Divergence verification and doubts about the theory of limits in calculus

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Abstract: In the two definitions of divergence, the flux-density divergence provides the physical definition of divergence, whereas the nabla-operator divergence serves as a mathematical tool for calculation and derivation. Using examples derived from Coulomb's law of electrostatics, and based on the theory of limits, this paper reveals that the flux-density divergence and the nabla-operator divergence are non-equivalent. The differential form of Gauss's flux theorem in Maxwell's equations does not hold. Furthermore, keeping or discarding relatively higher-order infinitesimals yields different results of the flux-density divergence, which contradicts the theory of limits in calculus. This paper proposes the effective order infinitesimal theorem to enhance the theory of limits. In the expression of the flux-density divergence, the effective order of its infinitesimal variables is 3. However, during the mathematical derivation of the nabla-operator divergence from the flux-density divergence, only 2nd-order infinitesimal variables are retained. Gauss's flux theorem in differential form does not hold, and the theory of limits in calculus exhibits deficiencies. Physics and mathematics are undergoing a revolutionary scientific transformation.

Keywords: The flux-density divergence, The nabla-operator divergence, Coulomb's law, Gauss's flux theorem, Theory of limits, The effective order infinitesimal theorem.

1. Introduction

Coulomb's law describes the electrostatic force acting between two point charges in a vacuum. For two point charges, q_1 and q_2, separated in space, the electric force exerted by q_1 on q_2 can be expressed as follows:

$$\boldsymbol{F} = \frac{q_1 q_2}{4\pi\,\varepsilon_0\, r^2}\boldsymbol{r} \qquad (1-1)$$

where r is the distance between q_1 and q_2, **r** is the unit vector directed from q_1 to q_2, and ε_0 is the vacuum permittivity.

The electric field intensity $\boldsymbol{E}$ is defined as the electric force per unit test charge. For a point charge Q at a distance r, the electric field intensity is expressed as follows:

$$\boldsymbol{E} = \frac{\mathrm{Q}}{4\pi\,\varepsilon_0\, r^2}\boldsymbol{r} \qquad (1-2)$$

Gauss's flux theorem follows directly from Coulomb's law of electrostatics. As shown in Fig. 1.1, for a differential surface element d**S** in three-dimensional space, the differential electric flux $d\Phi_E$ is determined by the product of the electric field intensity $\boldsymbol{E}(x, y, z)$ and the effective area of the surface element d**S**:

$$d\Phi_E = \boldsymbol{E}\,(x, y, z) \cdot d\mathbf{S} = E\, dS \,\mathrm{Cos}\,\theta$$

where θ is the angle between the electric field $\boldsymbol{E}(x, y, z)$ and the outward normal to the surface element d**S**.

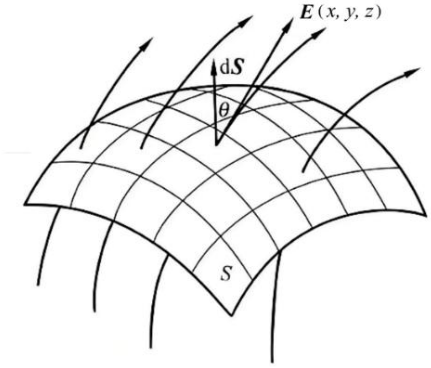

Fig. 1.1 Electric flux through a surface

In three-dimensional space, the electric flux Φ_E through a general surface S is given by the surface integral:

$$\Phi_E = \oiint_S \boldsymbol{E}(x, y, z) \bullet \mathrm{d}\mathbf{S} \qquad (1-3)$$

Gauss's flux theorem, derived from Coulomb's law of electrostatics, states that the electric flux through a closed surface S is proportional to the net charge Q enclosed by that surface. Mathematically, this relation is expressed as follows:

$$\oiint_S \boldsymbol{E}(x, y, z) \bullet \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0} \qquad (1-4)$$

$$\nabla \bullet \boldsymbol{E}(x, y, z) = \frac{\rho(x, y, z)}{\varepsilon_0} \qquad (1-5)$$

Equation (1-4) is Gauss's flux theorem in integral form, serves as a cornerstone of classical electromagnetism. Equation (1-5) represents the corresponding differential form. Gauss's flux theorem in differential form is also referred to as Gauss's divergence theorem, where ρ(x, y, z) denotes the charge density at a specific point in space.

