$

These metrics are not solutions for the vacuum equation (7). For a finite ensemble of metrics given on the same manifold with a fixed identification of points, the pointwise average is itself a tensor that lies in the non-uniqueness of this identification between different solutions (in the present model, this identification is fixed by adopting the same Kottler chart for all members of the ensemble, with the areal radius and the normalization of the static Killing field fixed in common) and by averaging over extended regions of spacetime [21–23].

We consider the consequences of non-invariant averaging. To do this, we introduced an axiomatic prohibition on transitions from the spherical coordinate system (t, r, θ, φ) to other systems. This means that the vacuum region under study was observed only from the outside. In this case, the average geodesic lines were rigidly tied to the average extent of the elastoplastic vacuum and followed its deformations such as the bending of lines drawn on the surface of a bent rubber sheet.

In this case, it is permissible to consider the average metric (21), but the hypothesis then ceases to be relativistically invariant and belongs to continuum mechanics using the parameters and mathematical apparatus of general relativity.

Let us substitute the components $\langle g_{ii}^{(+)} \rangle$ from the averaged metrics (21) into the expressions for the components of the 4-vector of the relative elongation of the vacuum [26]:

$$l_i^{(\pm)} = \sqrt{1 + \frac{g_{ii}^{(\pm)} - g_{iio}^{(\pm)}}{g_{iio}^{(\pm)}}} - 1, \quad (23)$$

where the components $g_{iio}^{(+)}$ and $g_{iio}^{(-)}$ are obtained from flat (initial) metrics (13) and (18), respectively. As a result, we obtain

$$\begin{aligned}
\overline{l_r^{(\pm)}} &= \left(\frac{\Delta r}{r} \right) = \sqrt{1 + \frac{\langle g_{rr}^{(\pm)} \rangle - g_{rr0}^{(\pm)}}{g_{rr0}^{(\pm)}}} - 1 = \sqrt{\langle g_{11}^{(\pm)} \rangle} - 1 = \\
&= \sqrt{\frac{1}{4} \left[\frac{1}{\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)} + \frac{1}{\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} + \frac{1}{\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} + \frac{1}{\left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)} \right]} - 1 = \sqrt{\frac{1 - \left(\frac{r_b}{r}\right)^2 - \left(\frac{r^2}{r_a^2}\right)^2}{\left(1 - \left(\frac{r_b}{r} + \frac{r^2}{r_a^2}\right)^2\right) \left(1 - \left(\frac{r_b}{r} - \frac{r^2}{r_a^2}\right)^2\right)}} - 1, \quad (24) \\
l_t^{(\pm)} &= 0, \quad l_\theta^{(\pm)} = 0, \quad l_\phi^{(\pm)} = 0.
\end{aligned}$$

The function $\overline{l_r^{(\pm)}}$ (24) shows the averaged relative elongation of the elastoplastic vacuum in the radial direction on the open interval between the inner and outer horizons, roughly $r_b < r < r_a$ (r_a and r_b have the dimension of length (m) because equation (7) maintains its form when the metric is multiplied by a constant factor C^2 and Λ is divided by C^2 ; they may be of any scale). For the values $r_a = 90$ m and $r_b = 1.5$ m, conventionally adopted for illustration, the graph of function (24) diverges at the Schwarzschild horizon near $r \approx r_b$ and at the de Sitter horizon near $r \approx r_a$, so this function is defined on the interval $r \in (1.5004; 89.2404)$ (see Fig. 1a).

