Title: Visual Perception and the Horizon
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Abstract: This paper examines two different explanations of what the horizon is and how it is formed. The first explanation is that the horizon is a physical location, more specifically it is an occluding edge of a spherical surface which obstructs objects below and beyond it from being viewed. The second explanation is that the horizon is an optical phenomenon involving human visual perception and the behavior of light.

Globe earth curvature calculations & the occluding edge horizon

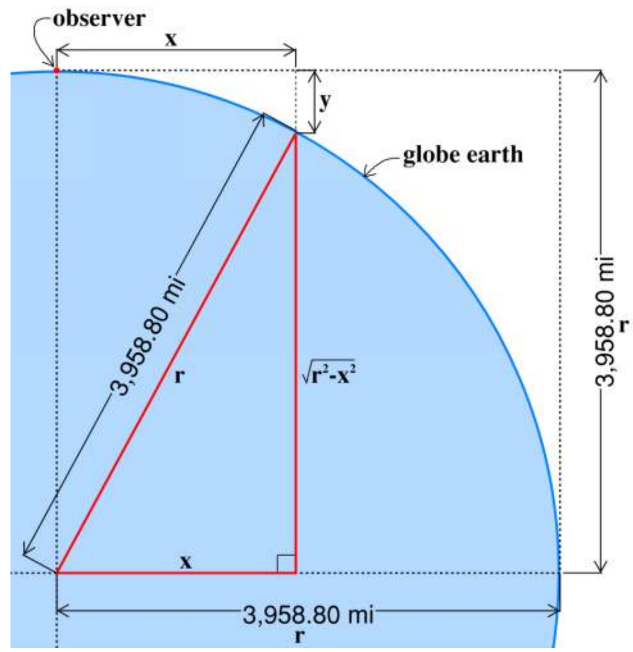
Radius of globe earth $r := 3958.80\ mi$

Horizontal distance (larger values) | Vertical drop

$$x := \begin{bmatrix} 100 \\ 500 \\ 1000 \\ 1500 \\ 2000 \\ 2500 \\ 3000 \\ 3500 \\ 3958.80 \end{bmatrix} mi \qquad y := r - \sqrt{r^2 - x^2} = \begin{bmatrix} 1.26 \\ 31.70 \\ 128.38 \\ 295.18 \\ 542.35 \\ 889.26 \\ 1375.76 \\ 2108.91 \\ 3958.80 \end{bmatrix} mi$$

Horizontal distance (smaller values) | Vertical drop

$$x := \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{bmatrix} mi \qquad y := r - \sqrt{r^2 - x^2} = \begin{bmatrix} 0.67 \\ 2.67 \\ 6.00 \\ 10.67 \\ 16.67 \\ 24.01 \\ 32.68 \\ 42.68 \\ 54.02 \\ 66.69 \end{bmatrix} ft$$



For x miles away from you, the surface of the earth should drop by a height of y. For example, in $x_{10} = 10\ mi$ the surface of the earth should drop $y_{10} = 66.69\ ft$.

The common rule of "8 inches (of vertical drop) multiplied by the square of the distance (x)" is a close approximation of the curvature.

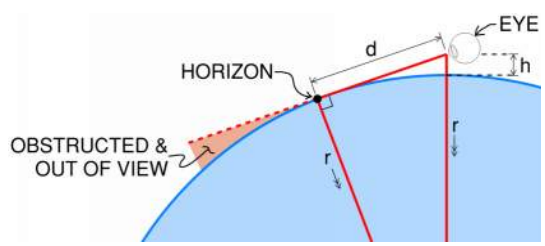
Using the globe earth model, the horizon is a result of earth's curvature. The horizon is an occluding edge which obstructs objects below and beyond it from being viewed by the observer.

For example, at a few observer heights:

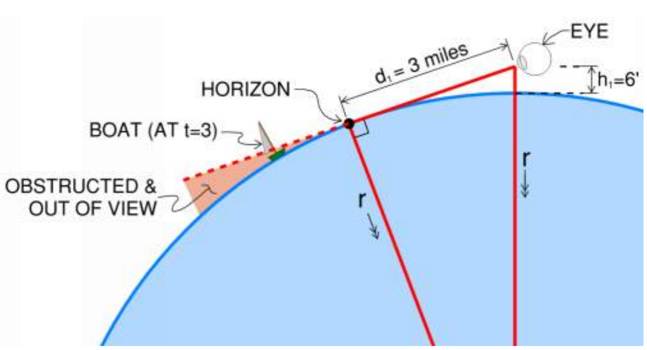
Observer (eye) height $h := \begin{bmatrix} 6 \\ 33000 \\ 120000 \end{bmatrix} ft \begin{bmatrix} \text{“person”} \\ \text{“airplane”} \\ \text{“high altitude balloon”} \end{bmatrix}$ (typ.)

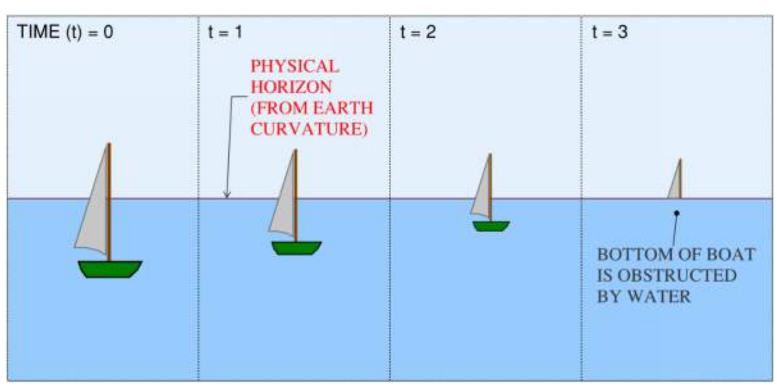
Radius of earth $r = 3958.8\ mi$

Distance to horizon $d := \sqrt{(r+h)^2 - r^2} = \begin{bmatrix} 3.00 \\ 222.54 \\ 424.81 \end{bmatrix} mi$



For the boat example below, say the observer's eye is $h_1 = 6\ ft$ above sea level. The horizon will be $d_1 = 3.00\ mi$ away. When the boat gets farther than 3 miles away (at time t = 3 in the diagrams below) a portion of the bottom of the boat will be obstructed by the water because the bottom of the boat is below and beyond the horizon. I suppose you are to think the surface is curving upwards in your perception and then downwards beyond the horizon.





For the next example an observer height of 120,000 feet will be used (which is the approximate maximum altitude of a high altitude balloon before it pops).

Radius of earth $\quad r = 3958.8\ mi$

Eye height $\quad h_{eye} := 120000\ ft = 22.72727\ mi$

Distance from (0,0) to base of object (along globe surface) $\qquad$ Corresponding coordinates

$$s := 40\ mi \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \\ 120 \\ 160 \\ 200 \\ 240 \\ 280 \\ 320 \\ 360 \\ 400 \\ 440 \\ 480 \\ 520 \\ 560 \end{bmatrix} mi \qquad x := r \cdot \sin\left(\frac{s}{r}\right) = \begin{bmatrix} 39.99932 \\ 79.99456 \\ 119.98162 \\ 159.95644 \\ 199.91493 \\ 239.85301 \\ 279.76661 \\ 319.65164 \\ 359.50404 \\ 399.31973 \\ 439.09466 \\ 478.82476 \\ 518.50598 \\ 558.13426 \end{bmatrix} mi \qquad y := -r + r \cdot \cos\left(\frac{s}{r}\right) = \begin{bmatrix} -0.20208 \\ -0.80830 \\ -1.81859 \\ -3.23286 \\ -5.05096 \\ -7.27270 \\ -9.89786 \\ -12.92617 \\ -16.35732 \\ -20.19096 \\ -24.42669 \\ -29.06409 \\ -34.10269 \\ -39.54196 \end{bmatrix} mi$$

(a few coordinates shown in diagram below)

Height of object $\quad h_{obj} := 10\ mi$

Radius of object $\quad r_{obj} := r + h_{obj} = 3968.8\ mi$

Distance from (0,10) to top of object (along circle at object height) $\qquad$ Corresponding coordinates

$$s_{obj} := \frac{r_{obj}}{r} \cdot s = \begin{bmatrix} 40.10104 \\ 80.20208 \\ 120.30312 \\ 160.40416 \\ 200.50520 \\ 240.60624 \\ 280.70729 \\ 320.80833 \\ 360.90937 \\ 401.01041 \\ 441.11145 \\ 481.21249 \\ 521.31353 \\ 561.41457 \end{bmatrix} mi \qquad x_{obj} := r_{obj} \cdot \sin\left(\frac{s_{obj}}{r_{obj}}\right) = \begin{bmatrix} 40.10036 \\ 80.19662 \\ 120.28470 \\ 160.36050 \\ 200.41992 \\ 240.45889 \\ 280.47330 \\ 320.45908 \\ 360.41215 \\ 400.32842 \\ 440.20382 \\ 480.03428 \\ 519.81573 \\ 559.54412 \end{bmatrix} mi \qquad y_{obj} := -r + r_{obj} \cdot \cos\left(\frac{s_{obj}}{r_{obj}}\right) = \begin{bmatrix} 9.79741 \\ 9.18966 \\ 8.17681 \\ 6.75897 \\ 4.93628 \\ 2.70893 \\ 0.07713 \\ -2.95882 \\ -6.39864 \\ -10.24196 \\ -14.48840 \\ -19.13751 \\ -24.18883 \\ -29.64184 \end{bmatrix} mi$$

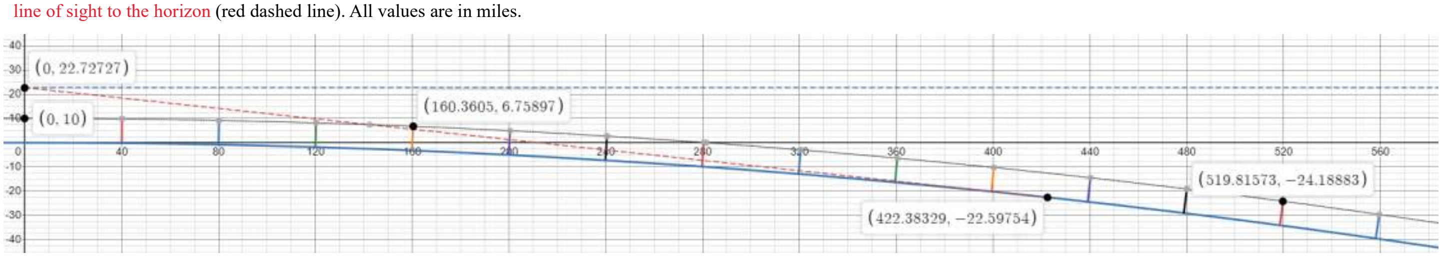
(a few coordinates shown in diagram below)

Distance to horizon $\quad d_{hor} := \sqrt{\left(r + h_{eye}\right)^2 - r^2} = 424.81\ mi \quad$ (red dashed line in diagram below)

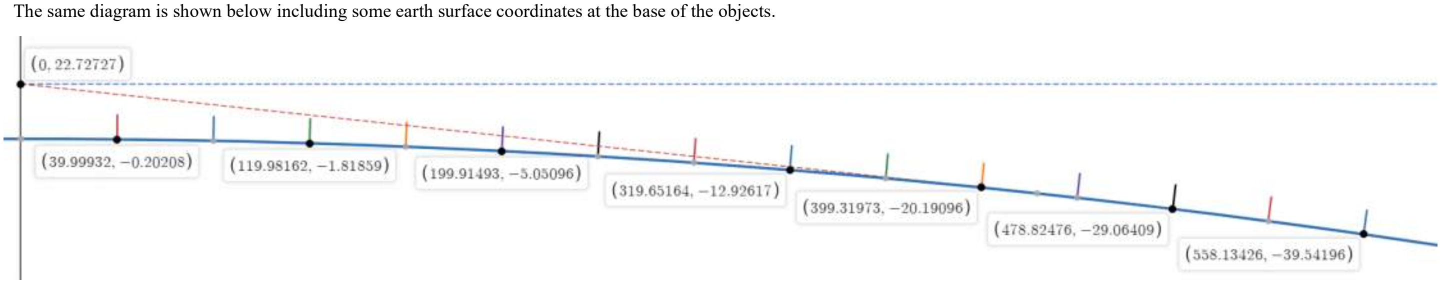
Angle downwards of eye (from eye level) required to look at the horizon $\quad \Theta_{eye} := 90\ deg - \operatorname{atan}\left(\frac{r}{d_{hor}}\right) = 6.12\ deg$

Geometric coordinates of the horizon $\quad y_{hor} := h_{eye} - d_{hor} \cdot \sin\left(\Theta_{eye}\right) = -22.59754\ mi \quad x_{hor} := \sqrt{d_{hor}^{\ 2} - \left(d_{hor} \cdot \sin\left(\Theta_{eye}\right)\right)^2} = 422.38329\ mi \quad$ (shown in diagram below)

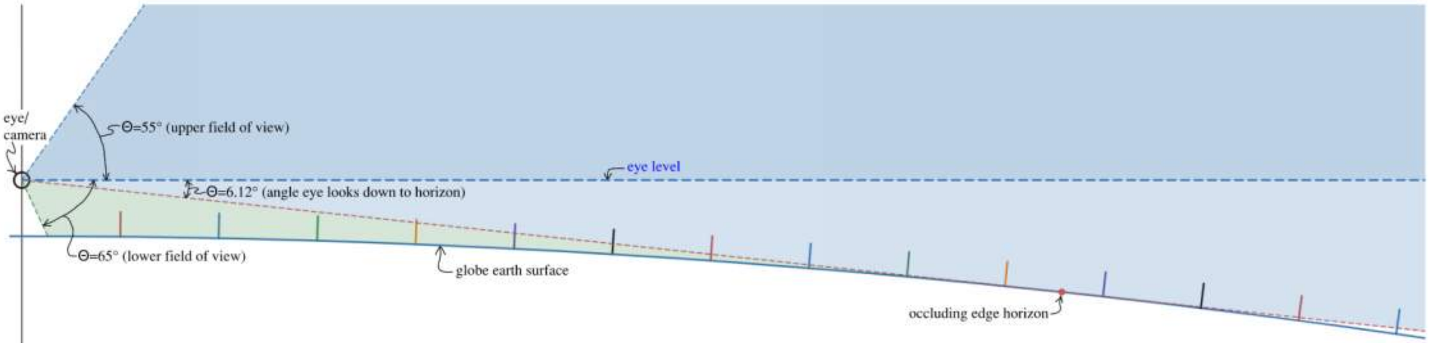
The diagram below (to scale) shows the surface of the globe earth (blue solid line), eye level of high altitude balloon (blue dashed line), 10 mile high vertical objects (multi-colored), and the line of sight to the horizon (red dashed line). All values are in miles.



The same diagram is shown below including some earth surface coordinates at the base of the objects.



In the vertical orientation, the visual field (what you see) can be divided into two portions: above eye level (upper field of view, approximately 55°) and below eye level (lower field of view, approximately 65°). The example below is the same as the previous page (a high altitude balloon camera at 120,000 feet).



A few more observer heights are included below.

Eye height (altitude of observer) $h_{eye} := \begin{bmatrix} 33000\ ft \\ 120000\ ft \\ 1369632\ ft \\ 1768800\ ft \end{bmatrix} = \begin{bmatrix} 6.25 \\ 22.73 \\ 259.40 \\ 335.00 \end{bmatrix} mi \quad \begin{bmatrix} \text{"airplane"} \\ \text{"high altitude balloon"} \\ \text{"alleged ISS orbital altitude"} \\ \text{"alleged Hubble telescope orbital altitude"} \end{bmatrix}$

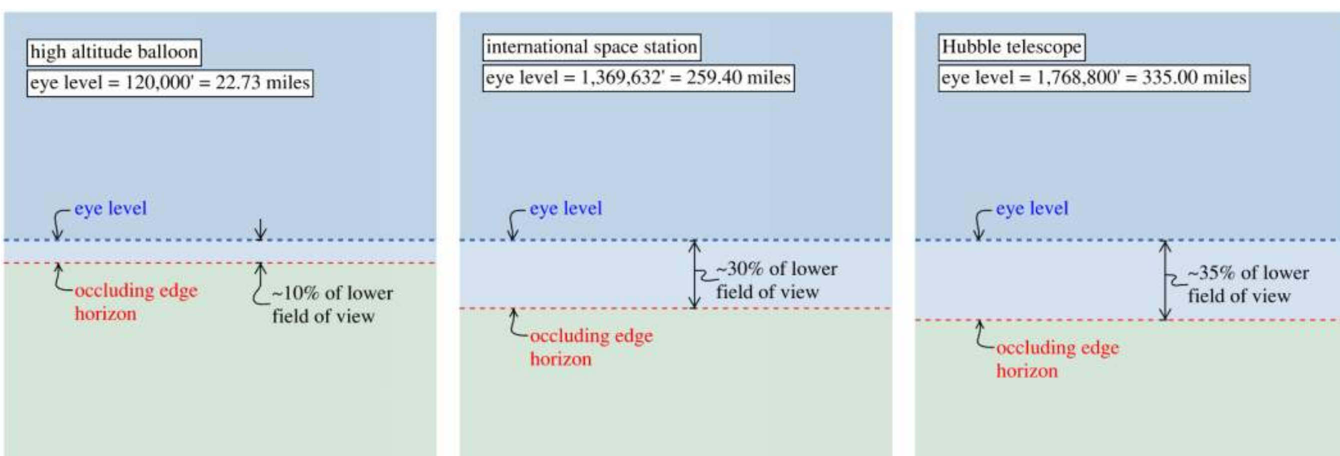
Distance to horizon $d_{hor} := \sqrt{(r + h_{eye})^2 - r^2} = \begin{bmatrix} 222.54 \\ 424.81 \\ 1456.40 \\ 1662.71 \end{bmatrix} mi$

Angle downwards of eye (from eye level) required to look at the horizon $\Theta_{eye} := 90\ deg - \operatorname{atan}\left(\frac{r}{d_{hor}}\right) = \begin{bmatrix} 3.22 \\ 6.12 \\ 20.20 \\ 22.78 \end{bmatrix} deg$

Upper field of view angular size $\Theta_{upper} := 55\ deg$

Lower field of view angular size $\Theta_{lower} := 65\ deg$

Angle downwards of eye to look at horizon : angular size of lower field of view $r_{lower} := \frac{\Theta_{eye}}{\Theta_{lower}} = \begin{bmatrix} 4.95\% \\ 9.42\% \\ 31.07\% \\ 35.05\% \end{bmatrix} \quad \begin{bmatrix} \text{"airplane"} \\ \text{"high altitude balloon"} \\ \text{"alleged ISS orbital altitude"} \\ \text{"alleged Hubble telescope orbital altitude"} \end{bmatrix}$



Up until now the examples discussed are according to mainstream concepts: globe earth & the occluding edge horizon. Mainstream explanations of the horizon do not include any aspects of how images are formed (whether it be on an image sensor of a camera or the spherically curved retina of the human eye) and form your visual perception.