2. Definitions of divergence and the nabla-operator (∇) [1] [2]

Definition of the divergence of a vector *F*(*x, y, z*)

Let a closed surface S enclose a three-dimensional solid volume ΔV. As ΔV → 0, the divergence of the vector ***F***(*x, y, z*) is defined as the limit of the ratio of the flux of ***F***(*x, y, z*) through S to the volume ΔV:

$$\operatorname{div} \boldsymbol{F}(x, y, z) = \lim_{\Delta V \to 0} \frac{\oiint_S \boldsymbol{F}(x, y, z) \bullet dS}{\Delta V} \qquad (2-1)$$

Equation (2-1) defines vector divergence in terms of the flux volume density, which is referred to as the **flux-density divergence.**

Definition of the nabla-operator (∇)

In a rectangular coordinate system, the nabla-operator is defined as follows:

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} \qquad (2-2)$$

where ***i, j,*** and ***k*** are the unit vectors along the respective coordinate axes.

Using the nabla-operator, the divergence of the vector $\boldsymbol{F}(x, y, z)$ can be expressed as follows:

$$\text{div}\,\boldsymbol{F}(x,y,z) = \nabla \bullet \boldsymbol{F}(x,y,z) \qquad (2-3)$$

Equation (2-3) defines vector divergence in terms of the partial differential nabla-operator, which is referred to as the **nabla-operator divergence.**

For a three-dimensional vector

$$\boldsymbol{F}(x,y,z) = \boldsymbol{F}x\,\boldsymbol{i} + \boldsymbol{F}y\,\boldsymbol{j} + \boldsymbol{F}z\,\boldsymbol{k}$$

where $\boldsymbol{F}x$, $\boldsymbol{F}y$, and $\boldsymbol{F}z$ are the projections of the vector $\boldsymbol{F}(x, y, z)$ on the x, y, and z axes, respectively.

According to the definitions of divergence and the nabla-operator, the divergence of the vector $\boldsymbol{F}(x, y, z)$ can be expressed as follows:

$$\nabla \bullet \boldsymbol{F}(x,y,z) = \frac{\partial \boldsymbol{F}x}{\partial x} + \frac{\partial \boldsymbol{F}y}{\partial y} + \frac{\partial \boldsymbol{F}z}{\partial z}$$

$$\nabla \bullet \boldsymbol{F}(x,y,z) = \lim_{\Delta V \to 0} \frac{\oiint_S \boldsymbol{F}(x,y,z) \bullet dS}{\Delta V} \qquad (2-4)$$

In the two definitions of divergence presented in Equation (2-4), the right-hand side denotes the **flux-density divergence,** while the left-hand side denotes the **nabla-operator divergence,** The flux-density divergence provides the physical definition of divergence, whereas the nabla-operator divergence serves as a mathematical tool for calculation and derivation [3] [4] [5]. In theoretical physics, particularly in Maxwell's equations and electrodynamics, the nabla-operator divergence is generally employed. However, the recent research reveals that **the flux-density divergence** and the **nabla-operator divergence** are non-equivalent, which poses a serious challenge to theoretical physics and fundamental calculus.

3. Divergence verification and doubts about the theory of limits

The development of divergence is closely related to Gauss's flux theorem in electromagnetism. Next, using examples derived from Coulomb's law, we verify the divergence and the theory of limits by selectively discarding or retaining higher-order infinitesimals.

Consider a static point charge q in a vacuum. Choose an arbitrary point P_1 at distance x from q. Taking the point charge q as the origin O and the line passing through q and P_1 as the x-axis, the electric field at any point on the x-axis has only an x-component, yielding a one-dimensional electrostatic field. As illustrated in Figure 3.1, define an O-xyz coordinate system with q at the origin and consider a cubic infinitesimal element of volume $\Delta x\Delta y\Delta z$ ($\Delta x = \Delta y = \Delta z$). The face normal to the x-axis at the left is ΔS_1, centered at P_1; the corresponding right face is ΔS_2, centered at P_2. The four faces parallel to the x-axis are denoted ΔS_3, with a representative center P_3. The field at P_1 has only an x-component $\boldsymbol{E}(x)$, and at P_2 only $\boldsymbol{E}(x + \Delta x)$. The field at P_3 is $\boldsymbol{E}_3$ and makes an angle $\Delta\Omega/2$ with the x-axis. The components of the field at P_1 are $\boldsymbol{E}x = \boldsymbol{E}(x)$, $\boldsymbol{E}y = 0$, and $\boldsymbol{E}z = 0$.