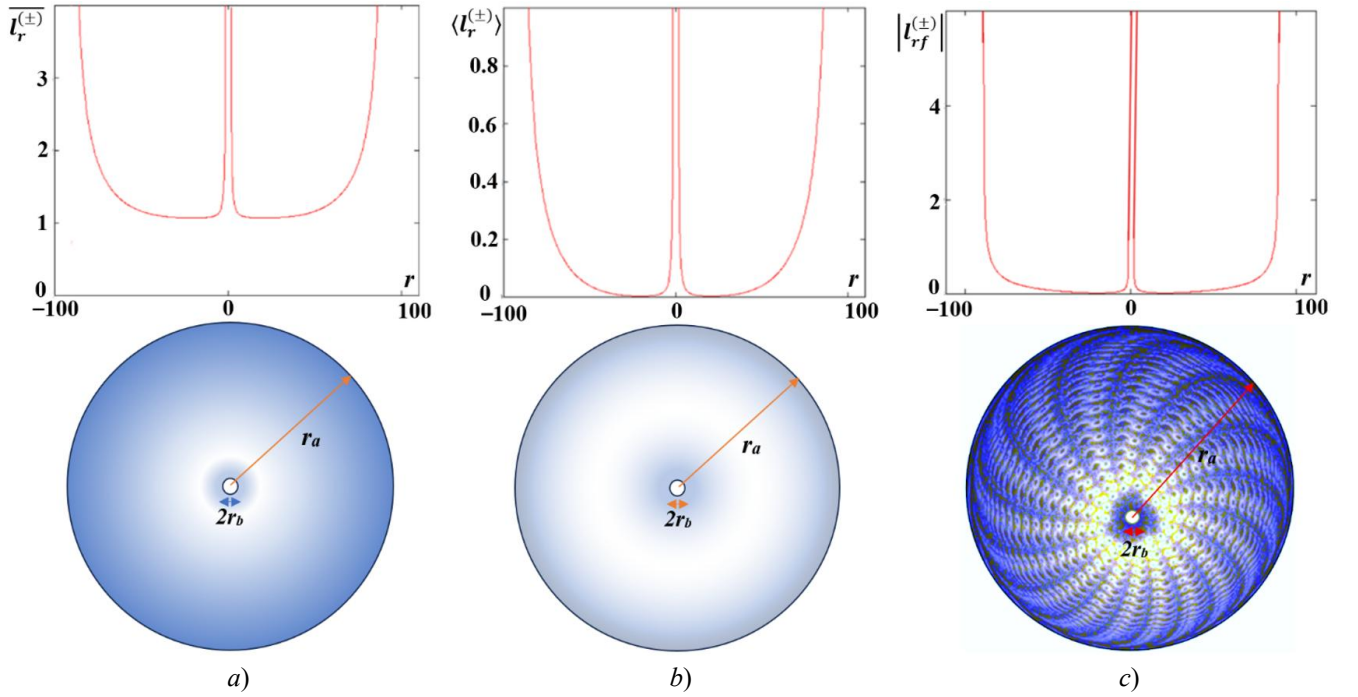


Fig. 1: Graphs of the relative elongation functions (illustrative): a) $\overline{l_r^{(\pm)}}$ (24); b) $\langle l_r^{(\pm)} \rangle$ (25); c) $\text{RMS}(l_{rf}^{(\pm)}) = |l_{rf}^{(\pm)}|$ (32), for the conventionally adopted de Sitter radius $r_a = 90$ m and Schwarzschild radius $r_b = 1.5$ m. The functions $\overline{l_r^{(\pm)}}$, $\langle l_r^{(\pm)} \rangle$ and $\text{RMS}(l_{rf}^{(\pm)})$ determine the radial relative elongation of spacetime (i.e., of the Einstein vacuum) averaged over the ensembles of Lorentzian metrics (9) – (12) and (14) – (17). The common analytical domain of all three graphs is the interval between the inner and outer horizons, approximately $1.5004 < r < 89.2404$ for the adopted values (roughly $r_b < r < r_a$); the left half of the graphs are their mirror images, added for easier perception of the spherical objects. The calculations were performed using Mathcad. The lower part of Fig. 1c is not the result of a calculation based on formulas, but an artistic illustration using fractal images taken from open Internet sources.

2.3. Methods of Invariant Averaging

Invariant averaging refers to a set of methods that allow for the correct averaging of physical quantities in General Relativity, independent of the choice of coordinate system. The core idea is to average scalar quantities (such as scalar curvature or curvature invariants) rather than the metric components.

Scalars are invariant under coordinate transformations; therefore, their values at different points can be consistently compared and averaged. One can then attempt to construct "effective" Einstein equations describing the behavior of these average quantities and interpret the resulting additional terms as "effective" matter or the "backreaction" backreaction' of inhomogeneities.

There are several key formalisms that implement this idea:

1] Buchert formalism: This is one of the most popular approaches. It is based on spatial averaging of the scalar parts of the Einstein equations within the 3+1 formalism (which splits spacetime into spatial slices and time). This yields equations for average scalars (such as the scale factor) that includes a "kinematic backreaction" term describing how inhomogeneities influence the overall expansion dynamics of the universe. This formalism is coordinate-independent (for scalars) and has been applied in cosmology to explain the observed accelerated expansion without invoking dark energy.

2] Zalaletdinov's Macroscopic Gravity: This ambitious and mathematically rigorous approach that aims for fully covariant averaging within four-dimensional spacetime. Zalaletdinov introduced a special parallel-transport bivector to compare tensors at different points, thereby enabling a consistent definition of the averaging operation for tensor fields. The result is a set of average Einstein equations modified by a correlation tensor, a new geometric object describing the influence of small-scale inhomogeneities on large-scale geometry.

3] The Gasperini, Marozzi, and Veneziano approach: This method builds upon the concept of gauge-invariant averaging for tensor fields. They introduce a specific "window function" to define an averaging procedure independent of the choice of gauge, a feature particularly important for cosmological perturbations.

4] Isaacson averaging: This approach was applied to gravitational waves. Isaacson demonstrated how to average the energy-momentum of high-frequency gravitational waves to obtain an effective energy-momentum tensor that "curves" the background space time. A key aspect is averaging over a region that is much larger than the wavelength, which renders the result gauge-invariant.

In this study, the problem of averaging is particularly acute, as attempting to combine the ten metric solutions of the Kottler family into a unified system via simple summation or component-wise averaging yields a non-physical result.

The invariant averaging of the scalars and curvature invariants associated with each solution of the Kottler family aligns with the methods described above.

3. Research results

3.1 Arithmetic Averaging Scalars and Curvature Invariants

Let us consider two possible ways of solving the "averaging problem" for the Kottler family of Lorentzian metric solutions (9) – (18) of vacuum equation (7).

As proposed in [21–25], we separately averaged the curvature invariants and scalars associated with the ensembles of metric solutions (9) – (12) and (14) – (17).

For example, the radial component of expression (23), $l_{ri}^{(\pm)} = \frac{\Delta r}{r} = \sqrt{g_{11i}^{(\pm)}} - 1$, is the ratio of the proper radial length to the differential of the areal radius, that is, in this coordinate form a chart-dependent quantity rather than a curvature invariant. However, it can be written as a true scalar *.

**Let R_A denote the areal radius, defined geometrically by the area $4\pi R_A^2$ of the orbits of spherical symmetry. For each metric $g_{(i)}$, we define the scalar $X_i = |g_{(i)}^{\mu\nu} \partial_\mu R_A \partial_\nu R_A|$ and $l_{ri} = X_i^{-1/2} - 1$. In the coordinates of (9) – (12), $R_A = r$, and on the domain considered here, where $f_i(r) > 0$, $X_i = f_i(r)$, so $l_{ri} = 1/\sqrt{f_i} - 1$, which reproduces (25); at the same time, l_{ri} is a true scalar, independent of the choice of coordinates.*

The areal radius r is geometric; the decomposition into "proper radial length" is not unique if the four metrics (9) – (12) or (14) – (17) are allowed to have different $f(r)$ values. Therefore, we postulate that all the four metrics are defined in the same coordinate chart. The angular part is the same ($r^2 d\Omega^2$) in all four metrics (9) – (12) and all four metrics (14) – (17); therefore, r is the same everywhere and is defined geometrically through the area of the sphere $A = 4\pi r^2$. In this case, the quantities subject to averaging are the invariant quantities (23), computed separately for each metric of ensembles (9) – (12) and (14) – (17) **.

$$\langle l_r^{(\pm)} \rangle = \frac{1}{4} \left[\left(\sqrt{\frac{1}{(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2})}} - 1 \right) + \left(\sqrt{\frac{1}{(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2})}} - 1 \right) + \left(\sqrt{\frac{1}{(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2})}} - 1 \right) + \left(\sqrt{\frac{1}{(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2})}} - 1 \right) \right], \quad (25)$$

$$l_t^{(\pm)} = 0, \quad l_\theta^{(\pm)} = 0, \quad l_\phi^{(\pm)} = 0. \quad (26)$$

***In this article, we also postulate that each of the four metrics of the ensembles (9) – (12) and (14) – (17) participates in the common system with the same weight $1/4$. However, this does not follow from equation (7); therefore, expression (25) is the unweighted arithmetic mean over four Lorentzian metrics defined on the same chart.*

The results of the calculations using Equation (25), which are identical for the ensembles of metrics (9) – (12) and (14) – (17), are shown in Fig. 1b. For the conventionally adopted values $r_a = 90$ m and $r_b = 1.5$ m, the graph of function (25) tends to infinity near the horizons $r \approx r_b$ and $r \approx r_a$, so it is defined on the interval $r \in (1.5004; 89.2404)$.