The shape (for example: flat, concave, convex) of the occluding edge horizon formed on the retina is also discussed farther down in this document.

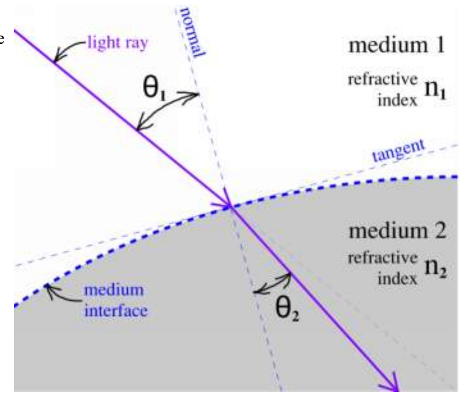
The next section will discuss human visual perception and the optical horizon.

Visual perception and the optical horizon.

Light from the physical world enters the eye and an image is formed on the spherically curved surface of the retina. Because of this, we perceive the world in five-point perspective, meaning there are convergence points straight out along eye level in our field of view, to the left and right, and to the top and bottom (these will be shown and discussed farther down in this document). Robert Hansen discussed this in 1973 in a paper titled: This Curving World: Hyperbolic Linear Perspective[1]. He called it five-point hyperbolic perspective.

In his paper, Hansen writes: "It is true that in many situations, when I am at a comfortable distance from the focus of my attention, this curvature is negligible. But if I am quite close to a large cube or rectangular solid, or, as is the case with most of us the greater part of every day, enclosed within the walls of a box, the lines of that box swell and bend on all sides-especially if I enlarge my focus and give attention to the whole of my field of vision. The curves are most subtle directly in front of my eyes, but they are emphatic at the periphery of vision". You can test this yourself and practice paying attention to your periphery to see these curves.

The examples within this document that deal with light rays forming images on the retina ignore the redirection of light rays due to refraction through the cornea and lens. This page is included to discuss this refraction briefly. The exclusion of this refraction in the later examples is not of consequence to the concepts discussed.



Refraction (change in direction) of a light ray traveling from less dense medium 1 (for example, air) into a denser medium 2 (for example, an acrylic prism).

Typical values of lateral visual field limits $\Theta_{left} := 75\ deg$ $\Theta_{right} := 75\ deg$

Refractive indices of mediums $n_{air} := 1.000$ (air)

$n_{cornea} := 1.377$ (cornea) (indices provided are assumed, values shown are close to commonly cited values found online)

$n_{lens} := 1.411$ (lens)

Relationship (Snell's law) between light ray incidence angle (θ) and refractive index of medium (see diagram to the right) $n_1 \cdot sin(\theta_1) = n_2 \cdot sin(\theta_2)$ --> $\theta_2 = asin\left(\frac{n_1 \cdot sin(\theta_1)}{n_2}\right)$

Some light rays are shown in the diagrams to the bottom right to illustrate how the light ray's direction towards the retina is affected by going through the cornea and lens.

Light ray incidence angles $\theta_1 := \begin{bmatrix} 10^{-9} \\ 5 \\ 15 \\ 25 \end{bmatrix} deg$ $\begin{bmatrix} \text{“ray 1”} \\ \text{“ray 2”} \\ \text{“ray 3”} \\ \text{“ray 4”} \end{bmatrix}$ (the diagram to the bottom right uses the third angle in this matrix, 15°, as an example)

Refracted angle into cornea $\theta_2 := \text{asin}\left(\frac{n_{air} \cdot \sin(\theta_1)}{n_{cornea}}\right) = \begin{bmatrix} 0.00 \\ 3.63 \\ 10.83 \\ 17.87 \end{bmatrix} deg$ Difference $\overrightarrow{|\theta_2 - \theta_1|} = \begin{bmatrix} 0.00 \\ 1.37 \\ 4.17 \\ 7.13 \end{bmatrix} deg$

Refracted angle into lens $\theta_3 := \text{asin}\left(\frac{n_{cornea} \cdot \sin(\theta_2)}{n_{lens}}\right) = \begin{bmatrix} 0.00 \\ 3.54 \\ 10.57 \\ 17.43 \end{bmatrix} deg$ Difference $\overrightarrow{|\theta_3 - \theta_2|} = \begin{bmatrix} 0.00 \\ 0.09 \\ 0.26 \\ 0.44 \end{bmatrix} deg$

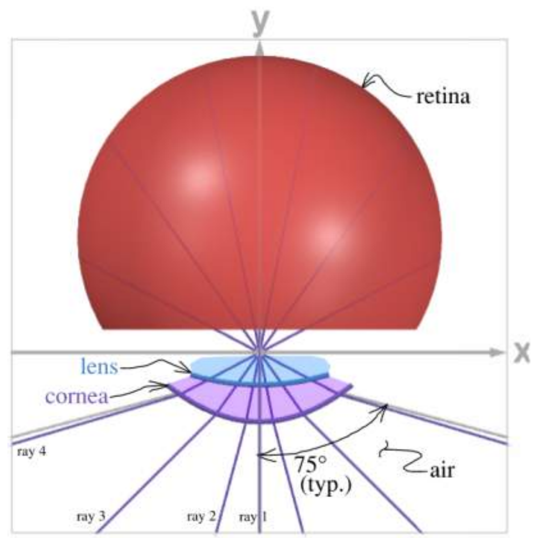
Total redirection $\Delta_{angle} := \overrightarrow{|\theta_3 - \theta_1|} = \begin{bmatrix} 0.00 \\ 1.46 \\ 4.43 \\ 7.57 \end{bmatrix} deg$ $\begin{bmatrix} \text{“ray 1”} \\ \text{“ray 2”} \\ \text{“ray 3”} \\ \text{“ray 4”} \end{bmatrix}$

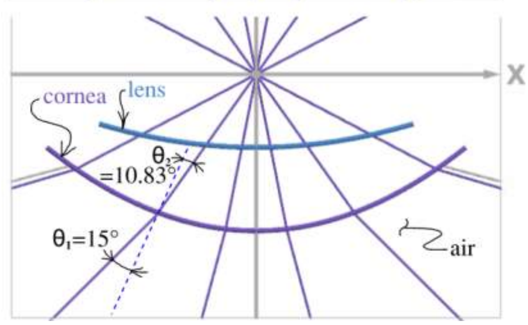
As can be seen in the calculations above, most of the refraction occurs when the light rays travel from the air into the cornea, about $\frac{\overrightarrow{|\theta_2 - \theta_1|}}{\Delta_{angle}} = \begin{bmatrix} 93.99\% \\ 94.00\% \\ 94.04\% \\ 94.13\% \end{bmatrix}$ of the total refraction. Note that in the diagrams to the right the slight refraction occurring at the lens is not included graphically since it is so small.

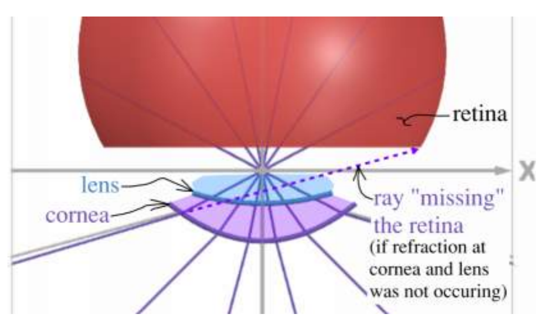
In this example calculation it can be seen that as you move away from the eye's focus point (along the -y axis) and towards the periphery, the incidence angles of the light rays increase resulting in a higher redirection of the light rays. Since the light ray is redirected, it no longer represents (in your visual perception) the true location of the object (or whatever the light ray is coming from). This difference between your visual perception and actual physical space is called a distortion.

If this refraction was not occurring (the refractive indices of the air, cornea, and lens were the same) then the two light rays (at each extreme of the light ray "fan", representing the limits of the lateral field of view) would "miss" the retina and not be in your perception (see very bottom right diagram). The light still exists and it still travels through your cornea and lens, but it might "land" inside the eye on a surface adjacent to the retina thereby not contributing to perception formation.

In summary, the refraction results in a peripheral distortion (difference between the actual object location in physical space and the object location in your perception). It can be thought of as an effect that serves to "move" (redirect the light of) the peripheral object's location in physical space forward and "into" your perception allowing it to be perceived and producing a wider field of view.





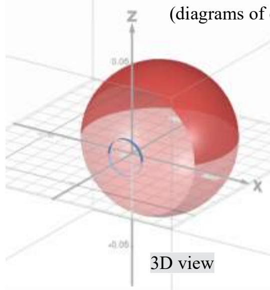


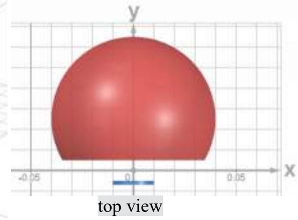
Shown below are calculations of light rays hitting the curved surface of the retina.

Diameter of retina $dia_{ret} := 24\ mm = 0.08\ ft$

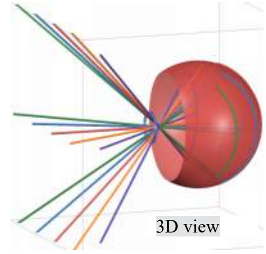
Radius of retina $r_{ret} := \dfrac{dia_{ret}}{2} = 0.04\ ft$

(diagrams of eye: retina and pupil shown)





(full diagram with light rays)



Equation of a spherical retina $(x-h)^2+(y-k)^2+(z-l)^2 = r_{ret}{}^2$

Equations of light ray in parametric form $x = x_0 + a\cdot t \qquad y = y_0 + b\cdot t \qquad z = z_0 + c\cdot t$

Example calculation for one light ray

Center of sphere $\begin{bmatrix} h \\ k \\ l \end{bmatrix} := \begin{bmatrix} 0 \\ 0.025 \\ 0 \end{bmatrix} ft$ Origin of light ray $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} := \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} ft$ Direction vector of light ray $\begin{bmatrix} a \\ b \\ c \end{bmatrix} := \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ Angle of light ray (off -y axis to x axis) $\Theta_x := \operatorname{atan}\left(\dfrac{x_0}{y_0}\right) = -45\ deg$

Substitute parametric lines into spherical retina equation $(x_0+a\cdot t-h)^2+(y_0+b\cdot t-k)^2+(z_0+c\cdot t-l)^2 = r_{ret}{}^2$

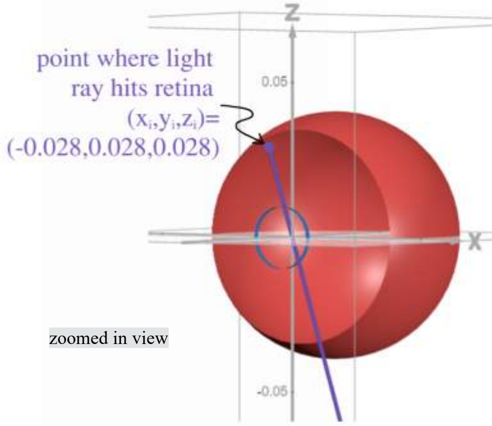
Expansion $(a^2+b^2+c^2)\cdot t^2 + 2\cdot(a\cdot(x_0-h)+b\cdot(y_0-k)+c\cdot(z_0-l))\cdot t + (x_0-h)^2+(y_0-k)^2+(z_0-l)^2 - r_{ret}{}^2 = 0$

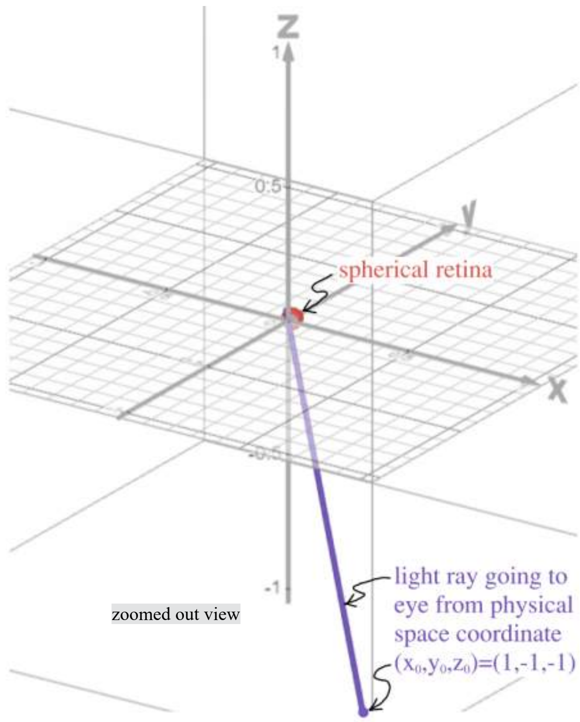
Discriminant $\Delta := (2\cdot(a\cdot(x_0-h)+b\cdot(y_0-k)+c\cdot(z_0-l)))^2 - 4\cdot(a^2+b^2+c^2)\cdot((x_0-h)^2+(y_0-k)^2+(z_0-l)^2-r_{ret}{}^2) = 0.01\ ft^2$

Parameter corresponding to intersection point on retina $t_1 := \dfrac{-(2\cdot(a\cdot(x_0-h)+b\cdot(y_0-k)+c\cdot(z_0-l)))+\sqrt{\Delta}}{2\cdot(a^2+b^2+c^2)} = 1.03\ ft$

Parameter corresponding to intersection point on "front of retinal sphere" $t_2 := \dfrac{-(2\cdot(a\cdot(x_0-h)+b\cdot(y_0-k)+c\cdot(z_0-l)))-\sqrt{\Delta}}{2\cdot(a^2+b^2+c^2)} = 0.99\ ft$

Intersection point of light ray at retina $\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} := \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}(t_1) = \begin{bmatrix} -0.028 \\ 0.028 \\ 0.028 \end{bmatrix} ft$

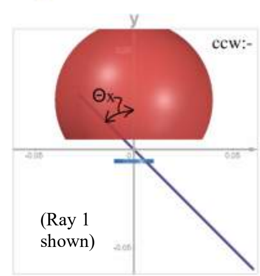




Matrices are used to perform the same calculation for multiple light rays

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{bmatrix} := \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}\cdot ft = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} ft \qquad p1 := \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{bmatrix}$$

$x_0 := p1^{\widehat{1}} = [1\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1]\ ft \qquad y_0 := p1^{\widehat{2}} = [-1\ \ -1\ \ -1\ \ -1\ \ -1\ \ -1]\ ft \qquad z_0 := p1^{\widehat{3}} = [-1\ \ 0\ \ 1\ \ -1\ \ 0\ \ 1]\ ft$



Angle of light rays (off y axis to x axis)

$$\Theta_x := \overrightarrow{\operatorname{atan}\left(\frac{x_0}{y_0}\right)} = [-45.00\ \ -45.00\ \ -45.00\ \ 45.00\ \ 45.00\ \ 45.00]\ deg$$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} := \frac{p1}{ft}\cdot -1 = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 \end{bmatrix} \qquad p2 := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix}$$

$a := p2^{\widehat{1}} = [-1\ \ -1\ \ -1\ \ 1\ \ 1\ \ 1] \qquad b := p2^{\widehat{2}} = [1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1] \qquad c := p2^{\widehat{3}} = [1\ \ 0\ \ -1\ \ 1\ \ 0\ \ -1]$

$$\begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot ft \qquad O := \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \end{bmatrix}$$

$$h := O^{\langle 1 \rangle} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]\, ft \qquad k := O^{\langle 2 \rangle} = [0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025]\, ft \quad l := O^{\langle 3 \rangle} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]\, ft$$

$$(x_0 + a \cdot t - h)^2 + (y_0 + b \cdot t - k)^2 + (z_0 + c \cdot t - l)^2 = r^2$$

$$(a^2 + b^2 + c^2) \cdot t^2 + 2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l)) \cdot t + (x_0 - h)^2 + (y_0 - k)^2 + (z_0 - l)^2 - r^2 = 0$$

$$\Delta := \overrightarrow{(2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l)))^2 - 4 \cdot (a^2 + b^2 + c^2) \cdot ((x_0 - h)^2 + (y_0 - k)^2 + (z_0 - l)^2 - r_{ret}{}^2)} = [0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01]\, ft^2$$

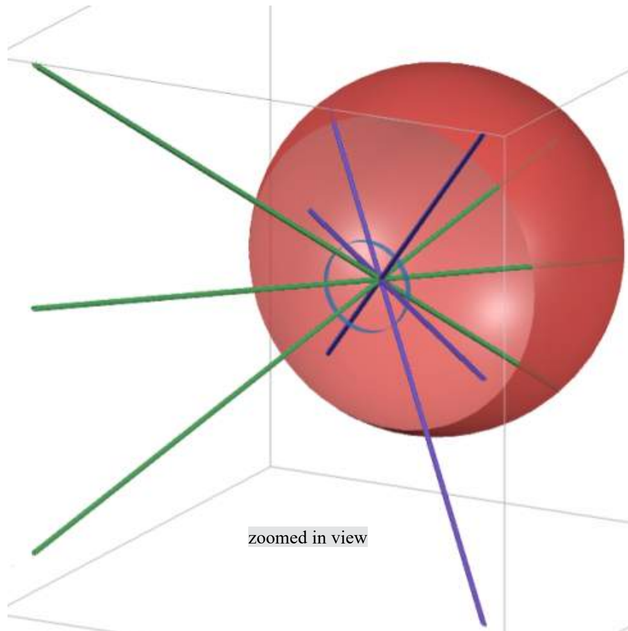
$$t_1 := \overrightarrow{\frac{-(2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l))) + \sqrt{\Delta}}{2 \cdot (a^2 + b^2 + c^2)}} = [1.03 \;\; 1.04 \;\; 1.03 \;\; 1.03 \;\; 1.04 \;\; 1.03]\, ft$$

