A Valuation Criterion for Totient-Periodic Stirling Transforms

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Abstract

Stirling transforms inherit modular periodicity from their fixed columns only when the weights suppress the additional prime-power factors in the column periods. We establish a local p-adic valuation criterion for an integer-weight Stirling transform and prove that, for every modulus greater than two, Euler's totient is an eventual period. The proof involves truncating the transform modulo each prime power, applying Kwong's fixed-column period bounds and handling the exceptional third column at the prime two on its own. We then apply the criterion to the signed binomial-recurrence sequence; the formal exponential generating function of this sequence yields integral Stirling weights such that the odd and even prime valuations satisfy the criterion, thereby proving Bala's modular-periodicity conjecture for this sequence. For every integral substitution $G(e^x - 1)$, we obtain the stronger universal eventual period given by the Carmichael function.

1 Introduction

For an integer sequence (v_n), a modulus $m \geq 1$, and an integer $T \geq 1$, we say that T is an *eventual period modulo m* if there is an integer $N \geq 0$ such that

$$v_{n+T} \equiv v_n \pmod{m} \qquad (n \geq N).$$

The least eventual period, when needed, divides every eventual period.

We write $S(n, r) = \left\{ {n \atop r} \right\}$ for the Stirling number of the second kind, namely the number of partitions of an n-element set into r nonempty blocks. Kwong determined the minimum periods of the fixed-column sequences $S(n, r)$ modulo prime powers and, in particular, established the divisibility bounds used below [1]. His results describe a single Stirling column; an infinite weighted sum requires a separate argument because the number of columns grows with n.

Peter Bala proposed a broader periodicity principle in connection with the Fubini numbers, also called the ordered Bell numbers and with the Stirling transform of the distinct-part partition numbers [2, 3]. The principle asserts that an integer sequence with exponential generating function $G(e^x - 1)$, where $G \in \mathbb{Z}[[x]]$, should be eventually periodic modulo every positive integer m, with a period dividing Euler's totient $\varphi(m)$.

Also, Prabha Sivaramannair recorded the integer sequence defined by $a_0 = 1$ and, for $n > 0$, by the signed binomial recurrence [4]

$$a_n = (-1)^n + \frac{1}{2}\sum_{j=1}^{n}\left(1 - (-1)^j - (-2)^j\right)\binom{n}{j} a_{n-j}. \tag{1}$$

Its first terms are

$$1,1,3,16,105,856,8433,96916,1272225,\ldots.$$

We therefore call it the *signed binomial-recurrence sequence*. Bala conjectured that, for every integer $m > 2$, its reduction modulo m is eventually periodic with a period dividing $\varphi(m)$ [4].

Schreyer was able to show that the Fubini numbers and their r-Fubini generalisations have eventual periods that are determined by the Carmichael function [5]. Although this result resolves an important class of sequences, it does not prove Bala's general principle based on integral substitution and fails to include the signed binomial-recurrence sequence, since the natural auxiliary series associated with it has a nonintegral coefficient. The main tool is a criterion for a more general class of sequences. If

$$u_n = \sum_{r=0}^{n} d_r\, S(n,r),$$

then sufficiently strong p-adic growth of d_r truncates the sum modulo each prime power. A second, logarithmic valuation bound cancels the extra power of p in the known period bound for the rth Stirling column. This leaves precisely the prime-power factor of Euler's totient. The only boundary obstruction occurs for $S(n,3)$ modulo powers of 2; Kwong's sharp formula for this column resolves it.

In section 3 we prove the valuation transfer theorem. In section 4 we prove Bala's integral-substitution conjecture with the stronger eventual period $\lambda(m)$, where λ is the Carmichael function. In section 5 we derive the Stirling transform of the signed binomial-recurrence sequence and verify the valuation hypotheses prime by prime. The coefficient $3/2$ of its auxiliary series at degree three explains why the integral-substitution argument alone is insufficient.

2 Periods of fixed Stirling columns

Let us write $v_p(c)$ for the p-adic valuation of a nonzero integer c, with $v_p(0) = +\infty$. For p prime and $r \geq 2$, we define

$$\ell_p(r) = \min\{b \geq 1 : r \leq p^b\}; \qquad p^{\ell_p(r)-1} < r \leq p^{\ell_p(r)}. \tag{2}$$

We use the following consequence of Kwong's study of minimum periods of Stirling numbers [1, Corollary 3.4 and Theorem 3.6].