In this case, the axiomatic prohibition on changing the reference frame can be lifted. This argument favors this approach because the averaging results remain invariant.

Clearly, the results of formulas (24) and (25) differ, that is, averaging the components of the metric tensors and averaging scalars does not yield the same result.

The complete list of all scalars and polynomial curvature invariants of the metric solutions is practically infinite. Below, we provide the standard independent basis of scalars and invariants up to the second-order curvature for the ensemble of metrics (9) – (12) (Table 2) and for the ensemble of metrics (14) – (17) (Table 3).

Table 2. Scalars and curvature invariants of the ensemble of metrics (9) – (12) with signature $(+ - - -)$

Squared interval	Killing vector norm $\xi_\mu \xi^\mu = g_{00}$	Kretschmann scalar, $K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$	Time component of the Ricci tensor*, R_{00}	Scalar curvature, $R = 4\Lambda$	Weyl invariant $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$	$R_0^0 = g^{00} R_{00}$	Squared Ricci tensor $R_{\mu\nu} R^{\mu\nu}$
$ds_1^{(+2)}$	$1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}$	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$\frac{3c^2}{r_a^2} \left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)$	$+12/r_a^2$	$\frac{12 r_b^2}{r^6}$	$+\frac{3}{r_a^2}$	$\frac{36}{r_a^4}$
$ds_2^{(+2)}$	$1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}$	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$-\frac{3c^2}{r_a^2} \left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)$	$-12/r_a^2$	$\frac{12 r_b^2}{r^6}$	$-\frac{3}{r_a^2}$	$\frac{36}{r_a^4}$
$ds_3^{(+2)}$	$1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}$	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$-\frac{3c^2}{r_a^2} \left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)$	$-12/r_a^2$	$\frac{12 r_b^2}{r^6}$	$-\frac{3}{r_a^2}$	$\frac{36}{r_a^4}$
$ds_4^{(+2)}$	$1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}$	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$\frac{3c^2}{r_a^2} \left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)$	$+12/r_a^2$	$\frac{12 r_b^2}{r^6}$	$+\frac{3}{r_a^2}$	$\frac{36}{r_a^4}$

*Here R_{00} denotes the scalar $R_{\mu\nu}\xi^\mu\xi^\nu$ for the static Killing field $\xi = \partial_t$, with the normalization of ξ fixed by the common Kottler chart used throughout this paper; strictly speaking, it is a scalar associated with the Killing field rather than a curvature invariant.

Table 3. Scalars and curvature invariants of the ensemble of metrics (14) – (17) with signature $(- + + +)$

Squared interval	Killing vector norm $\xi_\mu \xi^\mu = g_{00}$	Kretschmann scalar, $K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$	Time component of the Ricci tensor, R_{00}	Scalar curvature, $R = 4\Lambda$	Weyl invariant $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$	$R_0^0 = g^{00} R_{00}$	Squared Ricci tensor $R_{\mu\nu} R^{\mu\nu}$
$ds_1^{(-2)}$	$-\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)$	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$\frac{3c^2}{r_a^2} \left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)$	$-12/r_a^2$	$\frac{12 r_b^2}{r^6}$	$-\frac{3}{r_a^2}$	$\frac{36}{r_a^4}$
$ds_2^{(-2)}$	$-\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)$	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$-\frac{3c^2}{r_a^2} \left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)$	$+12/r_a^2$	$\frac{12 r_b^2}{r^6}$	$+\frac{3}{r_a^2}$	$\frac{36}{r_a^4}$
$ds_3^{(-2)}$	$-\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)$	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$-\frac{3c^2}{r_a^2} \left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)$	$+12/r_a^2$	$\frac{12 r_b^2}{r^6}$	$+\frac{3}{r_a^2}$	$\frac{36}{r_a^4}$
$ds_4^{(-2)}$	$-\left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)$	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$\frac{3c^2}{r_a^2} \left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)$	$-12/r_a^2$	$\frac{12 r_b^2}{r^6}$	$-\frac{3}{r_a^2}$	$\frac{36}{r_a^4}$

A comparative analysis of Tables 2 and 3 shows that certain curvature scalars and invariants for metrics with signatures $(+ - - -)$ and $(- + + +)$ coincide only up to a sign, for example, $\xi_\mu \xi^\mu$, R and R_0^0 . In the present model, we adopt the distinction between these two sign-related branches as an additional hypothesis: metric solutions (9) – (12) and (14) – (17), which have mutually opposite signatures, describe different objects. In addition, it was shown in [17,18] that metric spaces with mutually opposite signatures $(+ - - -)$ and $(- + + +)$ were rotated relative to each other by 90° .

Note also that such curvature scalars and invariants are the total curvature (i.e., the Kretschmann scalar) $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, the Weyl invariant $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ and the squared Ricci tensor $R_{\mu\nu}R^{\mu\nu}$ are the same for all metric solutions (9) – (12) and (14) – (17).

Under the equal-weight ($1/4$) hypothesis, the arithmetic mean of the scalars and curvature invariants of the Lorentzian metrics (9) – (12) (from Table 2) almost completely coincided with the arithmetic mean of the scalars and curvature invariants of metrics (14) – (17) (Table 3) (see Table 4).

Table 4. Mean scalars and curvature invariants of the ensembles of metrics (9) – (12) and (14) – (17)

Mean (interval) ²	Killing vector norm $\langle \xi_\mu \xi^\mu \rangle = \langle g_{00} \rangle$	Kretschmann scalar, $\langle K \rangle = \langle R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \rangle$	Time component of the Ricci ten- sor, $\langle R_{00} \rangle$	Scalar curvature, $\langle R \rangle = \langle 4\Lambda \rangle$	Weyl invari- ant $\langle C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \rangle$	$\langle R^0_0 \rangle = \langle g^{00}R_{00} \rangle$	Squared Ricci tensor $\langle R_{\mu\nu}R^{\mu\nu} \rangle$
$\langle ds_i^{(\pm)2} \rangle$	± 1 + for (+ + + +) - for (- + + +)	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$\frac{3c^2r^2}{r_a^4}$	0	$\frac{12r_b^2}{r^6}$	0	$\frac{36}{r_a^4}$

The averaged scalars and curvature invariants of the ensembles of metrics (9) – (12) and (14) – (17) presented in Table 4 are quantities that do not change with a change in the coordinate system. Therefore, they reflect the intrinsic geometric properties of the curved region of the averaged vacuum under study with the averaged relative elongations (25) (Fig. 1b).