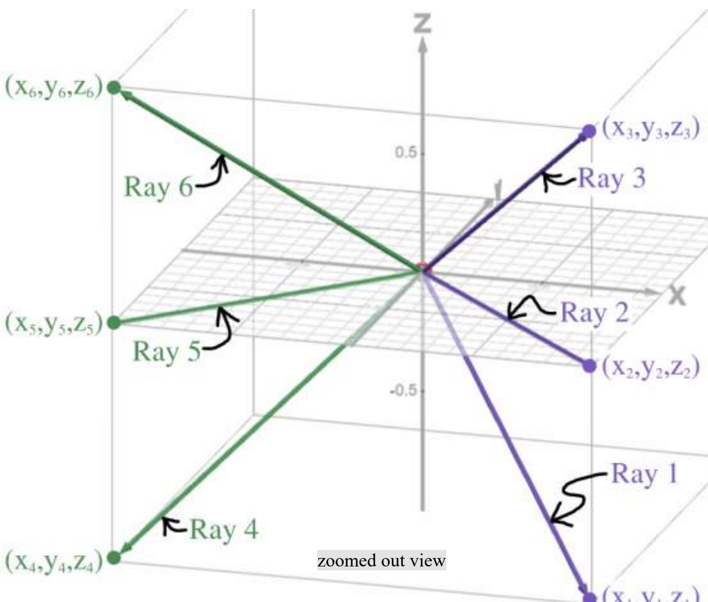
$$t_2 := \overrightarrow{\frac{-(2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l))) - \sqrt{\Delta}}{2 \cdot (a^2 + b^2 + c^2)}} = [0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99]\, ft$$

Intersection points of light rays at retina $\quad x_i := \overrightarrow{x_0 + a \cdot t_1} \quad y_i := \overrightarrow{y_0 + b \cdot t_1} \quad z_i := \overrightarrow{z_0 + c \cdot t_1}$

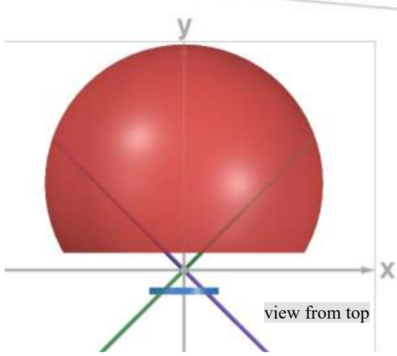
Ray 1 Ray 2 Ray 3 Ray 4 Ray 5 Ray 6

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} [-0.028 \;\; -0.037 \;\; -0.028 \;\; 0.028 \;\; 0.037 \;\; 0.028] \\ [0.028 \;\; 0.037 \;\; 0.028 \;\; 0.028 \;\; 0.037 \;\; 0.028] \\ [0.028 \;\; 0.000 \;\; -0.028 \;\; 0.028 \;\; 0.000 \;\; -0.028] \end{bmatrix} ft$$

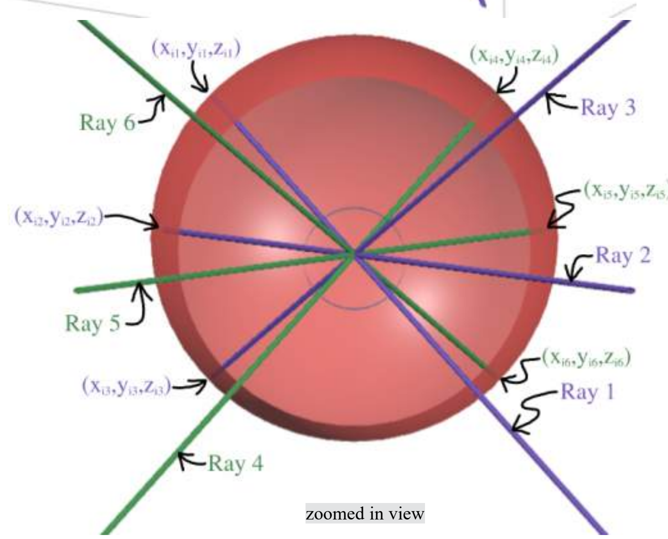

zoomed in view



zoomed out view

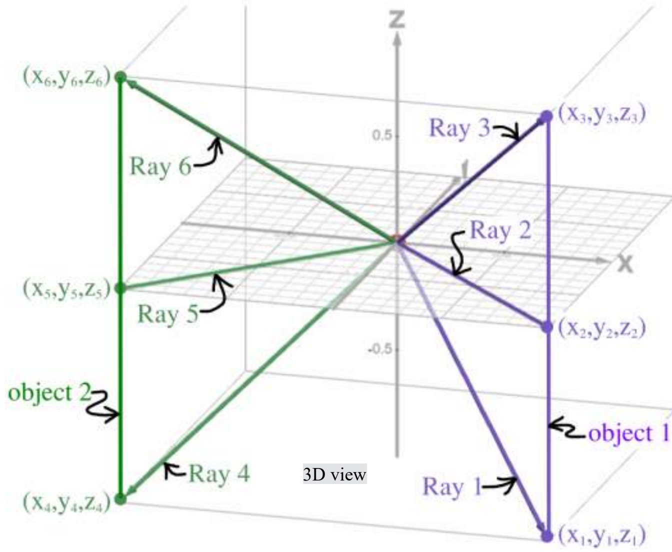


view from top

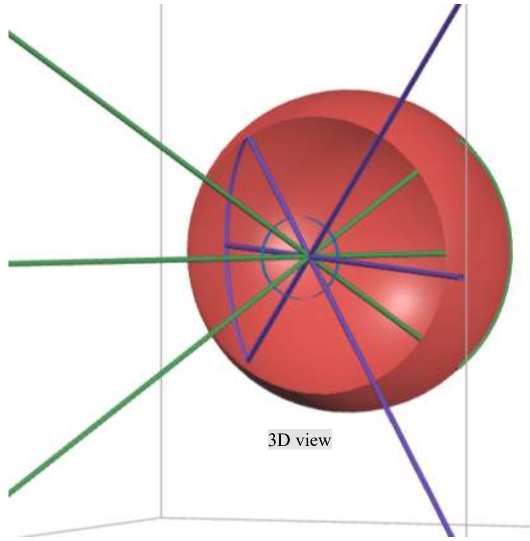


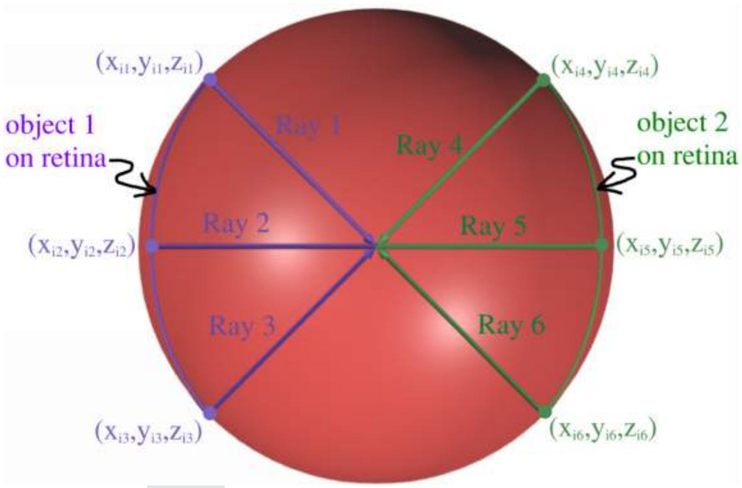
zoomed in view

Refer to zoomed out view, from point (x_1,y_1,z_1) (bottom right) to point (x_3,y_3,z_3) (top right) there is a purple object and from point (x_4,y_4,z_4) (bottom left) to point (x_6,y_6,z_6) (top left) there is a green object. If the eye is focusing on something straight out (between the vertical objects), then due to the curvature of your retina the vertical physical objects will be displayed in the periphery of your visual perception as curved objects as shown in the front view diagram on the following page. In your perception this curvature is not as extreme or obvious as depicted in these diagrams because you are "in your perception", meaning your perception is created around the focal point of the light rays, point (0,0,0) in these diagrams, and you perceive from that point, not from a point observing these 2D representations of 3D models.

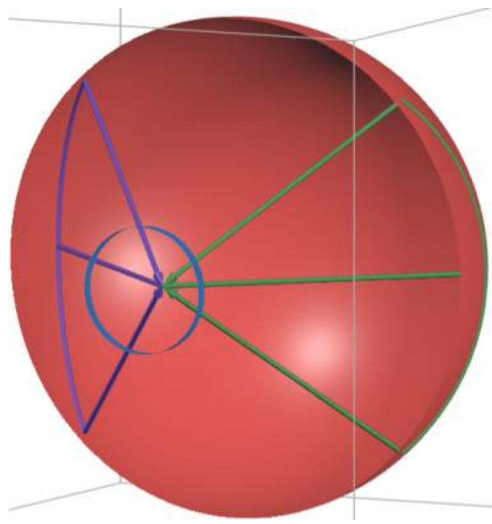


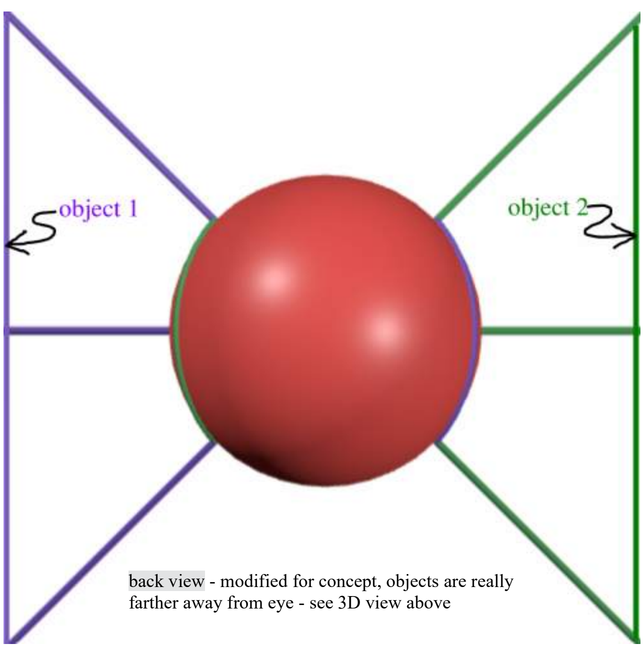
3D view


3D view



front view - modified for clarity, retina pulled back and light rays trimmed near the pupil (blue ring)


3D view - modified for clarity, retina pulled back and light rays trimmed near the pupil



back view - modified for concept, objects are really farther away from eye - see 3D view above

Calculating the arc length of object 1 on retina.

Intersection points of light rays at retina

$$\begin{array}{c} \\ \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \end{array} \begin{array}{cccccc} \text{Ray 1} & \text{Ray 2} & \text{Ray 3} & \text{Ray 4} & \text{Ray 5} & \text{Ray 6} \\ \end{array}$$

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} [-0.028 & -0.037 & -0.028 & 0.028 & 0.037 & 0.028] \\ [0.028 & 0.037 & 0.028 & 0.028 & 0.037 & 0.028] \\ [0.028 & 0.000 & -0.028 & 0.028 & 0.000 & -0.028] \end{bmatrix} \; ft$$

$$x_{i.f} := \overrightarrow{x_0 + a \cdot t_2} \quad y_{i.f} := \overrightarrow{y_0 + b \cdot t_2} \quad z_{i.f} := \overrightarrow{z_0 + c \cdot t_2}$$

Intersection points of light rays on front of sphere

$$\begin{array}{cccccc} \text{Ray 1} & \text{Ray 2} & \text{Ray 3} & \text{Ray 4} & \text{Ray 5} & \text{Ray 6} \end{array}$$

$$\begin{bmatrix} x_{i.f} \\ y_{i.f} \\ z_{i.f} \end{bmatrix} = \begin{bmatrix} [0.011 & 0.012 & 0.011 & -0.011 & -0.012 & -0.011] \\ [-0.011 & -0.012 & -0.011 & -0.011 & -0.012 & -0.011] \\ [-0.011 & 0.000 & 0.011 & -0.011 & 0.000 & 0.011] \end{bmatrix} \; ft$$

$n := 2$ (to select nth ray in matrix above)

Diameter of the circle along the sphere in which the intersection points are on (from here called: small circle)

$$dia_{cir.1} := \sqrt{{x_i}_{1,n}^{\,2} + {y_i}_{1,n}^{\,2} + {z_i}_{1,n}^{\,2}} + \sqrt{{x_{i.f}}_{1,n}^{\,2} + {y_{i.f}}_{1,n}^{\,2} + {z_{i.f}}_{1,n}^{\,2}} = 0.070 \; ft$$

Radius of the small circle

$$r_{cir.1} := \frac{dia_{cir.1}}{2} = 0.035 \; ft$$

Length from (0,0,0) to center of the small circle

$$l_{cir.1} := \sqrt{{x_i}_{1,n}^{\,2} + {y_i}_{1,n}^{\,2} + {z_i}_{1,n}^{\,2}} - r_{cir.1} = 0.018 \; ft$$

Coordinates of center of small circle

$$x_{cir.1} := \frac{\left|{x_i}_{1,n}\right| + \left|{x_{i.f}}_{1,n}\right|}{2} \cdot -1 + {x_{i.f}}_{1,n} = -0.013 \; ft \qquad y_{cir.1} := \frac{\left|{y_i}_{1,n}\right| + \left|{y_{i.f}}_{1,n}\right|}{2} + {y_{i.f}}_{1,n} = 0.013 \; ft$$

$$z_{cir.1} := \frac{\left|{z_i}_{1,n}\right| + \left|{z_{i.f}}_{1,n}\right|}{2} + {z_{i.f}}_{1,n} = 0.000 \; ft$$

$m := 1$ (to select mth ray in matrix above)

Angle subtended by arc length of object 1 on retina at center of small circle

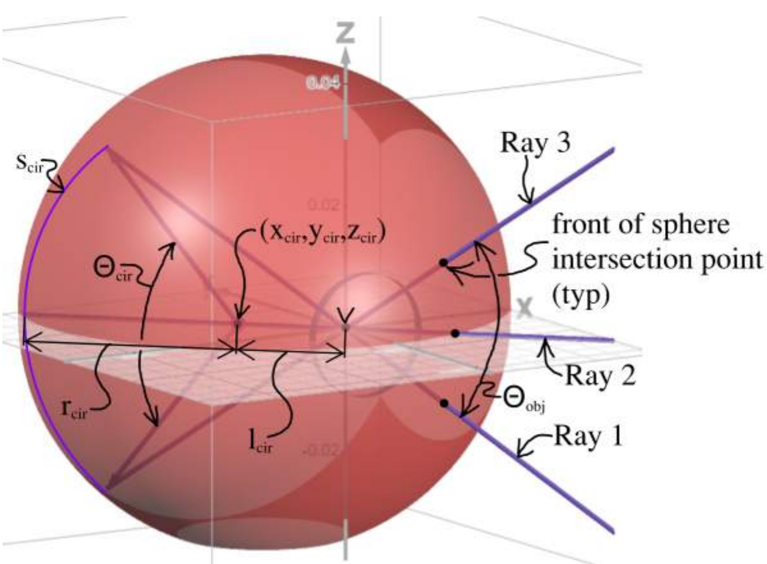
$$\Theta_{cir.1} := 2 \cdot \operatorname{atan}\left(\frac{\left|{z_i}_{1,m} - z_{cir.1}\right|}{\sqrt{\left(\left|{x_i}_{1,m} - x_{cir.1}\right|\right)^2 + \left(\left|{y_i}_{1,m} - y_{cir.1}\right|\right)^2}}\right) = 104.26 \; deg$$

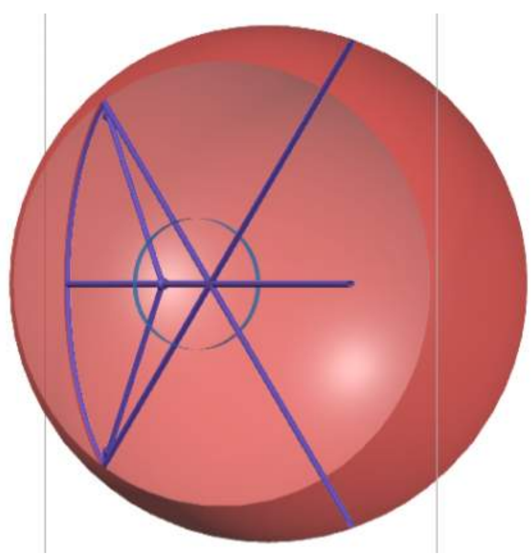
Arc length of object 1 on retina

$$s_{cir.1} := r_{cir.1} \cdot \Theta_{cir.1} = 0.064 \; ft$$

Angle subtended by object 1 in physical space at the eye

$$\Theta_{obj.1} := 2 \cdot \operatorname{atan}\left(\frac{\left|p1_{3,1} - 0\right|}{\sqrt{\left(\left|p1_{1,1} - 0\right|\right)^2 + \left(\left|p1_{2,1} - 0\right|\right)^2}}\right) = 70.53 \; deg$$





The height of the object 1 in physical space is $p1_{3,3} - p1_{3,1} = 2 \; ft$ and creates an arc length on the retina (in this model) of approximately $s_{cir.1} = 0.768 \; in$.

Shown below are calculations for more light rays - orange and blue rays (which are less in the periphery than the green and purple ones above).