Proposition 2.1 (Kwong). *Let p be prime, $q \geq 1$, $r \geq 2$, and $b = \ell_p(r)$. The fixed-column sequence $S(n,r)$ modulo p^q is eventually periodic with a period dividing*

$$(p-1)p^{q+b-2}. \tag{3}$$

For $p = 2$ the two small columns satisfy

$$\mu(1;2^q) = \mu(2;2^q) = 1$$

and

$$\mu(3;2^q) = \begin{cases} 2, & q = 1,2, \\ 2^{q-1}, & q \geq 3, \end{cases} \tag{4}$$

where $\mu(r;2^q)$ denotes the least eventual period of $S(n,r)$ modulo 2^q.

The divisibility form of (3), rather than the exact minimum period, is what is needed below. We need to notice that multiplication by an integer having p-adic valuation t reduces the effective modulus from p^{α} to $p^{\alpha-t}$. This elementary observation is the source of the valuation threshold in the next theorem.

3 A valuation transfer theorem

Theorem 3.1 (Totient-period transfer). *First let $(d_r)_{r\geq 0}$ be integers and we define*

$$u_n = \sum_{r=0}^{n} d_r\, S(n,r). \tag{5}$$

Suppose that the following conditions hold.

(i) *For every prime p, $v_p(d_r) \to +\infty$ as $r \to \infty$.*

(ii) *For every odd prime p and every $r \geq 2$,*

$$v_p(d_r) \geq \ell_p(r) - 1.$$

(iii) *For every $r \geq 4$,*

$$v_2(d_r) \geq \ell_2(r) - 1.$$

Then, for every integer $m > 2$, $\varphi(m)$ is an eventual period of (u_n) modulo m. In particular, the least eventual period divides $\varphi(m)$.

Proof. we fix a prime power $p^{\alpha} \| m$. By condition (i), there is an R such that

$$p^{\alpha} \mid d_r \qquad (r \geq R).$$

For $n \geq R$, the transform (5) therefore reduces modulo p^{α} to the fixed finite sum

$$u_n \equiv \sum_{r=0}^{R-1} d_r\, S(n,r) \pmod{p^{\alpha}}.$$

It is enough to assign to every nonzero summand an eventual period dividing the desired global period.

First suppose that p is odd. The columns $r = 0$ and $r = 1$ are eventually zero and constant, respectively. For $r \geq 2$, put

$$t = v_p(d_r).$$

If $t \geq \alpha$, the summand vanishes modulo p^{α}. Otherwise set $q = \alpha - t$ and $b = \ell_p(r)$. Proposition 2.1 shows that $d_r S(n,r)$ has an eventual period dividing

$$(p-1)p^{q+b-2} = (p-1)p^{\alpha-t+b-2}.$$

Condition (ii), namely $t \geq b - 1$, implies that this number divides

$$(p-1)p^{\alpha-1} = \varphi(p^{\alpha}).$$

Thus the entire finite sum modulo p^{α} has an eventual period dividing $\varphi(p^{\alpha})$.

Now let $p = 2$. The columns $r = 0,1,2$ are safe, $S(n, 0)$ is eventually zero, while $S(n, 1)$ is constant and $S(n, 2)$ is eventually constant modulo every power of 2. If $r \geq 4$, the preceding argument, using condition (iii) and (3), gives a period dividing $2^{\alpha-1} = \varphi(2^{\alpha})$ whenever the summand is nonzero.

It remains to consider $r = 3$. If $\alpha \geq 2$, multiplication by d_3 can only lower the effective exponent and the sharp period (4) always divides $2^{\alpha-1}$. If $\alpha = 1$, this column has period dividing 2. The excluded modulus $m = 2$ is the only case in which this would cause trouble: when $2 \| m$ and $m > 2$, m has an odd prime divisor and hence $\varphi(m)$ is even.

Finally, $\varphi(p^{\alpha})$ divides $\varphi(m)$ for every odd prime power $p^{\alpha} \| m$, and the same is true for $p = 2$ when $\alpha \geq 2$. The exceptional local period 2 when $2 \| m$ also divides $\varphi(m)$. Taking the maximum of the finitely many transient thresholds and applying the Chinese remainder theorem gives

$$u_{n+\varphi(m)} \equiv u_n \pmod{m}$$

for all sufficiently large n.

Remark 3.2. *The restriction $m > 2$ is structural. The theorem deliberately imposes no condition on d_3 at the prime 2, and $S(n, 3)$ has eventual period 2 modulo 2, whereas $\varphi(2) = 1$.*

4 Integral substitutions and the Carmichael bound

Bala's integral-substitution conjecture has a direct proof that is stronger than Theorem 3.1 in its period bound. For $m \geq 2$, let $\lambda(m)$ denote the exponent of the finite group $(\mathbb{Z}/m\mathbb{Z})^{\times}$, and set $\lambda(1) = 1$.