We note the following features. The norms of the Killing vector $\xi_\mu \xi^\mu$ for metrics (9) – (12) and (14) – (17) are equal to their zeroth (i.e., time) components (see Tables 2 and 3). These norms, as is well known, are scalar functions of r . They depend on the radial coordinate, but the values of $\xi_\mu \xi^\mu$ are scalars at each point of the region under study for a fixed killing field ξ_μ . In this case, averaging the scalars $\xi_\mu \xi^\mu$ is an invariant operation that yields $\langle g_{00} \rangle = \langle g_{tt}^{(\pm)} \rangle = \pm 1$. Thus, in this chart, within the admissible domain of function (25), on average over the ensembles of solutions (9) – (12) and (14) – (17), the gravitational time dilation (i.e., the averaged redshift factor) cancels out.

3.2. Invariant Root-Mean-Square Averaging of Scalars and Curvature Invariants (the Texture Hypothesis)

We consider one more invariant way of combining the family of metric solutions of the vacuum equation (7) into a unified space–time system.

The Lorentzian intervals $ds_i^{(+2)}$ (9) – (12) and $ds_i^{(-2)}$ (14) – (17) are scalars for a given displacement dx^i ; therefore, their averaging is an invariant operation:

$$\langle s_{1-4}^{(+2)} \rangle = \frac{1}{4} (ds_1^{(+2)} + ds_2^{(+2)} + ds_3^{(+2)} + ds_4^{(+2)}), \quad (27)$$

$$\langle s_{1-4}^{(-2)} \rangle = \frac{1}{4} (ds_1^{(-2)} + ds_2^{(-2)} + ds_3^{(-2)} + ds_4^{(-2)}). \quad (28)$$

Equations (27) and (28) can be represented as the scalar product of the complex-conjugate half-quaternions (i.e., quaternions divided by 2) [17, 18]:

$$\langle s_{1-4}^{(\pm)2} \rangle = \frac{1}{2} (ds_1^{(\pm)} + id s_2^{(\pm)} + j ds_3^{(\pm)} + k ds_4^{(\pm)}) \frac{1}{2} (ds_1^{(\pm)} - id s_2^{(\pm)} - j ds_3^{(\pm)} - k ds_4^{(\pm)}), \quad (29)$$

where $ds_i^{(\pm)}$ are absolute invariants;

i, j, k are the imaginary units that form an orthonormal basis of the algebra of quaternions H , which is isomorphic to the even Clifford subalgebra $Cl^0(3)$

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \quad (30)$$

The properties of quaternions suggest, and we adopted this as a modeling assumption that the conventionally “colored” segments of lines (i.e., “colored threads”) $ds_1^{(\pm)}, ds_2^{(\pm)}, ds_3^{(\pm)}, ds_4^{(\pm)}$ are mutually perpendicular: $ds_1^{(\pm)} \perp ds_2^{(\pm)} \perp ds_3^{(\pm)} \perp ds_4^{(\pm)}$. This is possible if the “colored” invariant threads $s_1^{(\pm)}, s_2^{(\pm)}, s_3^{(\pm)}, s_4^{(\pm)}$ (where $s_{nd}^{(\pm)} = \sqrt{g_{ikd}^{(\pm)} dx^i dx^k}$, $d = 1, 2, 3, 4$) are interwoven into 4-strand helices that form an elastoplastic vacuum fabric (i.e., texture) [16–20] (see Fig. 1c).

Representing a vacuum conventionally as a flexible elastoplastic fabric has the following consequences. The averaged relative elongations for the ensembles of metrics (9) – (12) and (14) – (17) take the form of half-quaternions (i.e., quaternions divided by two):

$$l_{rf}^{(\pm)} = \left[\frac{1}{2} \left(\sqrt{\frac{1}{\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)}} - 1 \right) + i \frac{1}{2} \left(\sqrt{\frac{1}{\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)}} - 1 \right) + j \frac{1}{2} \left(\sqrt{\frac{1}{\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)}} - 1 \right) + k \frac{1}{2} \left(\sqrt{\frac{1}{\left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)}} - 1 \right) \right], \quad (31)$$

and the corresponding scalars may be the moduli of these quaternions, which coincide with the root-mean-square values.

$$|l_{rf}^{(\pm)}| = \text{RMS}(l_{rf}^{(\pm)}) = \frac{1}{2} \sqrt{\left(\sqrt{\frac{1}{\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)}} - 1 \right)^2 + \left(\sqrt{\frac{1}{\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)}} - 1 \right)^2 + \left(\sqrt{\frac{1}{\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)}} - 1 \right)^2 + \left(\sqrt{\frac{1}{\left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)}} - 1 \right)^2}. \quad (32)$$

The graphs of functions (32) coincide for the conventionally adopted values $r_a = 90$ m and $r_b = 1.5$ m, and are shown in Fig. 1c. At the boundaries of the admissible interval $r \in (r_b \approx 1.5004; r_a \approx 89.2404)$ Equation (32) tends to infinity.

Using the data from Tables 2 and 3, we similarly calculated the root-mean-square (RMS) values of the scalars and curvature invariants of the ensembles of metrics (9) – (12) and (14) – (17) using the formula

$$\text{RMS}(a) = \sqrt{\frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{4}} = \frac{1}{2} \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}.$$

The results are presented in Table 5.

Table 5. Identical root-mean-square curvature invariants of the ensembles of metrics (9) – (12) and (14)–(17)

RMS (interval) ² $ds_1^{(+2)}$	RMS of the Killing vector norm $\xi_\mu \xi^\mu = g_{00}$	RMS of the Kretsch- mann scalar, $K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$	RMS of the time compo- nent of the Ricci tensor, R_{00}	RMS of the scalar curvature, $R = 4\Lambda$	RMS of the Weyl invariant $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$	RMS $R_0^0 = g^{00} R_{00}$	RMS of the squared Ricci tensor $R_{\mu\nu} R^{\mu\nu}$
$ds_\Sigma^{(+2)}$	$\sqrt{1 + \frac{r_b^2}{r^2} + \frac{r^4}{r_a^4}}$	$\frac{12r_b^2}{r^6} + \frac{24}{r_a^4}$	$\frac{3c^2}{r_a^2} \sqrt{1 + \frac{r_b^2}{r^2} + \frac{r^4}{r_a^4}}$	$\frac{12}{r_a^2}$	$\frac{12 r_b^2}{r^6}$	$\frac{3}{r_a^2}$	$\frac{36}{r_a^4}$

Notably, the root-mean-square scalars and curvature invariants of the ensembles of metrics (9) – (12) and (14) – (17) in Table 5 coincide completely.