Matrices are used to perform the same calculation for multiple light rays

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{bmatrix} := \begin{bmatrix} 0.5 & 0.5 & 0.5 & -0.5 & -0.5 & -0.5 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \cdot ft \qquad p1 := \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{bmatrix}$$

$$x_0 := p1^{\langle 1 \rangle} = [0.5 \;\; 0.5 \;\; 0.5 \;\; -0.5 \;\; -0.5 \;\; -0.5]\, ft \qquad y_0 := p1^{\langle 2 \rangle} = [-1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1]\, ft \qquad z_0 := p1^{\langle 3 \rangle} = [-1 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \;\; 1]\, ft$$

Angle of light rays (off -y axis to x axis)

$$\Theta_x := \overrightarrow{\operatorname{atan}\left(\frac{x_0}{y_0}\right)} = [-26.57 \;\; -26.57 \;\; -26.57 \;\; 26.57 \;\; 26.57 \;\; 26.57]\, deg$$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} := \frac{p1}{ft} \cdot -1 = \begin{bmatrix} -0.5 & -0.5 & -0.5 & 0.5 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 \end{bmatrix} \qquad p2 := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix}$$

$$a := p2^{\langle 1 \rangle} = [-0.5 \;\; -0.5 \;\; -0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5] \qquad b := p2^{\langle 2 \rangle} = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1] \qquad c := p2^{\langle 3 \rangle} = [1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; -1]$$

$$\begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot ft \qquad O := \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \end{bmatrix}$$

$$h := O^{\langle 1 \rangle} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]\, ft \qquad k := O^{\langle 2 \rangle} = [0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025]\, ft \qquad l := O^{\langle 3 \rangle} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]\, ft$$

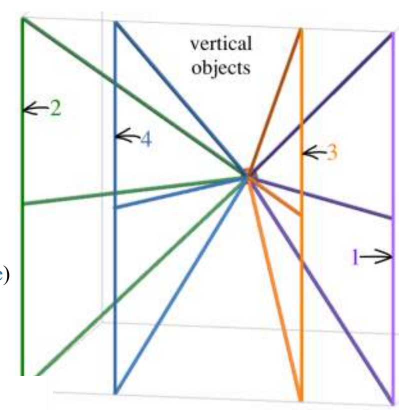
$$(x_0 + a \cdot t - h)^2 + (y_0 + b \cdot t - k)^2 + (z_0 + c \cdot t - l)^2 = r^2$$

$$(a^2 + b^2 + c^2) \cdot t^2 + 2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l)) \cdot t + (x_0 - h)^2 + (y_0 - k)^2 + (z_0 - l)^2 - r^2 = 0$$

$$\Delta := \overrightarrow{(2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l)))^2 - 4 \cdot (a^2 + b^2 + c^2) \cdot ((x_0 - h)^2 + (y_0 - k)^2 + (z_0 - l)^2 - r_{ret}^2)} = [0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01]\, ft^2$$

$$t_1 := \overrightarrow{\frac{-(2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l))) + \sqrt{\Delta}}{2 \cdot (a^2 + b^2 + c^2)}} = [1.03 \;\; 1.05 \;\; 1.03 \;\; 1.03 \;\; 1.05 \;\; 1.03]\, ft$$

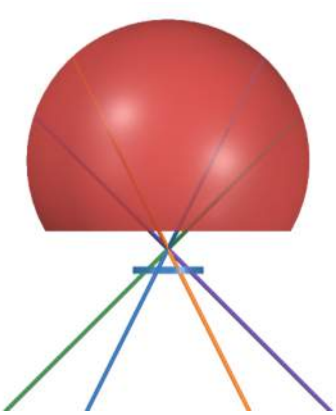
$$t_2 := \overrightarrow{\frac{-(2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l))) - \sqrt{\Delta}}{2 \cdot (a^2 + b^2 + c^2)}} = [0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99]\, ft$$

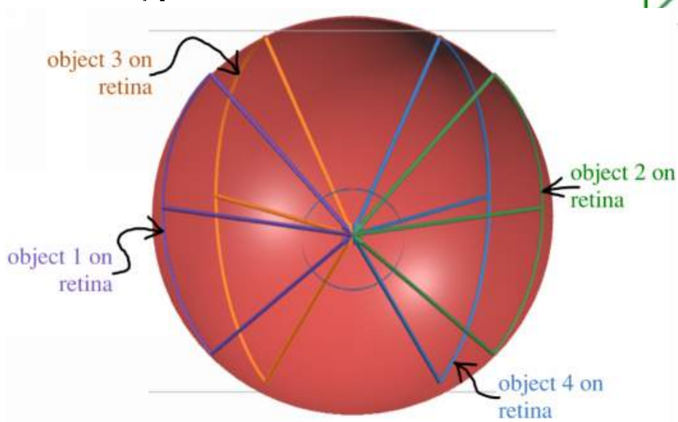


Intersection points of light rays at retina $\quad x_i := \overrightarrow{x_0 + a \cdot t_1} \qquad y_i := \overrightarrow{y_0 + b \cdot t_1} \qquad z_i := \overrightarrow{z_0 + c \cdot t_1}$

Ray 7 Ray 8 Ray 9 Ray 10 Ray 11 Ray 12 (rays 7 - 9 are orange, rays 10 -12 are blue)

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} [-0.017 \;\; -0.027 \;\; -0.017 \;\; 0.017 \;\; 0.027 \;\; 0.017] \\ [0.034 \;\; 0.054 \;\; 0.034 \;\; 0.034 \;\; 0.054 \;\; 0.034] \\ [0.034 \;\; 0.000 \;\; -0.034 \;\; 0.034 \;\; 0.000 \;\; -0.034] \end{bmatrix} ft$$





Objects more in the periphery are the purple and green objects, therefore there is more curvature to the corresponding "images" of these objects on the spherical retina.
Objects less in the periphery are the orange and blue objects therefore there is less curvature to the corresponding "images" of these objects on the spherical retina.

In this example, the distance from the eye to the middle of the purple and green vertical objects is $\sqrt{1\, ft^2 + 1\, ft^2} = 1.41\, ft$ & the distance from the eye to the middle of the orange and blue vertical objects is $\sqrt{0.5\, ft^2 + 1\, ft^2} = 1.22\, ft$

Calculating the arc length of object 3 on retina.

Intersection points of light rays at retina

$$\begin{array}{cccccc} \text{Ray 7} & \text{Ray 8} & \text{Ray 9} & \text{Ray 10} & \text{Ray 11} & \text{Ray 12} \end{array}$$

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -0.017 & -0.027 & -0.017 & 0.017 & 0.027 & 0.017 \end{bmatrix} \\ \begin{bmatrix} 0.034 & 0.054 & 0.034 & 0.034 & 0.054 & 0.034 \end{bmatrix} \\ \begin{bmatrix} 0.034 & 0.000 & -0.034 & 0.034 & 0.000 & -0.034 \end{bmatrix} \end{bmatrix} \boldsymbol{ft}$$

$$x_{i.f} := \overrightarrow{x_0 + a \cdot t_2} \qquad y_{i.f} := \overrightarrow{y_0 + b \cdot t_2} \qquad z_{i.f} := \overrightarrow{z_0 + c \cdot t_2}$$

Intersection points of light rays on front of sphere

$$\begin{array}{cccccc} \text{Ray 7} & \text{Ray 8} & \text{Ray 9} & \text{Ray 10} & \text{Ray 11} & \text{Ray 12} \end{array}$$

$$\begin{bmatrix} x_{i.f} \\ y_{i.f} \\ z_{i.f} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0.006 & 0.007 & 0.006 & -0.006 & -0.007 & -0.006 \end{bmatrix} \\ \begin{bmatrix} -0.012 & -0.014 & -0.012 & -0.012 & -0.014 & -0.012 \end{bmatrix} \\ \begin{bmatrix} -0.012 & 0.000 & 0.012 & -0.012 & 0.000 & 0.012 \end{bmatrix} \end{bmatrix} \boldsymbol{ft}$$

$n := 2$ (to select nth ray in matrix above)

Diameter of the circle along the sphere in which the intersection points are on (from here called: small circle)

$$dia_{cir.3} := \sqrt{{x_i}_{1,n}^{\,2} + {y_i}_{1,n}^{\,2} + {z_i}_{1,n}^{\,2}} + \sqrt{{x_{i.f}}_{1,n}^{\,2} + {y_{i.f}}_{1,n}^{\,2} + {z_{i.f}}_{1,n}^{\,2}} = 0.075 \; \boldsymbol{ft}$$

Radius of the small circle

$$r_{cir.3} := \frac{dia_{cir.3}}{2} = 0.038 \; \boldsymbol{ft}$$

Length from (0,0,0) to center of the small circle

$$l_{cir.3} := \sqrt{{x_i}_{1,n}^{\,2} + {y_i}_{1,n}^{\,2} + {z_i}_{1,n}^{\,2}} - r_{cir.3} = 0.022 \; \boldsymbol{ft}$$

Coordinates of center of small circle

$$x_{cir.3} := \frac{\left|{x_i}_{1,n}\right| + \left|{x_{i.f}}_{1,n}\right|}{2} \cdot -1 + {x_{i.f}}_{1,n} = -0.010 \; \boldsymbol{ft} \qquad y_{cir.3} := \frac{\left|{y_i}_{1,n}\right| + \left|{y_{i.f}}_{1,n}\right|}{2} + {y_{i.f}}_{1,n} = 0.020 \; \boldsymbol{ft}$$

$$z_{cir.3} := \frac{\left|{z_i}_{1,n}\right| + \left|{z_{i.f}}_{1,n}\right|}{2} + {z_{i.f}}_{1,n} = 0.000 \; \boldsymbol{ft}$$

$m := 1$ (to select mth ray in matrix above)

Angle subtended by arc length of object 3 on retina at center of small circle

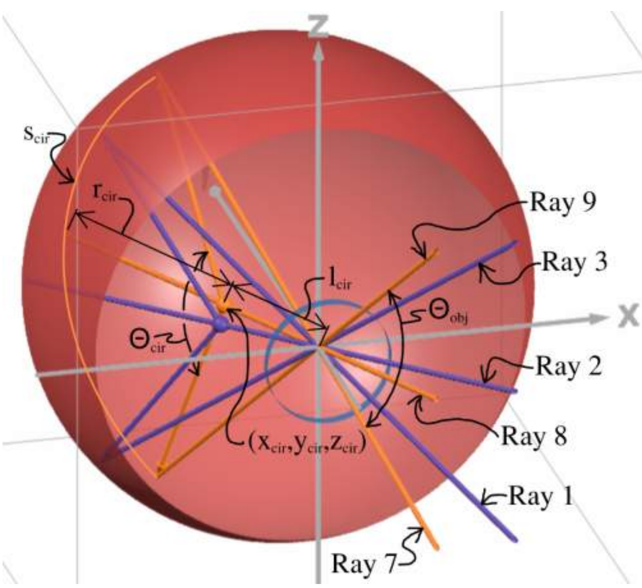
$$\Theta_{cir.3} := 2 \cdot \operatorname{atan}\left(\frac{\left|{z_i}_{1,m} - z_{cir.3}\right|}{\sqrt{\left(\left|{x_i}_{1,m} - x_{cir.3}\right|\right)^2 + \left(\left|{y_i}_{1,m} - y_{cir.3}\right|\right)^2}}\right) = 130.14 \; \boldsymbol{deg}$$

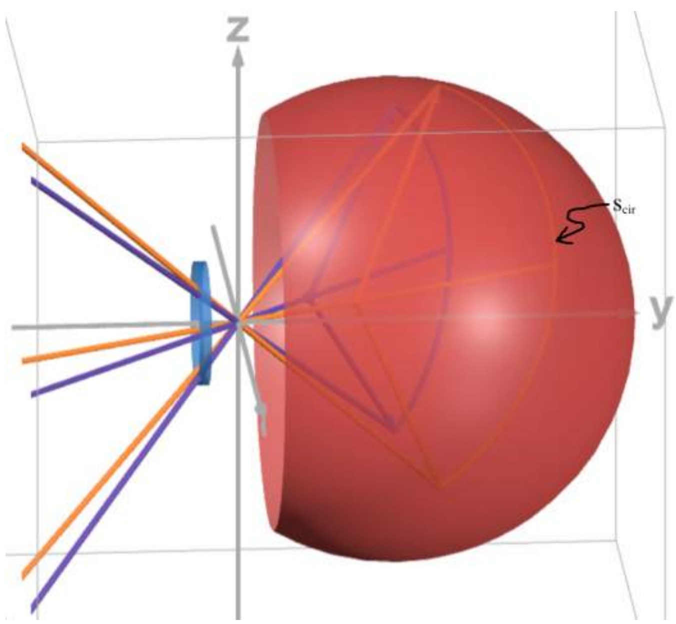
Arc length of object 3 on retina

$$s_{cir.3} := r_{cir.3} \cdot \Theta_{cir.3} = 0.086 \; \boldsymbol{ft}$$

Angle subtended by object 3 in physical space at the eye

$$\Theta_{obj.3} := 2 \cdot \operatorname{atan}\left(\frac{\left|p1_{3,1} - 0\right|}{\sqrt{\left(\left|p1_{1,1} - 0\right|\right)^2 + \left(\left|p1_{2,1} - 0\right|\right)^2}}\right) = 83.62 \; \boldsymbol{deg}$$





The height of the object 3 in physical space is $p1_{3,3} - p1_{3,1} = 2 \; \boldsymbol{ft}$ and creates an arc length on the retina (in this model) of approximately $s_{cir.3} = 1.029 \; \boldsymbol{in}$.

Additional light rays are now added representing an object straight out from the eye. This is object 5.

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \cdot ft = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix} ft \qquad p1 := \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$x_0 := p1^{\widehat{1}} = [0 \;\; 0 \;\; 0]\, ft \qquad y_0 := p1^{\widehat{2}} = [-1 \;\; -1 \;\; -1]\, ft \qquad z_0 := p1^{\widehat{3}} = [-1 \;\; 0 \;\; 1]\, ft$$

Angle of light rays (off -y axis to x axis)

$$\Theta_x := \overrightarrow{\operatorname{atan}\left(\frac{x_0}{y_0}\right)} = [0.00 \;\; 0.00 \;\; 0.00]\, deg$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} := \frac{p1}{ft} \cdot -1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \qquad p2 := \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$a := p2^{\widehat{1}} = [0 \;\; 0 \;\; 0] \qquad b := p2^{\widehat{2}} = [1 \;\; 1 \;\; 1] \qquad c := p2^{\widehat{3}} = [1 \;\; 0 \;\; -1]$$

$$\begin{bmatrix} h_1 & h_2 & h_3 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 \\ 0.025 & 0.025 & 0.025 \\ 0 & 0 & 0 \end{bmatrix} \cdot ft \qquad O := \begin{bmatrix} h_1 & h_2 & h_3 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

$$h := O^{\widehat{1}} = [0 \;\; 0 \;\; 0]\, ft \qquad k := O^{\widehat{2}} = [0.025 \;\; 0.025 \;\; 0.025]\, ft \qquad l := O^{\widehat{3}} = [0 \;\; 0 \;\; 0]\, ft$$

$$(x_0 + a \cdot t - h)^2 + (y_0 + b \cdot t - k)^2 + (z_0 + c \cdot t - l)^2 = r^2$$

$$(a^2 + b^2 + c^2) \cdot t^2 + 2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l)) \cdot t + (x_0 - h)^2 + (y_0 - k)^2 + (z_0 - l)^2 - r^2 = 0$$

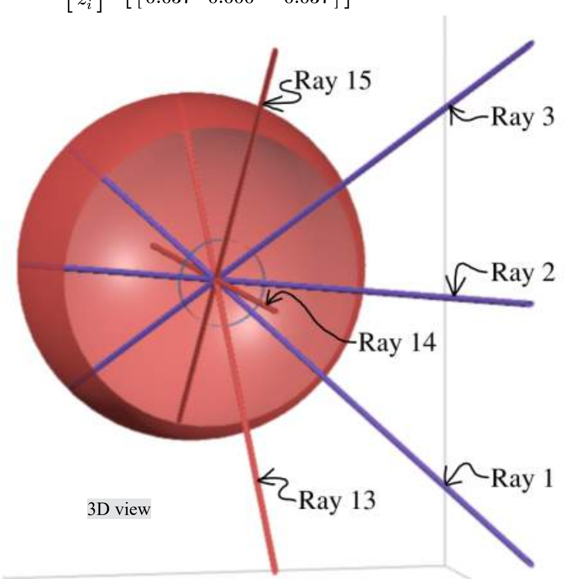
$$\Delta := \overrightarrow{(2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l)))^2 - 4 \cdot (a^2 + b^2 + c^2) \cdot ((x_0 - h)^2 + (y_0 - k)^2 + (z_0 - l)^2 - r_{ret}^{\,2})} = [0.01 \;\; 0.01 \;\; 0.01]\, ft^2$$

$$t_1 := \overrightarrow{\frac{-(2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l))) + \sqrt{\Delta}}{2 \cdot (a^2 + b^2 + c^2)}} = [1.04 \;\; 1.06 \;\; 1.04]\, ft$$

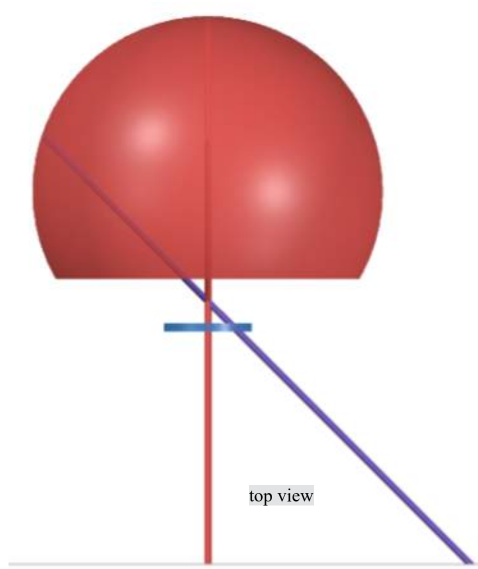
$$t_2 := \overrightarrow{\frac{-(2 \cdot (a \cdot (x_0 - h) + b \cdot (y_0 - k) + c \cdot (z_0 - l))) - \sqrt{\Delta}}{2 \cdot (a^2 + b^2 + c^2)}} = [0.99 \;\; 0.99 \;\; 0.99]\, ft$$

Intersection points of light rays at retina $\quad x_i := \overrightarrow{x_0 + a \cdot t_1} \quad y_i := \overrightarrow{y_0 + b \cdot t_1} \quad z_i := \overrightarrow{z_0 + c \cdot t_1}$

$$\begin{array}{cc} & \text{Ray 13} \quad \text{Ray 14} \quad \text{Ray 15} \\ \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = & \begin{bmatrix} [0.000 \;\; 0.000 \;\; 0.000] \\ [0.037 \;\; 0.064 \;\; 0.037] \\ [0.037 \;\; 0.000 \;\; -0.037] \end{bmatrix} ft \end{array}$$



3D view



top view

Calculating the arc length of object 5 on retina.