Theorem 4.1 (Integral substitution). We *let*

$$G(z) = \sum_{r \geq 0} g_r \, z^r \in \mathbb{Z}[[z]],$$

and define the coefficients (w_n) by

$$\sum_{n \geq 0} w_n \frac{x^n}{n!} = G(e^x - 1). \tag{6}$$

Then $w_n \in \mathbb{Z}$ for every $n \geq 0$. For every $m \geq 1$, the Carmichael function $\lambda(m)$ is an eventual period of (w_n) modulo m. Consequently the least eventual period divides $\lambda(m)$ and hence divides $\varphi(m)$.

Proof. The standard formal identity

$$(e^x - 1)^r = r! \sum_{n \geq r} S(n, r) \frac{x^n}{n!} \tag{7}$$

gives

$$w_n = \sum_{r=0}^{n} g_r \, r! \, S(n, r). \tag{8}$$

This finite sum proves first that w_n is an integer. For $r \geq m$ one has $m \mid r!$, so modulo m the sum may be truncated at $r = m - 1$. Moreover,

$$r! \, S(n, r) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^n. \tag{9}$$

For every fixed integer j, the sequence j^n has $\lambda(m)$ as an eventual period modulo m. Indeed, modulo a prime power $p^\alpha \| m$, the powers j^n eventually vanish if $p \mid j$; if $p \nmid j$, then $j^{\lambda(m)} \equiv 1 \pmod{p^\alpha}$ because $\lambda(p^\alpha) \mid \lambda(m)$. The Chinese remainder theorem gives the assertion modulo m. Equations (8) and (9) express w_n modulo m as a finite integer linear combination of these power sequences. Finally, $\lambda(m) \mid \varphi(m)$ because the exponent of a finite group divides its order.

Corollary 4.2. *Bala's modular-periodicity conjecture for integral substitutions $G(e^x - 1)$ holds, with the stronger eventual period $\lambda(m)$ in place of $\varphi(m)$.*

Remark 4.3. *Schreyer proved a Carmichael-period upper bound for Fubini and r-Fubini numbers by related finite exponential-sum methods [5]. Theorem 4.1 isolates the general formal-series mechanism. The signed binomial-recurrence sequence lies just outside this integral setting because its auxiliary series belongs to $\mathbb{Z}[1/2][[z]]$, not to $\mathbb{Z}[[z]]$.*

5 The signed binomial-recurrence sequence

We now derive a Stirling transform for (1) and verify the hypotheses of the Theorem 3.1.

5.1 The exponential generating function

Let us

$$A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}.$$

Multiplying (1) by $x^n/n!$ and summing over $n \geq 1$ gives

$$A(x) - 1 = e^{-x} - 1 + \frac{1}{2}(e^x - e^{-x} - e^{-2x} + 1)A(x).$$

Solving for $A(x)$ yields the recorded generating function

$$A(x) = \frac{2e^x}{1 + e^x + e^{2x} - e^{3x}}. \tag{10}$$

We set $z = e^x - 1$. A direct simplification of (10) gives

$$A(x) = C(e^x - 1), \qquad C(z) = \frac{1+z}{1 - z^2 - z^3/2} = \sum_{r \geq 0} c_r z^r. \tag{11}$$

We define

$$d_r = r!\, c_r. \tag{12}$$

Using (7) in (11), and comparing coefficients of $x^n/n!$, gives the exact identity

$$a_n = \sum_{r=0}^{n} d_r S(n, r). \tag{13}$$

5.2 Integrality of the weights

From (11),

$$c_0 = c_1 = c_2 = 1, \qquad c_r = c_{r-2} + \frac{1}{2} c_{r-3} \qquad (r \geq 3).$$

After multiplication by $r!$, this becomes

$$d_r = r(r-1)d_{r-2} + \frac{r(r-1)(r-2)}{2} d_{r-3} \qquad (r \geq 3), \tag{14}$$

with $d_0 = 1, d_1 = 1, d_2 = 2$. Both coefficients on the right of (14) are integers. Hence $d_r \in \mathbb{Z}$ for every r. The first few values are

$$1,1,2,9,36,240,1620,13860,131040,1406160, \ldots.$$

In particular, (13) also proves directly that every a_n is an integer, despite the factor $1/2$ in the defining recurrence.