Comparing the data in Tables 2, 3 and 5, we found that the following parameters coincided with the sign: $\text{RMS}(K) = K$, $\text{RMS}(R) = \pm R$, $\text{RMS}(R_0^0) = \pm R_0^0$ and $\text{RMS}(R_{\mu\nu} R^{\mu\nu}) = R_{\mu\nu} R^{\mu\nu}$.

The root-mean-square curvature invariants of the ensembles of metrics (9) – (12) and (14) – (17) presented in Table 5 are quantities that do not change with changes in the coordinate system. Therefore, they reflect the intrinsic geometric properties of the curved vacuum fabric under study with root-mean-square relative elongation (32) (Fig. 1c).

3.3. The Kottler–Schwarzschild–de Sitter “Core” and “Anti-Core” Models

In modern theories based on general relativity, it is customary to regard the change in the sign of some scalars under the signature change $(+ - - -) \rightarrow (- + + +)$ (compare Tables 2 and 3) as a standard consequence of a change in conventions. However, in the authors’ opinion, one cannot categorically reject the hypothesis that the ensembles of Lorentzian metrics (9) – (12) and (14) – (17) with mutually opposite signatures define metric-dynamic models of two different opposite objects.

Moreover, the two-dimensional example in § 5.2 of [18] demonstrates that if coordinate charts with opposite signatures are located on opposite sides of the sheet, they are rotated by 90° relative to each other.

Taken together, the above considerations show that the ensembles of metric solutions (9) – (12) and (14) – (17) of the Einstein vacuum equation (7) describe, on the one hand, identical and, on the other hand, mutually opposite closed spherical objects rotated relative to each other by 90° (however, owing to spherical symmetry, this rotation, or phase shift, between the objects is practically unnoticeable).

Considering the opposite signs of some scalars (Tables 2 and 3) and other properties, we have conventionally assumed that the ensemble of metrics (9) – (12) describes an averaged closed convexity * of spacetime (i.e., of the averaged Einstein vacuum), whereas the ensemble of metrics (14) – (17) describes the opposite averaged closed concavity * of the averaged vacuum.

**It is impossible to visualize a 3-dimensional (let alone a 4-dimensional) convexity or concavity of spacetime; here we attempt to synthesize a language for describing two opposite stable spherical curvatures of the multilayer Einstein vacuum on an auxiliary, figurative binary linguistic basis.*

From the graphs in Fig. 1b and 1c it can be seen that with either invariant arithmetic or root-mean-square averaging over the ensemble of metrics (9) – (12) with signature (+ – – –), one obtains a model of an almost hollow ball resembling the static de Sitter world with a strongly stretched periphery. Inside the large de Sitter ball, there is a strongly stretched neighborhood of a small spherical cavity, which we refer to as the Schwarzschild “nucleolus” (or “pit”). We conventionally call such a stable closed spherical vacuum formation a convex Kottler–Schwarzschild–de Sitter “core” (or “apple”) (abbreviated as the *KSdS* “core”) **.

***The divergence of the function $\sqrt{(1/f)} - 1$ at the roots of $f(r) = 0$ corresponds to the horizons of each individual Kottler metric, not to the wall of the averaged core, and the results obtained are only a set of mean scalars on the interval between the inner and outer horizons (roughly $r_b < r < r_a$). Nevertheless, the object under study has a pronounced spherical shape and a spherical boundary. Therefore, the illustrative term “core” is appropriate for the averaged mathematical model.*

The ensemble of metric solutions (14) – (17) with the opposite signature (– + + +), in turn, makes it possible to construct an identically structured averaged, but antipodal, closed spherical concavity of the vacuum (i.e., the *KSdS* “anti-core”).

The *KSdS* core with signature (+ – – –) and the *KSdS* anti-core with signature (– + + +) are called spherical convexity and spherical concavity of the vacuum, respectively, because they completely compensate for each other's manifestations: (+ – – –) + (– + + +) = (0 0 0 0), such as a particle and an antiparticle.

The core woven from the vacuum fabric (Figs. 1c and 2) with root-mean-square relative elongation (32) differs from the on-average linearly deformed core (Fig. 1b) in that the woven core resembles a thin-walled soap bubble.

Radii r_b and r_a are initially undetermined. That is, because equation (7) maintains its form when the metric is multiplied by a constant factor C^2 and Λ is divided by C^2 , solutions exist for any value of these radii under the condition $r_b \ll r_a$, more precisely, for $\frac{r_b}{r_a} < \frac{2}{3\sqrt{3}} \approx 0.3849$. Therefore, the aforementioned *KSdS* cores and *KSdS* anti-cores can be of any size.

This made it possible to develop the Hierarchical cosmological model [19, 20], in which the world is filled with a multitude of *KSdS* cores and anti-cores (which are geometrized analogues of particles and antiparticles) of various scales with a discrete set of r_{ai} and r_{bj} . The need to introduce the concepts of *KSdS* core and anti-core is justified in a series of articles under general ignorance of “Geometrized Vacuum Physics”, including [16, 20].

3.4. Extended Einstein Vacuum Equation

In the previous sections, it was shown that the Einstein vacuum equation with a Λ -term (7) has the potential to describe one *KSdS* core (i.e., a closed spherical convexity of the vacuum) and one *KSdS* anti-core (i.e., a closed spherical concavity of the vacuum).