Intersection points of light rays at retina

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{array}{c} \text{Ray 13} \quad \text{Ray 14} \quad \text{Ray 15} \\ \begin{bmatrix} [0.000 & 0.000 & 0.000] \\ [0.037 & 0.064 & 0.037] \\ [0.037 & 0.000 & -0.037] \end{bmatrix} \end{array} \; ft$$

$$x_{i.f} := \overrightarrow{x_0 + a \cdot t_2} \qquad y_{i.f} := \overrightarrow{y_0 + b \cdot t_2} \qquad z_{i.f} := \overrightarrow{z_0 + c \cdot t_2}$$

Intersection points of light rays on front of sphere

$$\begin{bmatrix} x_{i.f} \\ y_{i.f} \\ z_{i.f} \end{bmatrix} = \begin{array}{c} \text{Ray 13} \quad \text{Ray 14} \quad \text{Ray 15} \\ \begin{bmatrix} [0.000 & 0.000 & 0.000] \\ [-0.012 & -0.014 & -0.012] \\ [-0.012 & 0.000 & 0.012] \end{bmatrix} \end{array} \; ft$$

$n := 2$ (to select nth ray in matrix above)

Diameter of the circle along the sphere in which the intersection points are on (from here called: small circle)

$$dia_{cir.5} := \sqrt{{x_{i_{1,n}}}^2 + {y_{i_{1,n}}}^2 + {z_{i_{1,n}}}^2} + \sqrt{{x_{i.f_{1,n}}}^2 + {y_{i.f_{1,n}}}^2 + {z_{i.f_{1,n}}}^2} = 0.079 \; ft$$

Radius of the small circle

$$r_{cir.5} := \frac{dia_{cir.5}}{2} = 0.039 \; ft$$

(which is the same as the radius of the retina, $r_{ret} = 0.039 \; ft$)

Length from (0,0,0) to center of the small circle

$$l_{cir.5} := \sqrt{{x_{i_{1,n}}}^2 + {y_{i_{1,n}}}^2 + {z_{i_{1,n}}}^2} - r_{cir.5} = 0.025 \; ft$$

Coordinates of center of small circle

$$x_{cir.5} := \frac{\left|x_{i_{1,n}}\right| + \left|x_{i.f_{1,n}}\right|}{2} \cdot -1 + x_{i.f_{1,n}} = 0.000 \; ft \qquad y_{cir.5} := \frac{\left|y_{i_{1,n}}\right| + \left|y_{i.f_{1,n}}\right|}{2} + y_{i.f_{1,n}} = 0.025 \; ft$$

$$z_{cir.5} := \frac{\left|z_{i_{1,n}}\right| + \left|z_{i.f_{1,n}}\right|}{2} + z_{i.f_{1,n}} = 0.000 \; ft$$

(which are the same as the coordinates of the center of the spherical retina)

$m := 1$ (to select mth ray in matrix above)

Angle subtended by arc length of object 5 on retina at center of small circle

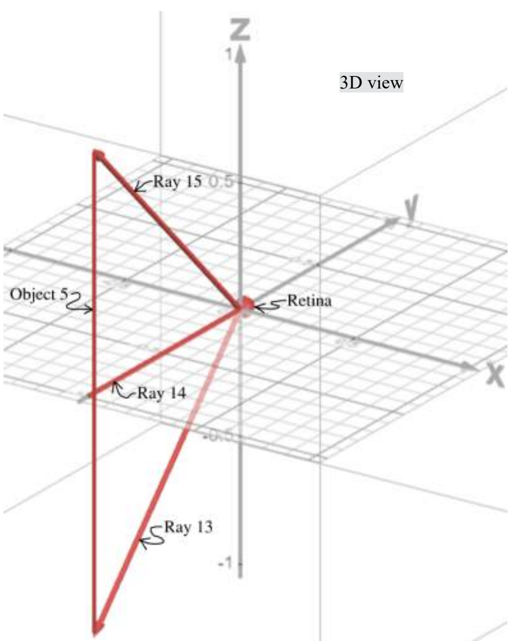
$$\Theta_{cir.5} := 2 \cdot \operatorname{atan}\left(\frac{\left| z_{i_{1,m}} - z_{cir.5} \right|}{\sqrt{\left(\left|x_{i_{1,m}} - x_{cir.5}\right|\right)^2 + \left(\left|y_{i_{1,m}} - y_{cir.5}\right|\right)^2}} \right) = 143.36 \; deg$$

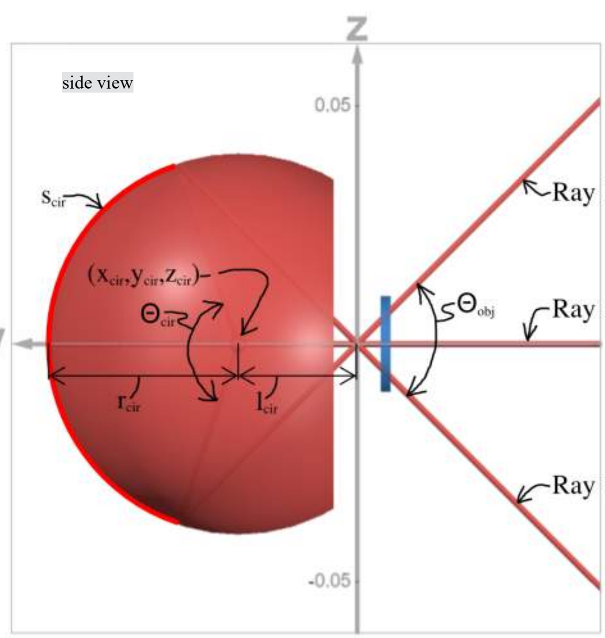
Arc length of object 5 on retina

$$s_{cir.5} := r_{cir.5} \cdot \Theta_{cir.5} = 0.099 \; ft$$

Angle subtended by object 5 in physical space at the eye

$$\Theta_{obj.5} := 2 \cdot \operatorname{atan}\left(\frac{\left| p1_{3,1} - 0 \right|}{\sqrt{\left(\left|p1_{1,1} - 0\right|\right)^2 + \left(\left|p1_{2,1} - 0\right|\right)^2}} \right) = 90 \; deg$$





The height of the object 5 in physical space is $p1_{3,3} - p1_{3,1} = 2 \; ft$ and creates an arc length on the retina (in this model) of approximately $s_{cir.5} = 1.182 \; in$.

Combining all light rays (1 - 15) into matrix form for summary.

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 & z_9 & z_{10} & z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \end{bmatrix} := \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0.5 & 0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \cdot ft$$

$$p1 := \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 & z_9 & z_{10} & z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \end{bmatrix}$$

$x_0 := p1^{\langle 1 \rangle} = [1 \;\; 1 \;\; 1 \;\; -1 \;\; -1 \;\; -1 \;\; 0.5 \;\; 0.5 \;\; -0.5 \;\; -0.5 \;\; -0.5 \;\; -0.5 \;\; 0 \;\; 0 \;\; 0]\, ft$ $\qquad y_0 := p1^{\langle 2 \rangle} = [-1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1 \;\; -1]\, ft$

$z_0 := p1^{\langle 3 \rangle} = [-1 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \;\; 1]\, ft$

Angle of light rays (off -y axis to x axis)

$$\Theta_x := \overrightarrow{\operatorname{atan}\left(\frac{x_0}{y_0}\right)} = [-45.00 \;\; -45.00 \;\; -45.00 \;\; 45.00 \;\; 45.00 \;\; 45.00 \;\; -26.57 \;\; -26.57 \;\; 26.57 \;\; 26.57 \;\; 26.57 \;\; 26.57 \;\; 0.00 \;\; 0.00 \;\; 0.00]\, deg$$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \end{bmatrix} := \frac{p1}{ft} \cdot -1 = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 1 & -0.5 & -0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \end{bmatrix}$$

$$p2 := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \end{bmatrix}$$

$a := p2^{\langle 1 \rangle} = [-1 \;\; -1 \;\; -1 \;\; 1 \;\; 1 \;\; 1 \;\; -0.5 \;\; -0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0 \;\; 0 \;\; 0]$ $\qquad b := p2^{\langle 2 \rangle} = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]$

$c := p2^{\langle 3 \rangle} = [1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; -1 \;\; 1 \;\; 0 \;\; -1]$

$$\begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & l_9 & l_{10} & l_{11} & l_{12} & l_{13} & l_{14} & l_{15} \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot ft$$

$$O := \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & l_9 & l_{10} & l_{11} & l_{12} & l_{13} & l_{14} & l_{15} \end{bmatrix}$$

$h := O^{\langle 1 \rangle} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]\, ft$ $\qquad k := O^{\langle 2 \rangle} = [0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025 \;\; 0.025]\, ft$

$l := O^{\langle 3 \rangle} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]\, ft$

$$\left(x_0 + a \cdot t - h\right)^2 + \left(y_0 + b \cdot t - k\right)^2 + \left(z_0 + c \cdot t - l\right)^2 = r^2$$

$$\left(a^2 + b^2 + c^2\right) \cdot t^2 + 2 \cdot \left(a \cdot \left(x_0 - h\right) + b \cdot \left(y_0 - k\right) + c \cdot \left(z_0 - l\right)\right) \cdot t + \left(x_0 - h\right)^2 + \left(y_0 - k\right)^2 + \left(z_0 - l\right)^2 - r^2 = 0$$

$$\Delta := \overrightarrow{\left(2 \cdot \left(a \cdot \left(x_0 - h\right) + b \cdot \left(y_0 - k\right) + c \cdot \left(z_0 - l\right)\right)\right)^2 - 4 \cdot \left(a^2 + b^2 + c^2\right) \cdot \left(\left(x_0 - h\right)^2 + \left(y_0 - k\right)^2 + \left(z_0 - l\right)^2 - r_{ret}^{\;2}\right)} = [0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01 \;\; 0.01]\, ft^2$$

$$t_1 := \overrightarrow{\frac{-\left(2 \cdot \left(a \cdot \left(x_0 - h\right) + b \cdot \left(y_0 - k\right) + c \cdot \left(z_0 - l\right)\right)\right) + \sqrt{\Delta}}{2 \cdot \left(a^2 + b^2 + c^2\right)}} = [1.03 \;\; 1.04 \;\; 1.03 \;\; 1.03 \;\; 1.04 \;\; 1.03 \;\; 1.03 \;\; 1.05 \;\; 1.03 \;\; 1.03 \;\; 1.05 \;\; 1.03 \;\; 1.04 \;\; 1.06 \;\; 1.04]\, ft$$

$$t_2 := \overrightarrow{\frac{-\left(2 \cdot \left(a \cdot \left(x_0 - h\right) + b \cdot \left(y_0 - k\right) + c \cdot \left(z_0 - l\right)\right)\right) - \sqrt{\Delta}}{2 \cdot \left(a^2 + b^2 + c^2\right)}} = [0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99 \;\; 0.99]\, ft$$

Intersection points of light rays at retina $\qquad x_i := \overrightarrow{x_0 + a \cdot t_1} \qquad y_i := \overrightarrow{y_0 + b \cdot t_1} \qquad z_i := \overrightarrow{z_0 + c \cdot t_1}$

Ray 1 Ray 2 Ray 3 Ray 4 Ray 5 Ray 6 Ray 7 Ray 8 Ray 9 Ray 10 Ray 11 Ray 12 Ray 13 Ray 14 Ray 15

$x_i = [-0.028 \;\; -0.037 \;\; -0.028 \;\; 0.028 \;\; 0.037 \;\; 0.028 \;\; -0.017 \;\; -0.027 \;\; 0.017 \;\; 0.017 \;\; 0.027 \;\; 0.017 \;\; 0.000 \;\; 0.000 \;\; 0.000]\, ft$

$y_i = [0.028 \;\; 0.037 \;\; 0.028 \;\; 0.028 \;\; 0.037 \;\; 0.028 \;\; 0.034 \;\; 0.054 \;\; 0.034 \;\; 0.034 \;\; 0.054 \;\; 0.034 \;\; 0.037 \;\; 0.064 \;\; 0.037]\, ft$

$z_i = [0.028 \;\; 0.000 \;\; -0.028 \;\; 0.028 \;\; 0.000 \;\; -0.028 \;\; 0.034 \;\; 0.000 \;\; -0.034 \;\; 0.034 \;\; 0.000 \;\; -0.034 \;\; 0.037 \;\; 0.000 \;\; -0.037]\, ft$

Ray 2 Ray 5 Ray 8 Ray 11 Ray 14

$x_{i.r.n} := \operatorname{augment}\left(x_{i_{1,2}}, x_{i_{1,5}}, x_{i_{1,8}}, x_{i_{1,11}}, x_{i_{1,14}}\right) = [-0.037 \;\; 0.037 \;\; -0.027 \;\; 0.027 \;\; 0.000]\, ft$

$y_{i.r.n} := \operatorname{augment}\left(y_{i_{1,2}}, y_{i_{1,5}}, y_{i_{1,8}}, y_{i_{1,11}}, y_{i_{1,14}}\right) = [0.037 \;\; 0.037 \;\; 0.054 \;\; 0.054 \;\; 0.064]\, ft$

$z_{i.r.n} := \operatorname{augment}\left(z_{i_{1,2}}, z_{i_{1,5}}, z_{i_{1,8}}, z_{i_{1,11}}, z_{i_{1,14}}\right) = [0.000 \;\; 0.000 \;\; 0.000 \;\; 0.000 \;\; 0.000]\, ft$

Calculating the arc length of objects on retina.

Intersection points of light rays on front of sphere $\qquad x_{i.f} := \overrightarrow{x_0 + a \cdot t_2} \qquad y_{i.f} := \overrightarrow{y_0 + b \cdot t_2} \qquad z_{i.f} := \overrightarrow{z_0 + c \cdot t_2}$

Ray 1 Ray 2 Ray 3 Ray 4 Ray 5 Ray 6 Ray 7 Ray 8 Ray 9 Ray 10 Ray 11 Ray 12 Ray 13 Ray 14 Ray 15

$x_{i.f} = [0.011 \;\; 0.012 \;\; 0.011 \;\; -0.011 \;\; -0.012 \;\; -0.011 \;\; 0.006 \;\; 0.007 \;\; -0.006 \;\; -0.006 \;\; -0.007 \;\; -0.006 \;\; 0.000 \;\; 0.000 \;\; 0.000]\, ft$

$y_{i.f} = [-0.011 \;\; -0.012 \;\; -0.011 \;\; -0.011 \;\; -0.012 \;\; -0.011 \;\; -0.012 \;\; -0.014 \;\; -0.012 \;\; -0.012 \;\; -0.014 \;\; -0.012 \;\; -0.012 \;\; -0.014 \;\; -0.012]\, ft$

$z_{i.f} = [-0.011 \;\; 0.000 \;\; 0.011 \;\; -0.011 \;\; 0.000 \;\; 0.011 \;\; -0.012 \;\; 0.000 \;\; 0.012 \;\; -0.012 \;\; 0.000 \;\; 0.012 \;\; -0.012 \;\; 0.000 \;\; 0.012]\, ft$

Ray 2 Ray 5 Ray 8 Ray 11 Ray 14

$$x_{i.f.n} := \text{augment}\left(x_{i.f_{1,2}}, x_{i.f_{1,5}}, x_{i.f_{1,8}}, x_{i.f_{1,11}}, x_{i.f_{1,14}}\right) = \begin{bmatrix} 0.012 & -0.012 & 0.007 & -0.007 & 0.000 \end{bmatrix} \boldsymbol{ft}$$

$$y_{i.f.n} := \text{augment}\left(y_{i.f_{1,2}}, y_{i.f_{1,5}}, y_{i.f_{1,8}}, y_{i.f_{1,11}}, y_{i.f_{1,14}}\right) = \begin{bmatrix} -0.012 & -0.012 & -0.014 & -0.014 & -0.014 \end{bmatrix} \boldsymbol{ft}$$

$$z_{i.f.n} := \text{augment}\left(z_{i.f_{1,2}}, z_{i.f_{1,5}}, z_{i.f_{1,8}}, z_{i.f_{1,11}}, z_{i.f_{1,14}}\right) = \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix} \boldsymbol{ft}$$

Diameter of the circle along the sphere in which the intersection points are on (from here called: small circle)

$$dia_{cir} := \overrightarrow{\sqrt{x_{i.r.n}^{\ 2} + y_{i.r.n}^{\ 2} + z_{i.r.n}^{\ 2}} + \sqrt{x_{i.f.n}^{\ 2} + y_{i.f.n}^{\ 2} + z_{i.f.n}^{\ 2}}} = \begin{bmatrix} 0.070 & 0.070 & 0.075 & 0.075 & 0.079 \end{bmatrix} \boldsymbol{ft}$$

Radius of the small circle

$$r_{cir} := \frac{dia_{cir}}{2} = \begin{bmatrix} 0.035 & 0.035 & 0.038 & 0.038 & 0.039 \end{bmatrix} \boldsymbol{ft}$$

Length from (0,0,0) to center of the small circle

$$l_{cir.3} := \overrightarrow{\sqrt{x_{i.r.n}^{\ 2} + y_{i.r.n}^{\ 2} + z_{i.r.n}^{\ 2}} - r_{cir}} = \begin{bmatrix} 0.018 & 0.018 & 0.022 & 0.022 & 0.025 \end{bmatrix} \boldsymbol{ft}$$

Coordinates of center of small circle

$$x_{cir} := \overrightarrow{\frac{\left|x_{i.r.n}\right| + \left|x_{i.f.n}\right|}{2} \cdot \begin{bmatrix} -1 & 1 & -1 & 1 & -1 \end{bmatrix} + x_{i.f.n}} = \begin{bmatrix} -0.013 & 0.013 & -0.010 & 0.010 & 0.000 \end{bmatrix} \boldsymbol{ft}$$

$$y_{cir} := \overrightarrow{\frac{\left|y_{i.r.n}\right| + \left|y_{i.f.n}\right|}{2} + y_{i.f.n}} = \begin{bmatrix} 0.013 & 0.013 & 0.020 & 0.020 & 0.025 \end{bmatrix} \boldsymbol{ft} \qquad z_{cir} := \overrightarrow{\frac{\left|z_{i.r.n}\right| + \left|z_{i.f.n}\right|}{2} + z_{i.f.n}} = \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix} \boldsymbol{ft}$$

Ray 1 Ray 4 Ray 7 Ray 10 Ray 13

$$x_{i.r.m} := \text{augment}\left(x_{i_{1,1}}, x_{i_{1,4}}, x_{i_{1,7}}, x_{i_{1,10}}, x_{i_{1,13}}\right) = \begin{bmatrix} -0.028 & 0.028 & -0.017 & 0.017 & 0.000 \end{bmatrix} \boldsymbol{ft}$$

$$y_{i.r.m} := \text{augment}\left(y_{i_{1,1}}, y_{i_{1,4}}, y_{i_{1,7}}, y_{i_{1,10}}, y_{i_{1,13}}\right) = \begin{bmatrix} 0.028 & 0.028 & 0.034 & 0.034 & 0.037 \end{bmatrix} \boldsymbol{ft}$$

$$z_{i.r.m} := \text{augment}\left(z_{i_{1,1}}, z_{i_{1,4}}, z_{i_{1,7}}, z_{i_{1,10}}, z_{i_{1,13}}\right) = \begin{bmatrix} 0.028 & 0.028 & 0.034 & 0.034 & 0.037 \end{bmatrix} \boldsymbol{ft}$$