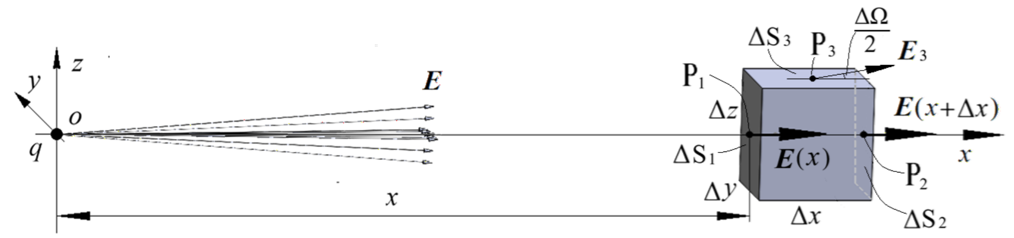

Figure 3.1 Divergence of the electric field in a cubic infinitesimal element

According to Coulomb's law, the electric field intensity at point P_1 is

$$\boldsymbol{E}(x) = \frac{q}{4\pi\varepsilon_0 x^2}$$

According to Equation (2-4), the nabla-operator divergence of the electric field at point P_1 is as follows:

$$\nabla \bullet \boldsymbol{E}(x) = \frac{\mathrm{d}\boldsymbol{E}(x)}{\mathrm{d}x} = \frac{q}{4\pi\varepsilon_0}\,\mathrm{d}\left(\frac{1}{x^2}\right)/\mathrm{d}x$$

$$\nabla \bullet \boldsymbol{E}(x) = -\frac{q}{2\pi\varepsilon_0 x^3} \qquad (3-1)$$

In Figure 3.1, the electric flux through ΔS_1 is as follows:

$$\boldsymbol{E}_1 \bullet \Delta S_1 = \frac{q\,\Delta x^2}{4\pi\varepsilon_0 x^2}$$

The electric flux through surface element ΔS_2 is as follows:

$$\boldsymbol{E}_2 \bullet \Delta S_2 = \frac{q\,\Delta x^2}{4\pi\varepsilon_0 (x+\Delta x)^2} = \frac{q\,\Delta x^2}{4\pi\varepsilon_0 (x^2 + 2x\Delta x + \Delta x^2)}$$

The field at P_3 is $\boldsymbol{E}_3$ and makes an angle $\Delta\Omega/2$ with the x-axis. Let $\mathrm{Sin}(\Delta\Omega/2) = \Delta x/2x$. Then the electric flux through the four surface elements ΔS_3 is as follows:

$$4\,\boldsymbol{E}_3\bullet\Delta S_3=\frac{4\,q\,\Delta x^2\,\mathrm{Sin}(\Delta\Omega/2)}{4\,\pi\,\varepsilon_0\,((x+\Delta x/2)^2+(\Delta x/2)^2)}=\frac{q\,\Delta x^3}{2\pi\,\varepsilon_0\,x(x^2+x\Delta x+\Delta x^2/2)}$$

Among these surface fluxes above, the fluxes through elements ΔS_1 and ΔS_2 are second-order infinitesimals, whereas the flux through ΔS_3 is a third-order infinitesimal.

Using the theory of limits, retain lower-order infinitesimals and discard relatively higher-order ones. In Figure 3.1, we keep the second-order infinitesimal electric fluxes through the surface elements ΔS_1 and ΔS_2, and discard the third-order infinitesimal flux through ΔS_3. Then the net electric flux through the closed surface of the cubic infinitesimal element $\Delta x\Delta y\Delta z$ is given by

$$\oiint_S \boldsymbol{E}\,(x,y,z)\bullet dS=\boldsymbol{E}_2\bullet\Delta S_2-\boldsymbol{E}_1\bullet\Delta S_1=\frac{q\,\Delta x^2}{4\pi\,\varepsilon_0\,(x^2+2x\Delta x+\Delta x^2)}-\frac{q\,\Delta x^2}{4\,\pi\,\varepsilon_0\,x^2}$$

$$=\frac{q\,\Delta x^2}{4\,\pi\,\varepsilon_0}\left(\frac{1}{(x^2+2x\Delta x+\Delta x^2)}-\frac{1}{x^2}\right)$$

$$=\frac{q\,\Delta x^2}{4\,\pi\,\varepsilon_0}\left(\frac{-2x\Delta x-\Delta x^2}{x^2(x^2+2x\Delta x+\Delta x^2)}\right)=\frac{q\,\Delta x^2}{4\,\pi\,\varepsilon_0}\left(\frac{-2x\Delta x}{x^2(x)^2}\right)$$

$$\oiint_S \boldsymbol{E}\,(x,y,z)\bullet dS=-\frac{q\,\Delta x^3}{2\,\pi\,\varepsilon_0\,x^3}\qquad(3-2)$$