5.3 Valuation estimates

Lemma 5.1. *The integers d_r defined by (12) satisfy*

$$v_p(d_r) \geq v_p(r!) \text{for every odd prime } p, \tag{15}$$

$$v_2(d_r) \geq v_2(r!) - \lfloor \tfrac{r}{3} \rfloor = r - s_2(r) - \lfloor \tfrac{r}{3} \rfloor, \tag{16}$$

where $s_2(r)$ is the number of ones in the binary expansion of r. Consequently $v_p(d_r) \to +\infty$ for every prime p.

Proof. The coefficients of $C(z)$ lie in $\mathbb{Z}[1/2]$, so $v_p(c_r) \geq 0$ for every odd prime p. Equation (15) follows from $d_r = r!\, c_r$.

For the prime 2, expand formally as

$$C(z) = (1+z) \sum_{h \geq 0} \left(z^2 + \frac{z^3}{2} \right)^h. \tag{17}$$

Suppose a monomial contributing to the coefficient of z^r uses k of the factors $z^3/2$. Its degree is at least $3k$, so $k \leq \lfloor r/3 \rfloor$. Thus

$$2^{\lfloor r/3 \rfloor} c_r \in \mathbb{Z},$$

which proves the inequality in (16). Its equality uses Legendre's formula $v_2(r!) = r - s_2(r)$.

The odd-prime lower bound tends to infinity. For $p = 2$, use $s_2(r) \leq 1 + \log_2 r$ in (16); the resulting lower bound tends to infinity as well.

Lemma 5.2. *The weights d_r satisfy*

$$v_p(d_r) \geq \ell_p(r) - 1 \qquad \text{for every odd prime } p \text{ and } r \geq 2,$$

$$v_2(d_r) \geq \ell_2(r) - 1 \qquad \text{for every } r \geq 4.$$

Proof. Let us p be odd and put $b = \ell_p(r)$. For $b = 1$ there is nothing to prove. If $b \geq 2$, then $r > p^{b-1}$, and Lemma 5.1 gives

$$v_p(d_r) \geq v_p(r!) \geq \lfloor \frac{r}{p} \rfloor \geq p^{b-2} \geq b - 1.$$

The last inequality follows immediately by induction from $p \geq 3$.

Now put $b = \ell_2(r)$. For $4 \le r \le 9$, the lower bound in (16) and the required value $b - 1$ are, respectively,

r	4	5	6	7	8	9
$r - s_2(r) - \lfloor r/3 \rfloor$	2	2	2	2	5	4
$b - 1$	1	2	2	2	2	3

so the claim holds. Suppose $r \ge 10$. If $b = 4$, then

$$r - s_2(r) - \lfloor \frac{r}{3} \rfloor \ge 10 - 4 - 3 = 3 = b - 1,$$

and $r - \lfloor r/3 \rfloor$ is nondecreasing, so this covers $10 \le r \le 16$.

Finally let $b \ge 5$. The elementary inequality $2^{b-1} \ge 3b$ holds for $b = 5$ and is preserved by induction. Since $r > 2^{b-1}$,

$$r - \lfloor \frac{r}{3} \rfloor = \lceil \frac{2r}{3} \rceil > \frac{2^b}{3} \ge 2b.$$

The left side is an integer, hence it is at least $2b + 1$. Since $s_2(r) \le b$, formula (16) gives

$$v_2(d_r) \ge b + 1 \ge b - 1,$$

as required.

Theorem 5.3 (Bala's signed-recurrence conjecture). *Let (a_n) be defined by (1). For every integer $m > 2$, $\varphi(m)$ is an eventual period of (a_n) modulo m. Consequently the least eventual period divides $\varphi(m)$.*

Proof. Equation (13) represents (a_n) in the form required by Theorem 3.1. Lemma 5.1 gives the p-adic tail condition, and Lemma 5.2 gives both logarithmic valuation conditions. The conclusion is therefore exactly that of Theorem 3.1.

Independent exact-arithmetic checks verified (13) for $0 \le n \le 80$, the valuation bounds for $p \in \{2,3,5,7,11,13,17,19\}$ and $2 \le r \le 80$ (with $r \ge 4$ when $p = 2$), and $a_{n+\varphi(m)} \equiv a_n \pmod m$ for $3 \le m \le 100$ and $200 \le n \le 400$.

Theorem 4.1 cannot be applied directly. There the factor $r!$ truncates the transform, and (9) reduces every surviving term to a power j^n. Here $c_3 = 3/2$. Estimate (16) controls this denominator, while Kwong's exact 2-adic period for $S(n, 3)$ supplies the exceptional local information.

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