This leads to the following problems:

- 1) The scaling property of equation (7) (the metric multiplied by C^2 , Λ divided by C^2) allows the *KSdS* core and anti-core to be of any size, but there are only two of them. Where is the vast diversity in the colossal numbers of cores (i.e., particle) and anti-cores (i.e., antiparticle)?
- 2) Within the model under consideration, based on the invariant averaging of scalars and curvature invariants of the ensembles of Lorentzian metric solutions (9) – (12) and (14) – (17) of the vacuum equation (7), it is impossible to describe the interaction between a *KSdS* core and a *KSdS* anti-core isolated from each other.
- 3) For the *KSdS* core and anti-core, the average relative elongation tended to be infinite at their spherical periphery and near the small central spherical cavity (Figs. 1b and 1c).
- 4) It is unclear what lies beyond the outer spherical de Sitter boundary of the *KSdS* core and anti-core and what lies inside the central Schwarzschild cavity. Within the mathematical model under consideration, there is nothing, not even a curved vacuum (Figs. 1b and 1c). This is unacceptable.
- 5) According to current understanding, metrics with opposite signatures (+ – – –) and (– + + +) are interchangeable and describe the same manifold using different notations. Within this axiomatic framework, the set of metrics (9) – (13) with signature (+ – – –) and the set of metrics (14) – (18) with signature (– + + +) describe the same spherical region of vacuum. That is, there is two-fold degeneracy. However, there is a difference between them, particularly in the opposite signs of some invariants (see Tables 2 and 3). Moreover, this degeneracy splits into two mutually opposite objects rotated by 90° relative to each other.

This leads to the following dilemma. On the one hand, the Kottler family of metrics (9) – (18) is extremely significant, since it is an interrelated set of solutions of one of the most fundamental equations of modern physics, (7). However, the invariant averaging of the two mutually opposite ensembles of these metrics leads to the problems listed above. Vacuum equation (7) offers no way out of this situation.

To solve these and many other problems, paper [19] proposed, as a hypothesis, an extended Einstein vacuum equation with a large number of positive and negative Λ -terms

$$R_{ik} - \frac{1}{2} g_{ik} (\sum_{m=1}^M \Lambda_m + \sum_{n=1}^N -\Lambda_n) = 0, \quad (33)$$

where $\Lambda_m = \frac{3}{r_m^2}$, $\Lambda_n = \frac{3}{r_n^2}$, r_m is the radius of the m -th core, r_n is the radius of the n -th anti-core.

M is the number of cores tending to ∞ and N is the number of anti-cores tending to ∞ .

Even if Λ_m and Λ_n decrease with increasing radii of the cores r_m and anti-cores r_n , the series $\sum_{m=1}^M \Lambda_m$ and $\sum_{n=1}^N \Lambda_n$ diverge (i.e., have no finite limit), but their difference may be finite *. This makes it possible to impose a condition on the extended Einstein vacuum equation (33).

$$\frac{1}{2} (\sum_{m=1}^M \Lambda_m + \sum_{n=1}^N -\Lambda_n) = \frac{1}{2} (\sum_{m=1}^M \Lambda_m - \sum_{n=1}^N \Lambda_n) = \Lambda_0 = \text{constant}. \quad (34)$$

*The difference in (34) is understood as the limit of the differences of partial sums under a specified common-cutoff prescription, provided that this limit exists; in the present model, the existence and value of this regularized limit are assumed as part of condition (34). Under this condition, Equation (33) reduces to Equation (7) with $\Lambda = \Lambda_0$, and its role consists in decomposing Λ_0 into contributions from individual cores and anti-cores, which leads to the interpretation (38) – (39) relating curvature to sources.

If $R_{ik} = \Lambda_0(x)g_{ik}$ holds on a connected manifold of dimension $n > 2$, the contracted Bianchi identity forces Λ_0 are constant (Schur's lemma; cf. [27]). Therefore, condition (34) is consistent with the field equation rather than an additional assumption. Taking the trace yields a constant scalar curvature

$$R = g^{ik} R_{ik} = g^{ik} g_{ik} \Lambda_0 = n \Lambda_0 = 4 \Lambda_0. \quad (35)$$

When condition (34) is satisfied, the extended vacuum equation (33) has the same properties as the Einstein vacuum equation (7) because, in this case, the covariant derivative of its left-hand side is also zero, similar to expression (6):

$$\nabla_j [R_{ik} - \frac{1}{2} g_{ik} (\sum_{m=1}^\infty \Lambda_m + \sum_{n=1}^\infty -\Lambda_n)] = \nabla_j [R_{ik} - g_{ik} \Lambda_0] = 0. \quad (36)$$

The extended vacuum equation (33) opens broad possibilities. For example, this equation can be expressed in the following form:

$$R_{ik} - \frac{1}{2} g_{ik} (\sum_{h=1}^H \Lambda_h + \sum_{p=1}^P -\Lambda_p) - \frac{1}{2} g_{ik} (\sum_{a=1}^A \Lambda_a + \sum_{b=1}^B -\Lambda_b) = 0, \quad (37)$$

where

$$\frac{1}{2} (\sum_{h=1}^H \Lambda_h + \sum_{p=1}^P -\Lambda_p) = T_s \quad \text{for cores and anti-cores of the microworld,} \quad (38)$$

$$\frac{1}{2} (\sum_{a=1}^A \Lambda_a + \sum_{b=1}^B -\Lambda_b) = \Lambda_B \quad \text{for cores and anti-cores of the macroworld,}$$

$$\text{and } (\Lambda_B + T_s) = \Lambda_0.$$

Then, considering the constancy of the scalar curvature (35), $R = 4\Lambda_0 = 4(\Lambda_B + T_s)$, equation (37) takes the form

$$R_{ik} - \Lambda_0 g_{ik} = R_{ik} - \frac{1}{2} R g_{ik} + \Lambda_0 g_{ik} = R_{ik} - \frac{1}{2} R g_{ik} + (T_s + \Lambda_B) g_{ik} = 0.$$

As a result, we obtain an equation with sources (38):

$$R_{ik} - \frac{1}{2} R g_{ik} + \Lambda_B g_{ik} = -T_s g_{ik}, \quad (39)$$

which, for $\Lambda_B = 1.089 \cdot 10^{-52} \text{ m}^{-2}$, has the form of an equation suitable for discussing cosmological problems.

Moving the constant term $T_s g_{ik}$ to the right-hand side corresponds to the standard interpretation of part of the cosmological term as the vacuum energy–momentum tensor, proportional to g_{ik} [28, 29].

4. Discussion of the results obtained

This study is devoted to unifying ten metric solutions of the Kottler family within a single system. The results obtained allowed several conclusions to be drawn and opened new directions for discussion.

4.1. Interpretation of the averaging results

The invariant averaging of scalars and curvature invariants over ensembles of Lorentzian metrics (9) – (12) and (14) – (17) yielded meaningful coordinate-independent quantities (Tables 4 and 5). These average values reflect the intrinsic geometric properties of the considered region of a curved vacuum, in contrast to noninvariant component-wise averaging, which leads to results that depend on the choice of coordinates and are physically ambiguous.