Substituting $\Delta V=\Delta x^3$ and the electric flux from Equation (3-2) into Equation (2-4), the flux-density divergence at point P_1 in a vacuum is expressed as:

$$\lim_{\Delta V\to 0}\frac{\oiint_S \boldsymbol{E}\,(x,y,z)\bullet dS}{\Delta V}=-\lim_{\Delta x\to 0}\frac{q\Delta x^3}{2\pi\,\varepsilon_0\,x^3\Delta x^3}$$

$$\lim_{\Delta V\to 0}\frac{\oiint_S \boldsymbol{E}\,(x,y,z)\bullet dS}{\Delta V}=-\frac{q}{2\pi\,\varepsilon_0\,x^3}\qquad(3-3)$$

The results of Equations (3-1) and (3-3) are identical and yield nonzero negative values. Therefore, Equation (2-4) is valid. This indicates that, upon discarding higher-order infinitesimals, the nabla-operator divergence can be derived from the flux-density divergence. However, because point P_1 lies in a vacuum where the charge density $\rho=0$, the differential form of Gauss's flux theorem in Equation (1 - 5) is not valid. Consequently, in Maxwell's equations, Gauss's flux theorem in differential form does not hold in the electrostatic field.

In the previous calculation of the flux through the closed surface of the cube, we discarded the higher-order infinitesimal contributions from the surface element ΔS_3. According to the theory of limits in calculus, discarding higher-order infinitesimal does not change the result. To confirm this explicitly, we now retain the higher-order infinitesimal associated with ΔS_3 when computing the net flux. Including the flux through ΔS_3, the net electric flux through the closed surface of the cubic infinitesimal element $\Delta x \Delta y \Delta z$ is given by

$$\oiint_S \boldsymbol{E}(x,y,z) \bullet dS = \boldsymbol{E}_2 \bullet \Delta S_2 + 4\,\boldsymbol{E}_3 \bullet \Delta S_3 - \boldsymbol{E}_1 \bullet \Delta S_1$$

$$= \frac{q\,\Delta x^2}{4\pi\varepsilon_0\,(x^2+2\,x\Delta x+\Delta x^2)} + \frac{q\,\Delta x^3}{2\pi\varepsilon_0\,x\,(x^2+x\Delta x+\Delta x^2/2)} - \frac{q\,\Delta x^2}{4\pi\varepsilon_0\,x^2}$$

$$= \frac{q\Delta x^2}{4\pi\varepsilon_0}\left(\frac{1}{(x^2+2\,x\Delta x+\Delta x^2)} + \frac{2\,\Delta x}{x\,(x^2+x\Delta x+\Delta x^2/2)} - \frac{1}{x^2}\right)$$

$$= \frac{q\,\Delta x^2}{4\pi\varepsilon_0}\left(\frac{x^2(x^2+x\Delta x+\Delta x^2/2)+(2x^3\Delta x+4\,x^2\Delta x^2+2x\Delta x^3)}{x^2(x^2+2\,x\Delta x+\Delta x^2)\,(x^2+x\Delta x+\Delta x^2/2)}\right)$$

$$-\frac{q\,\Delta x^2}{4\pi\varepsilon_0}\left(\frac{x^2(x^2+x\Delta x+\Delta x^2/2)+\ (2x^3\Delta x+\ 2x^2\Delta x^2+x\Delta x^3)}{x^2(x^2+2\,x\Delta x+\Delta x^2)\,(x^2+x\Delta x+\Delta x^2/2)}\right)$$

$$-\frac{q\,\Delta x^2}{4\pi\varepsilon_0}\left(\frac{(x^2\Delta x^2+x\Delta x^3+\Delta x^4/2)}{x^2(x^2+2\,x\Delta x+\Delta x^2)\,(x^2+x\Delta x+\Delta x^2/2)}\right)$$

$$= \frac{q\,\Delta x^2}{4\pi\varepsilon_0}\left(\frac{x^2\Delta x^2-\Delta x^4/2}{x^2(x^2+2\,x\Delta x+\Delta x^2)\,(x^2+x\Delta x+\Delta x^2/2)}\right) = \frac{q\,\Delta x^2}{4\pi\varepsilon_0}\left(\frac{x^2\Delta x^2}{x^6}\right)$$

$$\oiint_S \boldsymbol{E}\,(x,y,z) \bullet dS = \frac{q\,\Delta x^4}{4\pi\,\varepsilon_0\,x^4} \qquad (3-4)$$