Ray 1 Ray 4 Ray 7 Ray 10 Ray 13

$$x_{i.f.m} := \text{augment}\left(x_{i.f_{1,1}}, x_{i.f_{1,4}}, x_{i.f_{1,7}}, x_{i.f_{1,10}}, x_{i.f_{1,13}}\right) = \begin{bmatrix} 0.011 & -0.011 & 0.006 & -0.006 & 0.000 \end{bmatrix} \boldsymbol{ft}$$

$$y_{i.f.m} := \text{augment}\left(y_{i.f_{1,1}}, y_{i.f_{1,4}}, y_{i.f_{1,7}}, y_{i.f_{1,10}}, y_{i.f_{1,13}}\right) = \begin{bmatrix} -0.011 & -0.011 & -0.012 & -0.012 & -0.012 \end{bmatrix} \boldsymbol{ft}$$

$$z_{i.f.m} := \text{augment}\left(z_{i.f_{1,1}}, z_{i.f_{1,4}}, z_{i.f_{1,7}}, z_{i.f_{1,10}}, z_{i.f_{1,13}}\right) = \begin{bmatrix} -0.011 & -0.011 & -0.012 & -0.012 & -0.012 \end{bmatrix} \boldsymbol{ft}$$

Angle subtended by arc length of objects on retina at center of small circle

Obj 1 Obj 2 Obj 3 Obj 4 Obj 5

$$\Theta_{cir} := \overrightarrow{2 \cdot \text{atan}\left(\frac{\left|z_{i.r.m} - z_{cir}\right|}{\sqrt{\left(\left|x_{i.r.m} - x_{cir}\right|\right)^2 + \left(\left|y_{i.r.m} - y_{cir}\right|\right)^2}}\right)} = \begin{bmatrix} 104.26 & 104.26 & 130.14 & 130.14 & 143.36 \end{bmatrix} \boldsymbol{deg}$$

Arc length of objects on retina

Obj 1 Obj 2 Obj 3 Obj 4 Obj 5

$$s_{cir} := \overrightarrow{r_{cir} \cdot \Theta_{cir}} = \begin{bmatrix} 0.768 & 0.768 & 1.029 & 1.029 & 1.182 \end{bmatrix} \boldsymbol{in}$$

$$p1_{x.m} := \text{augment}\left(p1_{1,1}, p1_{1,4}, p1_{1,7}, p1_{1,10}, p1_{1,13}\right) = \begin{bmatrix} 1 & -1 & 0.5 & -0.5 & 0 \end{bmatrix} \boldsymbol{ft} \qquad p1_{y.m} := \text{augment}\left(p1_{2,1}, p1_{2,4}, p1_{2,7}, p1_{2,10}, p1_{2,13}\right) = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \end{bmatrix} \boldsymbol{ft}$$

$$p1_{z.m} := \text{augment}\left(p1_{3,1}, p1_{3,4}, p1_{3,7}, p1_{3,10}, p1_{3,13}\right) = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \end{bmatrix} \boldsymbol{ft}$$

Angle subtended by objects in physical space at the eye

Obj 1 Obj 2 Obj 3 Obj 4 Obj 5

$$\Theta_{obj} := \overrightarrow{2 \cdot \text{atan}\left(\frac{\left|p1_{z.m} - 0\right|}{\sqrt{\left(\left|p1_{x.m} - 0\right|\right)^2 + \left(\left|p1_{y.m} - 0\right|\right)^2}}\right)} = \begin{bmatrix} 70.53 & 70.53 & 83.62 & 83.62 & 90.00 \end{bmatrix} \boldsymbol{deg}$$

Angle of objects relative to straight out (off -y axis to x axis)

Obj 1 Obj 2 Obj 3 Obj 4 Obj 5

$$\Theta_{x.m} := \text{augment}\left(\Theta_{x_{1,1}}, \Theta_{x_{1,4}}, \Theta_{x_{1,7}}, \Theta_{x_{1,10}}, \Theta_{x_{1,13}}\right) = \begin{bmatrix} -45.00 & 45.00 & -26.57 & 26.57 & 0.00 \end{bmatrix} \boldsymbol{deg}$$

Summary

Angle subtended by object in physical space at the eye

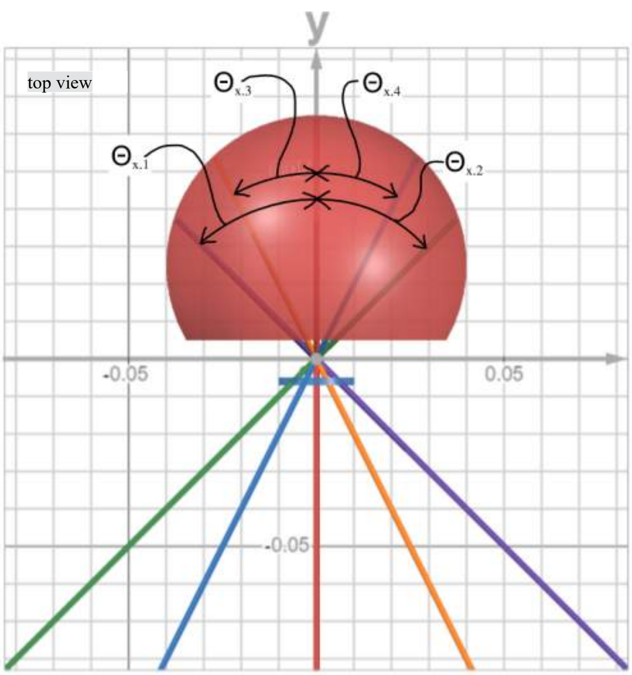
Obj 1 Obj 2 Obj 3 Obj 4 Obj 5

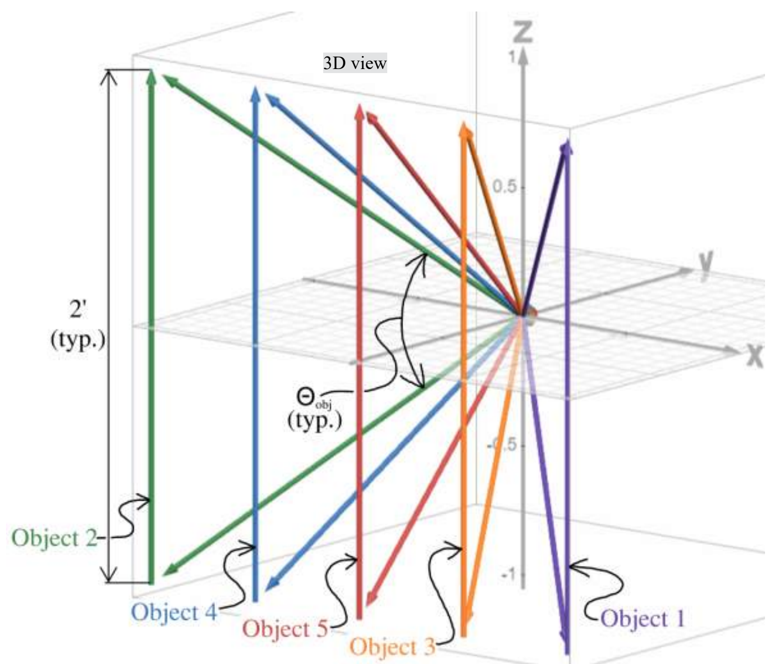
$$\Theta_{obj} = \begin{bmatrix} 70.53 & 70.53 & 83.62 & 83.62 & 90.00 \end{bmatrix} \boldsymbol{deg}$$

Angle of objects relative to straight out (off -y axis to x axis)

Obj 1 Obj 2 Obj 3 Obj 4 Obj 5

$$\Theta_{x.m} = \begin{bmatrix} -45.00 & 45.00 & -26.57 & 26.57 & 0.00 \end{bmatrix} \boldsymbol{deg}$$

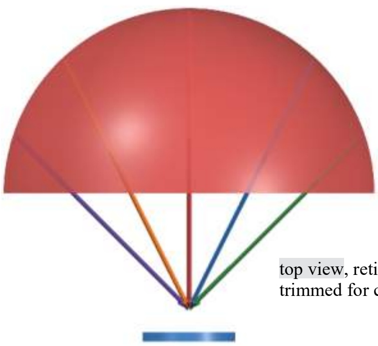




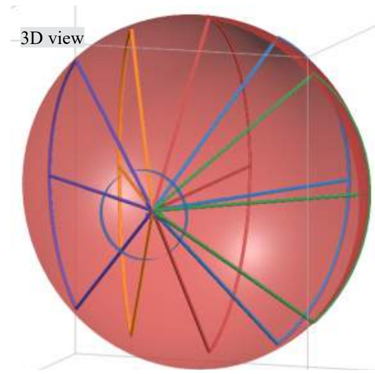
Summary (cont.)

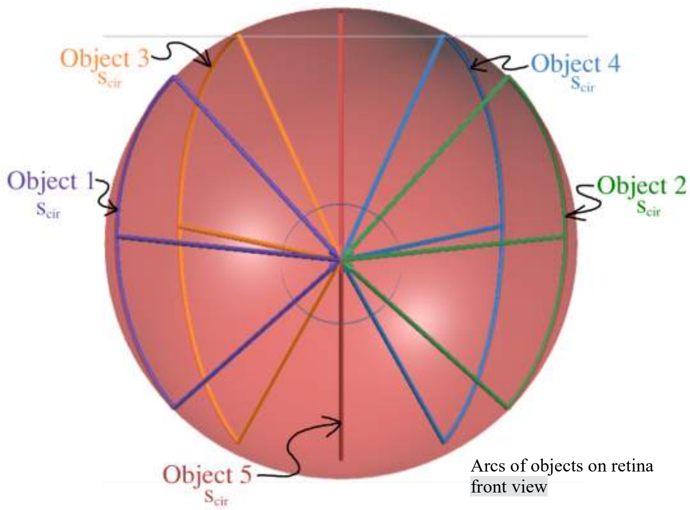
Arc length of object on retina
$$\begin{array}{c} \begin{array}{ccccc} \text{Obj 1} & \text{Obj 2} & \text{Obj 3} & \text{Obj 4} & \text{Obj 5} \end{array} \\ s_{cir} = \begin{bmatrix} 0.064 & 0.064 & 0.086 & 0.086 & 0.099 \end{bmatrix} \, \boldsymbol{ft} \end{array}$$
$$\begin{array}{c} \begin{array}{ccccc} \text{Obj 1} & \text{Obj 2} & \text{Obj 3} & \text{Obj 4} & \text{Obj 5} \end{array} \\ s_{cir} = \begin{bmatrix} 0.768 & 0.768 & 1.029 & 1.029 & 1.182 \end{bmatrix} \, \boldsymbol{in} \end{array}$$

Angle subtended by object arc on retina at center of small circle
$$\begin{array}{c} \begin{array}{ccccc} \text{Obj 1} & \text{Obj 2} & \text{Obj 3} & \text{Obj 4} & \text{Obj 5} \end{array} \\ \Theta_{cir} = \begin{bmatrix} 104.26 & 104.26 & 130.14 & 130.14 & 143.36 \end{bmatrix} \, \boldsymbol{deg} \end{array}$$

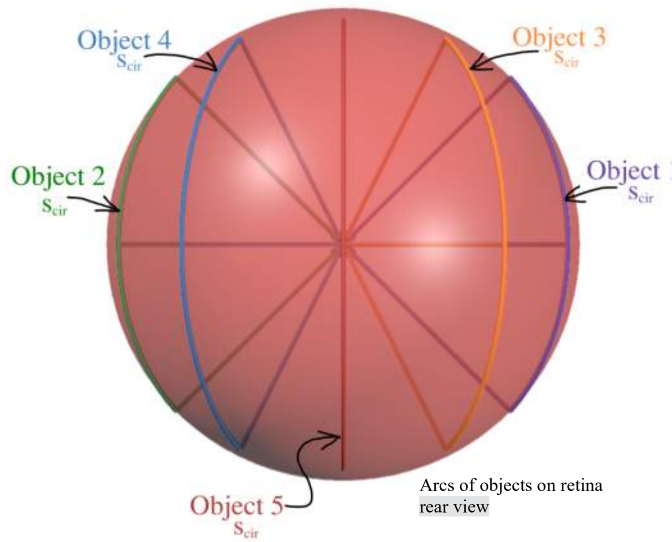


top view, retina pulled back and light rays trimmed for clarity

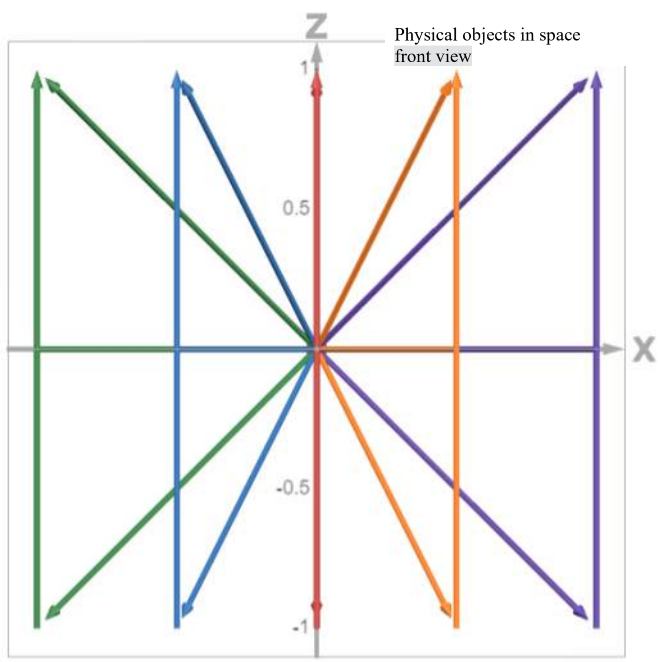




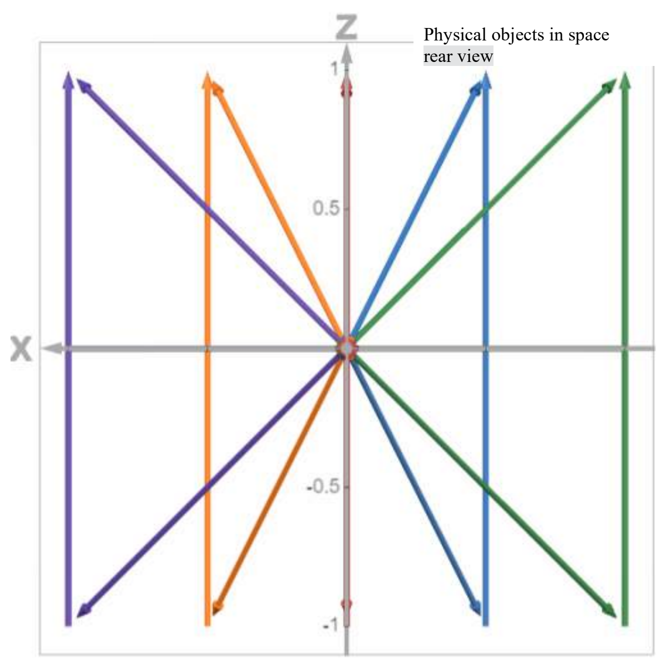
Arcs of objects on retina
front view



Arcs of objects on retina
rear view



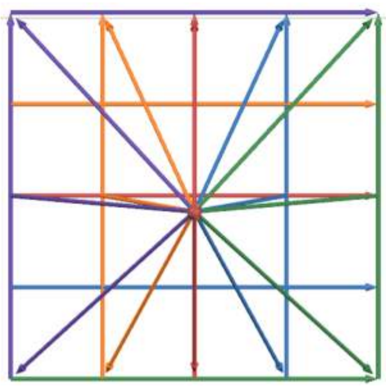
Physical objects in space
front view



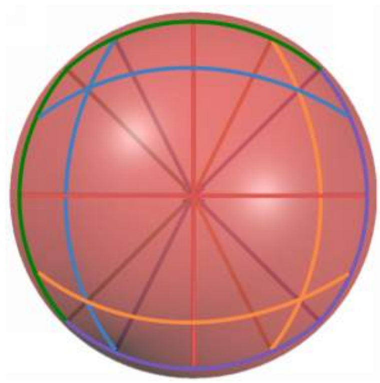
Physical objects in space
rear view

Horizontal objects are now added in physical space. Due to the symmetry of the objects in physical space and the nature of the spherical retina, calculations are not performed for the horizontal light rays. The vertically oriented arcs on the retina are simply rotated 90°.

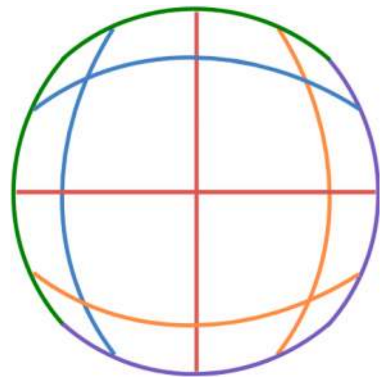
Physical objects in space rear view



Arcs of objects on retina rear view



Convergence lines of our visual perception



Note: this diagram represents the visual perception for an eye looking straight out (at intersection of red lines)

As can be seen, the arcs formed on the concave surface of the retina are creating a hyperbolic grid. In this example, a concave spherical retina is receiving planar information and creating a hyperbolic field of view.

Shown below are the equations for the vertically oriented arcs formed on the retina from the vertical physical objects (for plotting purposes).

Obj 1 Obj 2 Obj 3 Obj 4 Obj 5 (typ.)