It is important to emphasize that averaging curvature invariants and scalars is admissible even when averaging the components of the metric is impossible, as shown by Zalaletdinov et al. [21–23]. Our results confirm this principle: while the direct sum of all ten metrics leads to the zero metric (20), which, in general, is not a solution of the vacuum equation, the invariant averaging of scalars associated with these metrics gives well-defined, finite quantities characterizing the averaged vacuum region.

The coincidence of the root-mean-square curvature invariants for ensembles of metrics with opposite signatures (Table 5) is particularly noteworthy. This suggests that despite the difference in the signs of some scalars (Tables 2 and 3), the underlying geometric structure of the averaged vacuum possesses a certain symmetry between the two ensembles. This symmetry may have profound implications for understanding the connection between matter and antimatter in the context of the proposed Hierarchical cosmological model [16–20].

4.2. KSdS core and anti-core as mathematical models

The mathematical models of the KSdS (Kottler–Schwarzschild–de Sitter) core and anti-core, constructed based on invariant averaging of the metrics of the Kottler family, represent stable, spherically closed formations of the Einstein vacuum. These objects are characterized as follows:

- 1) a pronounced spherical shape with a small central cavity (Schwarzschild “nucleolus”) and an extended de Sitter periphery;
- 2) an average relative elongation tending to infinity on the horizons of the individual metrics, which reflects the strong curvature of the vacuum in these regions.
- 3) complete mutual compensation of their manifestations: the KSdS core with signature $(+ - - -)$ and the KSdS anti-core with signature $(- + + +)$ satisfy the relation $(+ - - -) + (- + + +) = (0 0 0 0)$, analogous to a particle–antiparticle pair.

The analogy with particle–antiparticle pairs is further strengthened by the observation that coordinate maps with opposite signatures are rotated relative to each other by 90° [18]. This phase shift between the KSdS core and anti-core may provide a geometric explanation for the baryon asymmetry of matter, as noted in the Conclusion section.

However, these models have several limitations. First, the divergence of the relative elongation function on the horizons indicates that the average description breaks down in these regions, and a more accurate approach is required. Second, the models do not describe the interaction between the KSdS core and KSdS anti-core, nor do they explain what lies beyond the external de Sitter boundary or inside the central Schwarzschild cavity. These shortcomings motivated the introduction of the extended vacuum-Einstein equation (33).

4.3. Extended Einstein vacuum equation

The extended vacuum Einstein equation (33), with a large number of positive and negative Λ -terms, is a natural generalization of the original vacuum equation (7): As shown in Section 3.4, this equation reduces to the classical Einstein equation with an effective cosmological constant Λ_0 when condition (34) is satisfied. The expansion of Λ_0 into contributions from individual KSdS cores and anti-cores provides a mechanism for describing a large variety of vacuum formations of various scales, thereby solving the first problem identified in Section 3.4.

Moreover, the extended equation (33) opens the possibility of describing the interaction between KSdS cores and anti-cores [20]. This equation, taking the form of the standard Einstein equation with sources, allows part of the cosmological constant to be interpreted as the energy-momentum tensor of the vacuum, in accordance with the ideas of Zeldovich [28] and Weinberg [29].

The connection between the extended vacuum equation and hierarchical cosmological model [19, 20] is especially promising. The discrete set of radii r_{ai} and r_{bj} for cores and anti-cores of different scales provides a natural basis for describing the hierarchical structure of the universe from elementary particles to galactic clusters. This approach is consistent with the Clifford–Einstein–Wheeler program for the complete geometrization of physics, as outlined in the Introduction.

4.4. Comparison with other approaches to the averaging problem

The method of invariant averaging used in this study is like several existing approaches to the averaging problem in general relativity; however, it also has distinctive features.

Buchert’s formalism [22] also emphasizes the averaging of scalar quantities and the emergence of backreaction terms describing the influence of inhomogeneities on overall dynamics. However, Buchert’s approach is applied mainly to

cosmological scenarios with a single metric, whereas our method deals with an ensemble of different metric solutions of the same equation. The averaging of scalars over a family of metrics, as performed here, represents a new extension of this idea.

Zalaletdinov's macroscopic gravity [21] provides a fully covariant formalism for averaging tensor fields by using a bi-local parallel propagator. Because it is mathematically more rigorous, this approach is also more complex and has not yet been widely applied to specific physical problems. By contrast, our method is simpler and more convenient for practical use, although at the cost of abandoning the full tensor covariance in favor of scalar invariance.

Isaacson's averaging [24, 25] for gravitational waves also involves averaging over regions larger than the characteristic wavelength, which provides an effective energy-momentum tensor. Although Isaacson's method is intended for high-frequency perturbations on a fixed background, our approach involves averaging over an ensemble of exact solutions, which is conceptually different.

The key novelty of our approach lies in applying invariant averaging to the Kottler family of metrics, which possesses different signatures and describes different geometric structures. The resulting models of the KSdS core and anti-core represent a new class of vacuum formations that have not been previously discussed in the literature.

4.5. Limitations and open questions

Several limitations of this study should be acknowledged:

- 1) **Equal-weight hypothesis:** The arithmetic mean (25) and root-mean-square mean (32) assume that each metric in the ensemble contributes equally. However, this assumption does not follow from the vacuum equation (7) and requires additional justification. A more general approach could include weighted averaging, in which weights are determined by dynamical or probabilistic considerations.
- 2) **Degeneration problem:** It is assumed that the zero metrics (20), arising from the formal sum of all ten metrics, are not solutions of the vacuum equation; however, the limit $g_{ik} \rightarrow 0$ requires more careful consideration. As noted in Section 3, solving this problem would make it possible to introduce a zero signature into Algebra of Signature, which could have significant consequences for the classification of metric spaces.
- 3) **Interaction problem:** The extended vacuum equation (33) provides a basis for describing a multitude of cores and anti-cores, but the detailed dynamics of their interactions still requires further study [16–20]. It is unclear how the source T_s in equation (39) should be interpreted physically and whether it can explain the observed properties of dark matter and dark energy.
- 4) **Observational status:** The KSdS core and anti-core are mathematical constructs derived from invariant averaging of the Kottler family. Their connection with observed phenomena remains speculative and requires further investigation (see [16–20]).
- 5) **Choice of averaging procedure:** We considered four different operations with metric solutions (formal zero-sum, non-invariant component-wise average, invariant arithmetic mean of scalars, and invariant semi-quaternionic root-mean-square average). Although we prefer the fourth operation, others require further detailed consideration. The question of which averaging procedure is the most adequate for a given physical problem remains open.