Substituting $\Delta V = \Delta x^3$ and the electric flux from Equation (3-4) into Equation (2-4), the flux-density divergence at point P_1 in a vacuum is expressed as:

$$\lim_{\Delta V\to 0}\frac{\oiint_S \boldsymbol{E}\,(x,y,z)\bullet dS}{\Delta V} = \lim_{\Delta x\to 0}\frac{q\,\Delta x^4}{4\pi\,\varepsilon_0\,x^4\Delta x^3} = \lim_{\Delta x\to 0}\frac{q\,\Delta x}{4\pi\,\varepsilon_0\,x^4}$$

$$\lim_{\Delta V\to 0}\frac{\oiint_S \boldsymbol{E}\,(x,y,z)\bullet dS}{\Delta V} = 0 \qquad (3-5)$$

The flux-density divergence in Equation (3-5) is zero, whereas the flux-density divergence in Equation (3-3) is a nonzero negative value. Keeping or discarding relatively higher-order infinitesimals yields different results of the flux-density divergence, which contradicts the theory of limits in calculus.

4. Summary and Discussion

In classical electromagnetism, Gauss's flux theorem in integral form states that the electric flux through a closed surface is proportional to the net charge enclosed by that surface.

In the two definitions of divergence， the flux-density divergence provides the physical definition of divergence, whereas the nabla-operator divergence serves as a mathematical tool for calculation and derivation. At any point in a vacuum, because no charge is present, the divergence of the electric field must be zero.

Using examples derived from Coulomb's law of electrostatics, and based on the theory of limits, this paper reveals that when the relatively higher-order infinitesimals are discarded, the flux-density divergence and the nabla-operator divergence coincide and yield a nonzero negative value. In other words, if discarding higher-order infinitesimals, the nabla-operator divergence can be mathematically derived from the flux-density divergence. However, in a vacuum, the charge density is zero, so this result contradicts Gauss's flux theorem.

Gauss's Flux is a surface integral of the electric field over an area, and its corresponding physical concept should be "the flux surface density". The flux volume density has no physical meaning. Moreover, mathematically, an infinitesimal three-dimensional volume can form an infinite surface, so the flux volume density lacks mathematical rigor. Therefore, the differential form of Gauss's flux theorem in Maxwell's equations does not hold.

Over more than two thousand years of mathematical development, three significant crises have arisen from deep contradictions or paradoxes in the foundations. The most consequential of these was the second crisis, which exposed logical deficiencies in the foundations of calculus. In the 17th century, Newton and Leibniz established calculus, which rapidly proved its power in physics and engineering. However, the notion of an "infinitesimal quantity" involved a logical contradiction (Berkeley's paradox), which led the mathematical community to question the logical foundation of calculus. This debate persisted for over a century. It was not until the 19th century that mathematicians such as Cauchy introduced the rigorous theory of limits and re-defined the infinitesimal as a "variable approaching zero", thereby closing the logical gap and establishing a solid foundation for modern calculus.

Using examples derived from Coulomb's law of electrostatics, keeping or discarding relatively higher-order infinitesimals yields different results of the flux-density divergence, which contradicts the theory of limits in calculus. Consequently, the theory of limits developed by Cauchy et al. contains intrinsic logical flaws and does not fully resolve the second mathematical crisis.

In Figure 3.1, the flux-density divergence described by Equation (2-4) indicates that ΔV is a third-order infinitesimal of Δx. Retaining the third-order infinitesimal flux through the surface element ΔS_3 results in a flux-density divergence of zero, consistent with Gauss's flux theorem.

In contrast, discarding the third-order infinitesimal flux through ΔS_3 results in a nonzero negative value for the flux-density divergence. A similar issue arises in the derivation of the nabla-operator divergence from the flux-density divergence: discarding the third-order infinitesimal flux through ΔS_3 leads to a nonzero negative value for the nabla-operator divergence, thereby contradicting Gauss's flux theorem.

In Figure 3.1, third-order infinitesimals are effective and must be retained in the derivations. Generalizing this observation yields the effective order infinitesimal theorem, which can be stated as follows: In the derivation process of infinitesimal mathematical equations, there exists an effective order for infinitesimals. Infinitesimals with an order lower than or equal to this effective order must be retained, while those with higher orders may be discarded.

For a specific example, such as Figure 3.1, we can identify the effective order of infinitesimals and retain terms of that order during the derivation process. However, in general mathematical derivations, how to identify the effective order of infinitesimals and retain the terms of that order during the derivation process is a significant challenge. In particular, when deriving the nabla-operator divergence from the flux-density divergence, the existing theory of limits fails to retain the third-order infinitesimals.

Gauss's flux theorem in differential form does not hold, and the theory of limits in calculus has defects. Theoretical physics and mathematics are undergoing a revolutionary scientific transformation.

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