Radius of the small circle $r_{cir} = [0.035 \quad 0.035 \quad 0.038 \quad 0.038 \quad 0.039]\ ft$

Angle of light rays (off -y axis to x axis) $\Theta_{x.n} := \text{augment}\left(\Theta_{x_{1,1}}, \Theta_{x_{1,4}}, \Theta_{x_{1,7}}, \Theta_{x_{1,10}}, \Theta_{x_{1,13}}\right) = [-45 \quad 45 \quad -26.57 \quad 26.57 \quad 0]\ deg$

Coordinates of center of small circle $x_{cir} = [-0.013 \quad 0.013 \quad -0.010 \quad 0.010 \quad 0.000]\ ft$

$y_{cir} = [0.013 \quad 0.013 \quad 0.020 \quad 0.020 \quad 0.025]\ ft$

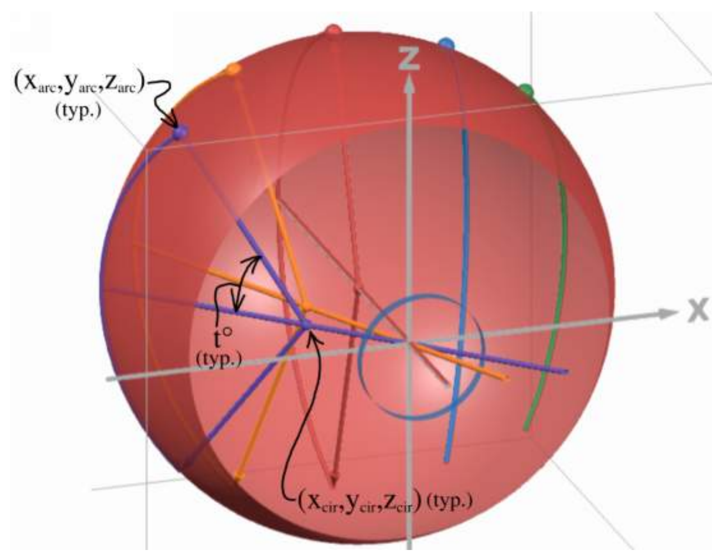
$z_{cir} = [0.000 \quad 0.000 \quad 0.000 \quad 0.000 \quad 0.000]\ ft$

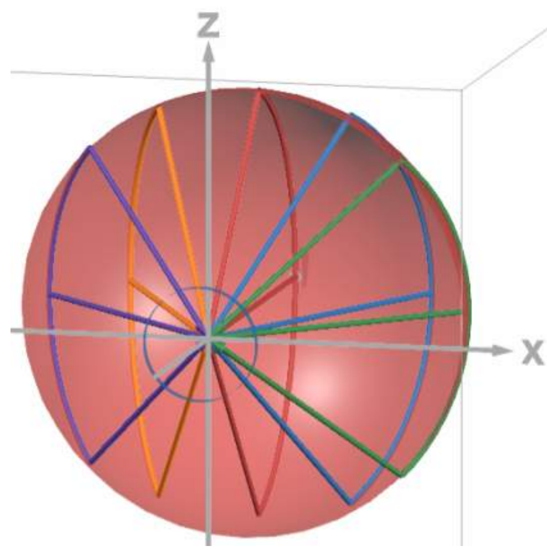
Select angle to arc from center of small circle (relative to x,y plane) $t := 0.5 \cdot \Theta_{cir} = [52.13 \quad 52.13 \quad 65.07 \quad 65.07 \quad 71.68]\ deg$ (for this example, choosing top of arc)

Coordinates of point along arc on retina $x_{arc} := \overrightarrow{r_{cir} \cdot \sin(\Theta_{x.n}) \cdot \cos(t) + x_{cir}} = [-0.028 \quad 0.028 \quad -0.017 \quad 0.017 \quad 0.000]\ ft$

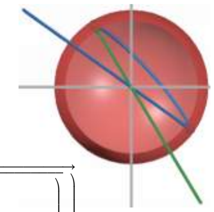
$y_{arc} := \overrightarrow{r_{cir} \cdot \cos(\Theta_{x.n}) \cdot \cos(t) + y_{cir}} = [0.028 \quad 0.028 \quad 0.034 \quad 0.034 \quad 0.037]\ ft$

$z_{arc} := \overrightarrow{r_{cir} \cdot \sqrt{\sin(\Theta_{x.n})^2 + \cos(\Theta_{x.n})^2} \cdot \sin(t) + z_{cir}} = [0.028 \quad 0.028 \quad 0.034 \quad 0.034 \quad 0.037]\ ft$





Up until now, objects which are symmetric about the x-y plane have been used, The following calculations will use an object without symmetry relative to the planes of the coordinate system to develop more general equations.



Coordinates of center of sphere $h := 0\ ft$ $k := 0.025\ ft$ $l := 0\ ft$

Functions to retrieve coordinate on retina from input of coordinate in physical space:

$$x_{ret}(x,y,z) := x + \left(\frac{x}{ft} \cdot -1\right) \cdot \left(\frac{-\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right) + \sqrt{\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right)^2 - 4 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right) \cdot \left((x-h)^2 + (y-k)^2 + (z-l)^2 - r_{ret}^2\right)}}{2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right)} \right)$$

$$y_{ret}(x,y,z) := y + \left(\frac{y}{ft} \cdot -1\right) \cdot \left(\frac{-\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right) + \sqrt{\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right)^2 - 4 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right) \cdot \left((x-h)^2 + (y-k)^2 + (z-l)^2 - r_{ret}^2\right)}}{2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right)} \right)$$

$$z_{ret}(x,y,z) := z + \left(\frac{z}{ft} \cdot -1\right) \cdot \left(\frac{-\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right) + \sqrt{\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right)^2 - 4 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right) \cdot \left((x-h)^2 + (y-k)^2 + (z-l)^2 - r_{ret}^2\right)}}{2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right)} \right)$$

Input coordinates in physical space $x_1 := -0.35\ ft$ $y_1 := -0.45\ ft$ $z_1 := 0.25\ ft$ $x_2 := 0.75\ ft$ $y_2 := -1.5\ ft$ $z_2 := -1.25\ ft$

Output coordinate on spherical retina

$p_{x.1} := x_{ret}(x_1, y_1, z_1) = 0.030\ ft$ $p_{x.2} := x_{ret}(x_2, y_2, z_2) = -0.019\ ft$

$p_{y.1} := y_{ret}(x_1, y_1, z_1) = 0.039\ ft$ $p_{y.2} := y_{ret}(x_2, y_2, z_2) = 0.038\ ft$

$p_{z.1} := z_{ret}(x_1, y_1, z_1) = -0.021\ ft$ $p_{z.2} := z_{ret}(x_2, y_2, z_2) = 0.032\ ft$

Functions to retrieve coordinate on "front" of spherical retina:

$$x_f(x,y,z) := x + \left(\frac{x}{ft} \cdot -1\right) \cdot \left(\frac{-\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right) - \sqrt{\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right)^2 - 4 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right) \cdot \left((x-h)^2 + (y-k)^2 + (z-l)^2 - r_{ret}^2\right)}}{2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right)} \right)$$

$$y_f(x,y,z) := y + \left(\frac{y}{ft} \cdot -1\right) \cdot \left(\frac{-\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right) - \sqrt{\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right)^2 - 4 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right) \cdot \left((x-h)^2 + (y-k)^2 + (z-l)^2 - r_{ret}^2\right)}}{2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right)} \right)$$

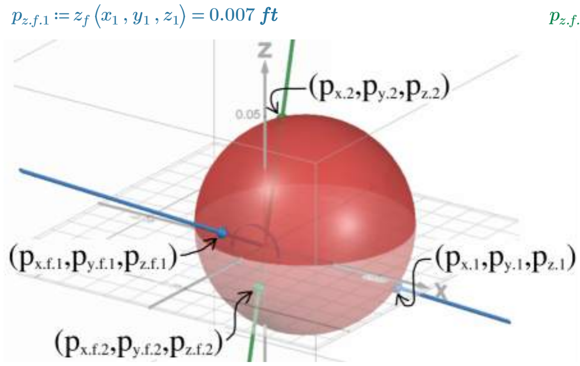
$$z_f(x,y,z) := z + \left(\frac{z}{ft} \cdot -1\right) \cdot \left(\frac{-\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right) - \sqrt{\left(2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right) \cdot (x-h) + \left(\frac{y}{ft} \cdot -1\right) \cdot (y-k) + \left(\frac{z}{ft} \cdot -1\right) \cdot (z-l)\right)\right)^2 - 4 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right) \cdot \left((x-h)^2 + (y-k)^2 + (z-l)^2 - r_{ret}^2\right)}}{2 \cdot \left(\left(\frac{x}{ft} \cdot -1\right)^2 + \left(\frac{y}{ft} \cdot -1\right)^2 + \left(\frac{z}{ft} \cdot -1\right)^2\right)} \right)$$

Output coordinate on "front" of spherical retina

$p_{x.f.1} := x_f(x_1, y_1, z_1) = -0.010\ ft$ $p_{x.f.2} := x_f(x_2, y_2, z_2) = 0.006\ ft$

$p_{y.f.1} := y_f(x_1, y_1, z_1) = -0.013\ ft$ $p_{y.f.2} := y_f(x_2, y_2, z_2) = -0.012\ ft$

$p_{z.f.1} := z_f(x_1, y_1, z_1) = 0.007\ ft$ $p_{z.f.2} := z_f(x_2, y_2, z_2) = -0.010\ ft$



$x_{avg} := \text{mean}\left(p_{x.1}, p_{x.2}\right) = 0.005\ ft$

$y_{avg} := \text{mean}\left(p_{y.1}, p_{y.2}\right) = 0.038\ ft$

$z_{avg} := \text{mean}\left(p_{z.1}, p_{z.2}\right) = 0.005\ ft$

$p_{x.bi.1} := x_{ret}\left(x_{avg}, y_{avg}, z_{avg}\right) = -0.002\ ft$

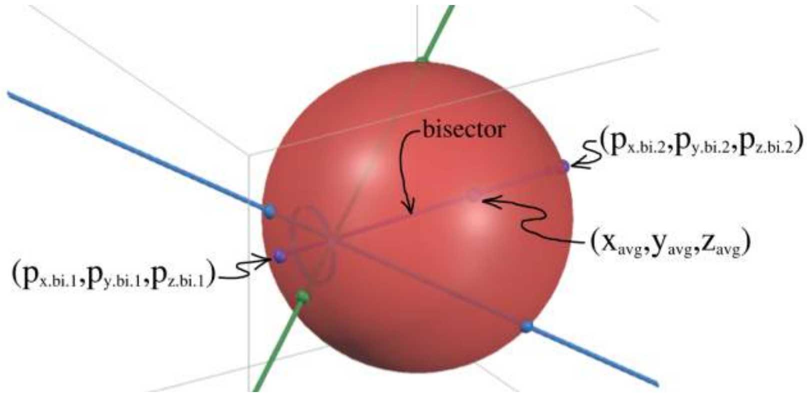
$p_{y.bi.1} := y_{ret}\left(x_{avg}, y_{avg}, z_{avg}\right) = -0.014\ ft$

$p_{z.bi.1} := z_{ret}\left(x_{avg}, y_{avg}, z_{avg}\right) = -0.002\ ft$

$p_{x.bi.2} := x_{ret}\left(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}\right) = 0.009\ ft$

$p_{y.bi.2} := y_{ret}\left(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}\right) = 0.062\ ft$

$p_{z.bi.2} := z_{ret}\left(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}\right) = 0.008\ ft$



$$r_{cir} := \frac{\sqrt{{p_{x.bi.1}}^2 + {p_{y.bi.1}}^2 + {p_{z.bi.1}}^2} + \sqrt{{p_{x.bi.2}}^2 + {p_{y.bi.2}}^2 + {p_{z.bi.2}}^2}}{2} = 0.039\ ft$$

$x_{cir} := \text{mean}\left(p_{x.bi.1}, p_{x.bi.2}\right) = 0.003436288072231\ ft$

$y_{cir} := \text{mean}\left(p_{y.bi.1}, p_{y.bi.2}\right) = 0.024072729676250\ ft$

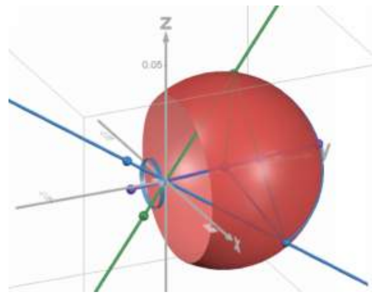
$z_{cir} := \text{mean}\left(p_{z.bi.1}, p_{z.bi.2}\right) = 0.003242507074024\ ft$

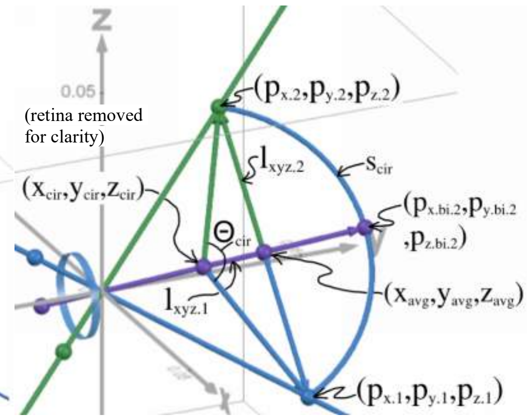
$l_{xyz.1} := \sqrt{\left(x_{avg} - x_{cir}\right)^2 + \left(y_{avg} - y_{cir}\right)^2 + \left(z_{avg} - z_{cir}\right)^2} = 0.01\ ft$

$l_{xyz.2} := \sqrt{\left(p_{x.2} - x_{avg}\right)^2 + \left(p_{y.2} - y_{avg}\right)^2 + \left(p_{z.2} - z_{avg}\right)^2} = 0.04\ ft$

$\Theta_{cir} := 2 \cdot \text{atan}\left(\frac{l_{xyz.2}}{l_{xyz.1}}\right) = 136.09\ deg$

$s_{cir} := r_{cir} \cdot \Theta_{cir} = 0.093\ ft$





Parametrizing the circle that the arc is on (for plotting purposes):

$$e_1 := \begin{bmatrix} p_{x.2} - p_{x.1} \\ p_{y.2} - p_{y.1} \\ p_{z.2} - p_{z.1} \end{bmatrix} = \begin{bmatrix} -0.04914 \\ -0.00046 \\ 0.05328 \end{bmatrix} ft$$

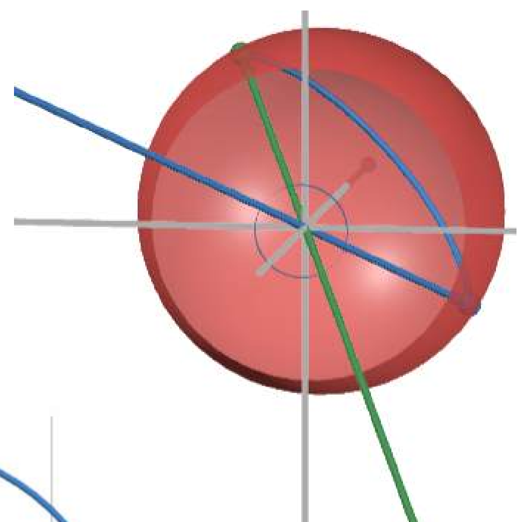
$$e_2 := \begin{bmatrix} p_{x.bi.2} - p_{x.1} \\ p_{y.bi.2} - p_{y.1} \\ p_{z.bi.2} - p_{z.1} \end{bmatrix} = \begin{bmatrix} -0.02114 \\ 0.02378 \\ 0.02987 \end{bmatrix} ft$$

$$n := e_1 \times e_2 = \begin{bmatrix} -0.00128 \\ 0.00034 \\ -0.00118 \end{bmatrix} ft^2$$

$$\check{n} := \frac{n}{\sqrt{{n_1}^2 + {n_2}^2 + {n_3}^2}} = \begin{bmatrix} -0.722156052380896 \\ 0.192574947301573 \\ -0.664383568190422 \end{bmatrix}$$

$$u := \frac{e_1}{\sqrt{{e_{1_1}}^2 + {e_{1_2}}^2 + {e_{1_3}}^2}} = \begin{bmatrix} -0.67797 \\ -0.0064 \\ 0.73507 \end{bmatrix}$$

$$v := \check{n} \times u = \begin{bmatrix} 0.1373 \\ 0.98126 \\ 0.13518 \end{bmatrix} \qquad u \cdot v = 0 \qquad \|u\| = \|v\| = 1$$

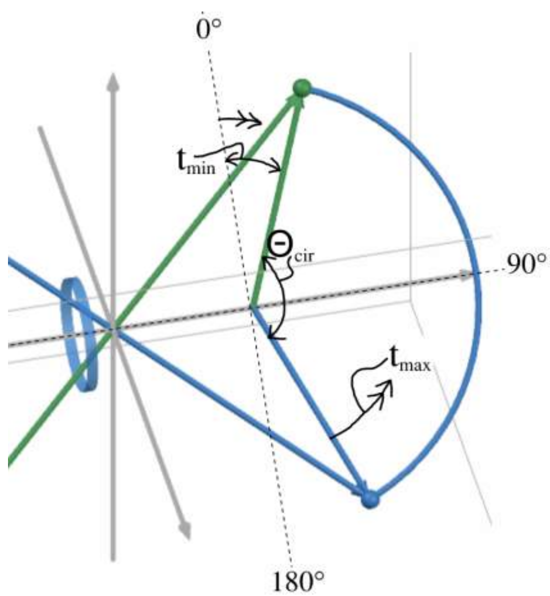


$$t := \begin{bmatrix} 90\ deg - 0.5 \cdot \Theta_{cir} \\ 90\ deg + 0.5 \cdot \Theta_{cir} \end{bmatrix} = \begin{bmatrix} 0.383 \\ 2.758 \end{bmatrix} rad \qquad \begin{bmatrix} \text{"min"} \\ \text{"max"} \end{bmatrix}$$

$$x_{arc} := \overrightarrow{x_{cir} + r_{cir} \cdot \left(u_1 \cdot \cos(t) + v_1 \cdot \sin(t)\right)} = \begin{bmatrix} -0.019 \\ 0.030 \end{bmatrix} ft$$

$$y_{arc} := \overrightarrow{y_{cir} + r_{cir} \cdot \left(u_2 \cdot \cos(t) + v_2 \cdot \sin(t)\right)} = \begin{bmatrix} 0.038 \\ 0.039 \end{bmatrix} ft$$

$$z_{arc} := \overrightarrow{z_{cir} + r_{cir} \cdot \left(u_3 \cdot \cos(t) + v_3 \cdot \sin(t)\right)} = \begin{bmatrix} 0.032 \\ -0.021 \end{bmatrix} ft$$



$$p_{arc} := \begin{bmatrix} x_{arc_1} & y_{arc_1} & z_{arc_1} \\ x_{arc_2} & y_{arc_2} & z_{arc_2} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} \text{x} & \text{y} & \text{z} \end{array} \\ \begin{bmatrix} -0.019 & 0.038 & 0.032 \\ 0.030 & 0.039 & -0.021 \end{bmatrix} \end{array} ft$$

Next, an example representing looking down a field with lines of lamp posts on either side of the observer will be analyzed.