4.6. Directions for future research

Several promising directions for further work follow from this study:

- **Development of the Algebra of Signature:** The introduction of a zero signature (neutral element) into Algebra signature [16, 17] would provide a more complete mathematical basis for describing the degenerate limit of a metric and could lead to new insights into the structure of spacetime.
- **Application to specific cosmological problems:** The Hierarchical cosmological model [19, 20], based on the extended vacuum equation (33), should be applied to specific cosmological problems, such as the formation of large-scale structures, nature of dark matter and dark energy, and baryon asymmetry of the universe.
- **Connection with quantum field theory:** The extended vacuum equation (33), with its many Λ -terms, is formally similar to the renormalization group equations in quantum field theory. Studying this connection could provide a bridge between general relativity and quantum mechanics in accordance with the Clifford–Einstein–Wheeler program.
- **Numerical modeling:** The models of the KSdS core and anti-core can be studied numerically to investigate their stability, interactions, and possible observational manifestations. This would require the development of numerical codes capable of working with an extended vacuum equation with many Λ -terms.
- **Generalization to other families of metrics:** The invariant averaging method developed here could be applied to other families of exact solutions of Einstein equations, such as the Kerr–Newman family or the Plebański–Demiański family, potentially leading to new insights into the structure of the vacuum.

In conclusion, invariant averaging of the Kottler family of metrics has led to significant results that open new perspectives for the averaging problem in general relativity and its applications to cosmology and elementary particle physics. Although several limitations and open questions remain, the approach developed here provides a solid basis for further research within the Geometrized Physics of Vacuum and the Clifford–Einstein–Wheeler program.

Conclusions

It is expected that different solutions of the same nonlinear differential equation with the same boundary conditions may be interrelated and describe a single system.

Excluding certain solutions to mathematical physics equations inevitably leads to dead ends, paradoxes, singularities, divergences, and other negative consequences. Conversely, considering all the possibilities provided by mathematics leads to logically complete models.

The purpose of this study is to show that considering all 10 possible Lorentzian metric solutions of the Kottler family (9) – (18) of the Einstein vacuum equation (7) within a single system yields meaningful results.

The main difficulties in unifying these metrics into a single system are: 1) the generally accepted statement that the sum of solutions of a nonlinear equation is not a solution of that equation; and 2) the “averaging problem,” which consists in the fact that averaging the components of the metric tensor g_{ik} over an extended region of spacetime is a non-invariant operation, while for an ensemble of metric solutions the result depends on how the points of different solutions are identified.

In this study, we propose metric solutions (9) – (18) in a single coordinate chart as being realized as parts of a unified system with a certain probability. In this case, referring to previous studies [21–23], we replaced the component-wise addition and averaging of the metric tensor components of the solutions of equation (7) with the arithmetic and root-mean-square averaging of the scalars and curvature invariants associated with the ensembles of metrics (9) – (12) and (14) – (17) (see Tables 4 and 5). With this approach, the averaged scalars and curvature invariants correspond to regions of spacetime (i.e., of the Einstein vacuum) averaged over ensembles of metrics.

This study considered four operations with metric solutions of the Kottler family (9) – (18): 1) the formal zero sum of intervals (19), 2) the non-invariant component-wise mean (21), 3) the invariant arithmetic mean of scalars (25) – (26), and 4) the invariant half-quaternion RMS mean (29) – (32). We give preference to the fourth operation; however, other operations require further detailed consideration.

As a result of the invariant averaging of the ensembles of metrics (9) – (12) with signature $(+---)$ and (14) – (17) with the opposite signature $(-+++)$, we obtained models of a stable spherically closed convexity of the vacuum and a stable spherically closed concavity of the vacuum, which we have conventionally called the *KSdS* core and *KSdS* anti-core, respectively, with the average relative elongations of the vacuum (25) or (32) (see Fig. 1b or 1c).

Because coordinate charts with opposite signatures located on either side of the metric sheet are perpendicular to each other, a core (analog of a particle) and an anti-core (analog of an antiparticle) are likewise rotated by 90° relative to one another (see the example of the two-sided case in §5.2 of [18]). It is possible that this phase shift underlies the explanation of the Sakharov conditions regarding the baryon asymmetry of matter.

The resulting models of the *KSdS* core and anti-core have several shortcomings and problems listed above, and [19] proposed an extended Einstein vacuum equation (EVE) with many $\pm\Lambda_i$ -terms (33):

$$R_{ik} - \frac{1}{2} g_{ik} (\sum_{m=1}^M \Lambda_m + \sum_{n=1}^N -\Lambda_n) = 0, \quad (33')$$

that considers all 16 ordered sign patterns of 4-dimensional metric spaces (signatures in the sense of the Algebra of Signatures [16, 17])

$$\text{sign}(ds^2) = \begin{pmatrix} (+ + + +) & (+ + + -) & (- + + -) & (+ + - +) \\ (- - - +) & (- + + +) & (- - + +) & (- + - +) \\ (+ - - +) & (+ + - -) & (+ - - -) & (+ - + +) \\ (- - + -) & (+ - + -) & (- + - -) & (- - - -) \end{pmatrix} \quad (40)$$

in our view, has the potential to outline paths for developing the Hierarchical Cosmological Model (“EVE project”) [16 – 20], aimed at advancing the Clifford–Einstein–Wheeler program for the geometrization of many branches of physics.

Author contributions

The author did all the research work for this study.

Competing interests

The author has declared that no competing interests exist.

Data access statement

Materials are available on request.

Ethics statement

All ethical standards are observed. Links to articles by other researchers are provided.

Funding statement

This research received no external funding.

Acknowledgments

I sincerely thank David Reid, Tatiana Levi, Georgy Usachev, and Eugene Etingin for their help, and I am grateful to Gennady Shipov and Evgeny Guborev, Academicians of the Russian Academy of Natural Sciences, for fruitful discussions of the issues considered here.

Generative AI tools were used in the preparation of this manuscript: Claude Opus 5 (Anthropic) for translation from Russian to English, language editing, and suggested wording of several technical clarifications, and GPT-5.6 Sol (OpenAI) for independent checking of the manuscript and clarification. The author reviewed all the AI-assisted content and took full responsibility for the final text.

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