Field of view vectors: (note: these vectors and input and output coordinates are for the grey lines in the diagram below - the field of view limits)

$\Theta_{upper} = 55\ deg$ $\quad \Theta_{lower} = 65\ deg$ $\quad \Theta_{left} := 75\ deg$ $\quad \Theta_{right} := 75\ deg$ $\quad y := -20\ ft$

$z_{upper} := |y| \cdot \tan\left(\Theta_{upper}\right) = 28.563\ ft$ $\quad z_{lower} := y \cdot \tan\left(\Theta_{lower}\right) = -42.89\ ft$ $\quad x_{left} := \dfrac{y}{\tan\left(90\ deg - \Theta_{left}\right)} = -74.641\ ft$ $\quad x_{right} := \dfrac{|y|}{\tan\left(90\ deg - \Theta_{left}\right)} = 74.641\ ft$

Input coordinates in physical space

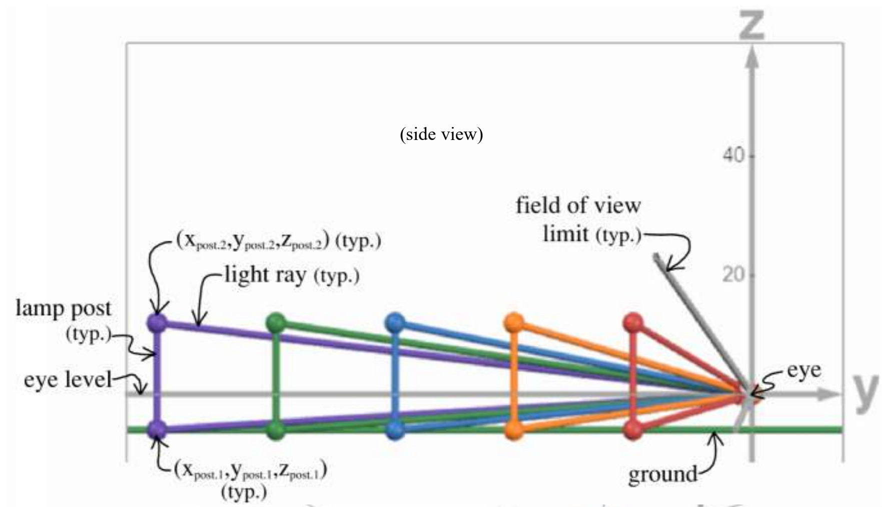
$$x_1 := \begin{bmatrix} x_{right} \\ x_{right} \\ x_{left} \\ x_{left} \\ 0\ ft \\ 0\ ft \end{bmatrix} = \begin{bmatrix} 74.641 \\ 74.641 \\ -74.641 \\ -74.641 \\ 0 \\ 0 \end{bmatrix} ft \qquad y_1 := \begin{bmatrix} y \\ y \\ y \\ y \\ y \\ y \end{bmatrix} = \begin{bmatrix} -20 \\ -20 \\ -20 \\ -20 \\ -20 \\ -20 \end{bmatrix} ft$$

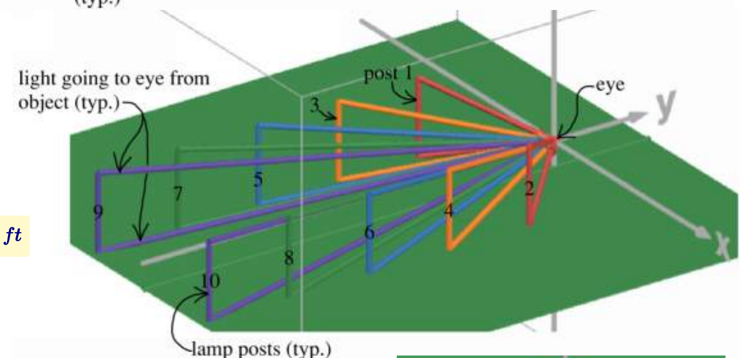
$$z_1 := \begin{bmatrix} z_{upper} \\ z_{lower} \\ z_{lower} \\ z_{upper} \\ z_{upper} \\ z_{lower} \end{bmatrix} = \begin{bmatrix} 28.563 \\ -42.890 \\ -42.890 \\ 28.563 \\ 28.563 \\ -42.890 \end{bmatrix} ft$$

Output coordinates on retina

$$x_{ret}\left(x_1, y_1, z_1\right) = \begin{bmatrix} -0.034 \\ -0.031 \\ 0.031 \\ 0.034 \\ 0.000 \\ 0.000 \end{bmatrix} ft \qquad y_{ret}\left(x_1, y_1, z_1\right) = \begin{bmatrix} 0.009 \\ 0.008 \\ 0.008 \\ 0.009 \\ 0.028 \\ 0.018 \end{bmatrix} ft$$

$$z_{ret}\left(x_1, y_1, z_1\right) = \begin{bmatrix} -0.013 \\ 0.018 \\ 0.018 \\ -0.013 \\ -0.039 \\ 0.039 \end{bmatrix} ft$$

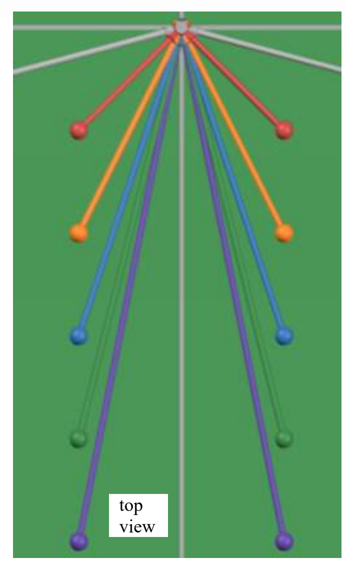




Lamp post coordinates:

$$x_{post.1} := 20\ ft \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ -20 \\ 20 \\ -20 \\ 20 \\ -20 \\ 20 \\ -20 \\ 20 \end{bmatrix} ft \qquad y_{post.1} := 20\ ft \begin{bmatrix} -1 \\ -1 \\ -2 \\ -2 \\ -3 \\ -3 \\ -4 \\ -4 \\ -5 \\ -5 \end{bmatrix} = \begin{bmatrix} -20 \\ -20 \\ -40 \\ -40 \\ -60 \\ -60 \\ -80 \\ -80 \\ -100 \\ -100 \end{bmatrix} ft \qquad z_{post.1} := -6\ ft$$

$$x_{post.2} := 20\ ft \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ -20 \\ 20 \\ -20 \\ 20 \\ -20 \\ 20 \\ -20 \\ 20 \end{bmatrix} ft \qquad y_{post.2} := 20\ ft \begin{bmatrix} -1 \\ -1 \\ -2 \\ -2 \\ -3 \\ -3 \\ -4 \\ -4 \\ -5 \\ -5 \end{bmatrix} = \begin{bmatrix} -20 \\ -20 \\ -40 \\ -40 \\ -60 \\ -60 \\ -80 \\ -80 \\ -100 \\ -100 \end{bmatrix} ft \qquad z_{post.2} := 12\ ft$$



$p_{x.1} := x_{ret}\left(x_{post.1}, y_{post.1}, z_{post.1}\right)$ $\quad p_{y.1} := y_{ret}\left(x_{post.1}, y_{post.1}, z_{post.1}\right)$ $\quad p_{z.1} := z_{ret}\left(x_{post.1}, y_{post.1}, z_{post.1}\right)$

$p_{x.2} := x_{ret}\left(x_{post.2}, y_{post.2}, z_{post.2}\right)$ $\quad p_{y.2} := y_{ret}\left(x_{post.2}, y_{post.2}, z_{post.2}\right)$ $\quad p_{z.2} := z_{ret}\left(x_{post.2}, y_{post.2}, z_{post.2}\right)$

$p_{x.f.1} := x_f\left(x_{post.1}, y_{post.1}, z_{post.1}\right)$ $\quad p_{y.f.1} := y_f\left(x_{post.1}, y_{post.1}, z_{post.1}\right)$ $\quad p_{z.f.1} := z_f\left(x_{post.1}, y_{post.1}, z_{post.1}\right)$

$p_{x.f.2} := x_f\left(x_{post.2}, y_{post.2}, z_{post.2}\right)$ $\quad p_{y.f.2} := y_f\left(x_{post.2}, y_{post.2}, z_{post.2}\right)$ $\quad p_{z.f.2} := z_f\left(x_{post.2}, y_{post.2}, z_{post.2}\right)$

$x_{avg.}\left(p_{x.1}, p_{x.2}\right) := \overrightarrow{\mathrm{mean}\left(p_{x.1}, p_{x.2}\right)}$ $\quad y_{avg.}\left(p_{y.1}, p_{y.2}\right) := \overrightarrow{\mathrm{mean}\left(p_{y.1}, p_{y.2}\right)}$ $\quad z_{avg.}\left(p_{z.1}, p_{z.2}\right) := \overrightarrow{\mathrm{mean}\left(p_{z.1}, p_{z.2}\right)}$

$x_{avg} := \overrightarrow{x_{avg.}\left(p_{x.1}, p_{x.2}\right)}$ $\quad y_{avg} := \overrightarrow{y_{avg.}\left(p_{y.1}, p_{y.2}\right)}$ $\quad z_{avg} := \overrightarrow{z_{avg.}\left(p_{z.1}, p_{z.2}\right)}$

$p_{x.bi.1} := x_{ret}\left(x_{avg}, y_{avg}, z_{avg}\right)$ $\quad p_{y.bi.1} := y_{ret}\left(x_{avg}, y_{avg}, z_{avg}\right)$ $\quad p_{z.bi.1} := z_{ret}\left(x_{avg}, y_{avg}, z_{avg}\right)$

$p_{x.bi.2} := x_{ret}\left(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}\right)$ $\quad p_{y.bi.2} := y_{ret}\left(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}\right)$ $\quad p_{z.bi.2} := z_{ret}\left(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}\right)$

$$r_{cir} := \frac{\sqrt{{p_{x.bi.1}}^2 + {p_{y.bi.1}}^2 + {p_{z.bi.1}}^2} + \sqrt{{p_{x.bi.2}}^2 + {p_{y.bi.2}}^2 + {p_{z.bi.2}}^2}}{2}$$

$x_{cir.}\left(p_{x.bi.1}, p_{x.bi.2}\right) := \overrightarrow{\mathrm{mean}\left(p_{x.bi.1}, p_{x.bi.2}\right)}$ $\quad y_{cir.}\left(p_{y.bi.1}, p_{y.bi.2}\right) := \overrightarrow{\mathrm{mean}\left(p_{y.bi.1}, p_{y.bi.2}\right)}$ $\quad z_{cir.}\left(p_{z.bi.1}, p_{z.bi.2}\right) := \overrightarrow{\mathrm{mean}\left(p_{z.bi.1}, p_{z.bi.2}\right)}$

$x_{cir} := \overrightarrow{x_{cir.}\left(p_{x.bi.1}, p_{x.bi.2}\right)}$ $\quad y_{cir} := \overrightarrow{y_{cir.}\left(p_{y.bi.1}, p_{y.bi.2}\right)}$ $\quad z_{cir} := \overrightarrow{z_{cir.}\left(p_{z.bi.1}, p_{z.bi.2}\right)}$

$l_{xyz.1} := \overrightarrow{\sqrt{\left(x_{avg} - x_{cir}\right)^2 + \left(y_{avg} - y_{cir}\right)^2 + \left(z_{avg} - z_{cir}\right)^2}}$ $\quad l_{xyz.2} := \overrightarrow{\sqrt{\left(p_{x.2} - x_{avg}\right)^2 + \left(p_{y.2} - y_{avg}\right)^2 + \left(p_{z.2} - z_{avg}\right)^2}}$ $\quad \Theta_{cir} := 2 \cdot \mathrm{atan}\left(\dfrac{l_{xyz.2}}{l_{xyz.1}}\right)$ $\quad s_{cir} := \overrightarrow{r_{cir} \cdot \Theta_{cir}}$

Parametrizing the circles that the arcs are on (used for plotting arcs):

$n := 1$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} -0.003 \\ -0.003 \\ -0.031 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.001 \\ -0.016 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.707 \\ -0.707 \\ 0.000 \end{bmatrix} \quad u_1 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} -0.101 \\ -0.101 \\ -0.990 \end{bmatrix} \quad v_1 := \hat{n} \times u_1 = \begin{bmatrix} 0.700 \\ 0.700 \\ -0.142 \end{bmatrix}$$

$n := 2$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.003 \\ -0.003 \\ -0.031 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} -0.001 \\ 0.001 \\ -0.016 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.707 \\ 0.707 \\ 0.000 \end{bmatrix} \quad u_2 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} 0.101 \\ -0.101 \\ -0.990 \end{bmatrix} \quad v_2 := \hat{n} \times u_2 = \begin{bmatrix} -0.700 \\ 0.700 \\ -0.142 \end{bmatrix}$$

$n := 3$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} -0.001 \\ -0.002 \\ -0.023 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.001 \\ -0.012 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.894 \\ -0.447 \\ 0.000 \end{bmatrix} \quad u_3 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} -0.046 \\ -0.091 \\ -0.995 \end{bmatrix} \quad v_3 := \hat{n} \times u_3 = \begin{bmatrix} 0.445 \\ 0.890 \\ -0.102 \end{bmatrix}$$

$n := 4$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.001 \\ -0.002 \\ -0.023 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.001 \\ -0.012 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.894 \\ 0.447 \\ 0.000 \end{bmatrix} \quad u_4 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} 0.046 \\ -0.091 \\ -0.995 \end{bmatrix} \quad v_4 := \hat{n} \times u_4 = \begin{bmatrix} -0.445 \\ 0.890 \\ -0.102 \end{bmatrix}$$

$n := 5$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ -0.001 \\ -0.017 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ -0.009 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.949 \\ -0.316 \\ 0.000 \end{bmatrix} \quad u_5 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} -0.024 \\ -0.071 \\ -0.997 \end{bmatrix} \quad v_5 := \hat{n} \times u_5 = \begin{bmatrix} 0.315 \\ 0.946 \\ -0.075 \end{bmatrix}$$

$n := 6$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ -0.001 \\ -0.017 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ -0.009 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.949 \\ 0.316 \\ 0.000 \end{bmatrix} \quad u_6 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} 0.024 \\ -0.071 \\ -0.997 \end{bmatrix} \quad v_6 := \hat{n} \times u_6 = \begin{bmatrix} -0.315 \\ 0.946 \\ -0.075 \end{bmatrix}$$

$n := 7$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ -0.001 \\ -0.014 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ -0.007 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.970 \\ -0.243 \\ 0.000 \end{bmatrix} \quad u_7 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} -0.014 \\ -0.056 \\ -0.998 \end{bmatrix} \quad v_7 := \hat{n} \times u_7 = \begin{bmatrix} 0.242 \\ 0.968 \\ -0.058 \end{bmatrix}$$

$n := 8$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ -0.001 \\ -0.014 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ -0.007 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.970 \\ 0.243 \\ 0.000 \end{bmatrix} \quad u_8 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} 0.014 \\ -0.056 \\ -0.998 \end{bmatrix} \quad v_8 := \hat{n} \times u_8 = \begin{bmatrix} -0.242 \\ 0.968 \\ -0.058 \end{bmatrix}$$

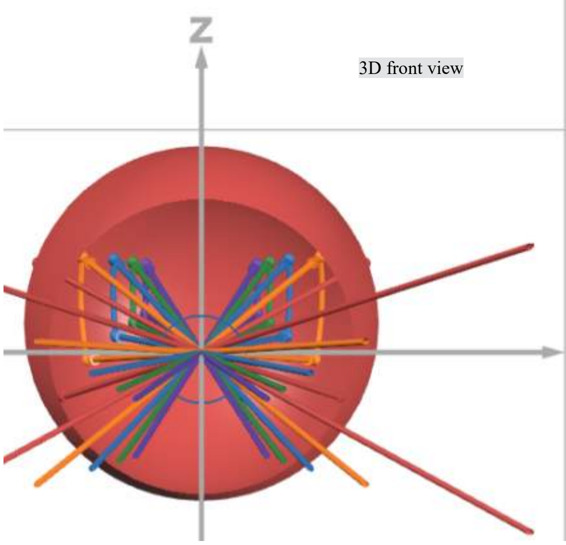
$n := 9$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ -0.001 \\ -0.011 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ -0.006 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.981 \\ -0.196 \\ 0.000 \end{bmatrix} \quad u_9 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} -0.009 \\ -0.047 \\ -0.999 \end{bmatrix} \quad v_9 := \hat{n} \times u_9 = \begin{bmatrix} 0.196 \\ 0.979 \\ -0.047 \end{bmatrix}$$

$n := 10$

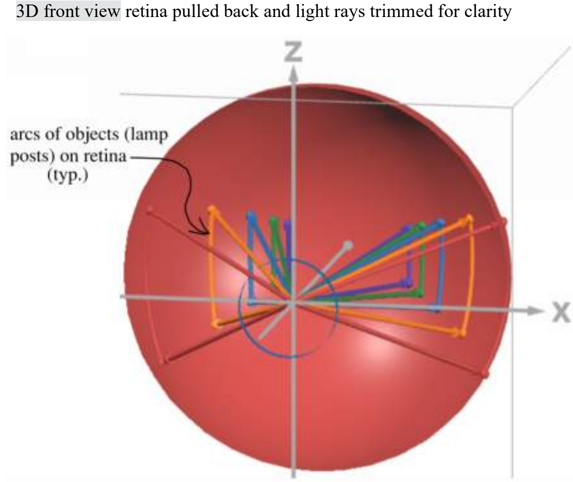
$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ -0.001 \\ -0.011 \end{bmatrix} ft \quad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ -0.006 \end{bmatrix} ft \quad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2 \quad \hat{n} := \frac{n}{\|n\|} = \begin{bmatrix} 0.981 \\ 0.196 \\ 0.000 \end{bmatrix} \quad u_{10} := \frac{e_1}{\|e_1\|} = \begin{bmatrix} 0.009 \\ -0.047 \\ -0.999 \end{bmatrix} \quad v_{10} := \hat{n} \times u_{10} = \begin{bmatrix} -0.196 \\ 0.979 \\ -0.047 \end{bmatrix}$$

$$x_{arc} = \overrightarrow{x_{cir} + r_{cir} \cdot \left(u_n \cdot cos(t) + v_n \cdot sin(t)\right)} \quad y_{arc} = \overrightarrow{y_{cir} + r_{cir} \cdot \left(u_n \cdot cos(t) + v_n \cdot sin(t)\right)} \quad z_{arc} = \overrightarrow{z_{cir} \cdot 0 + r_{cir} \cdot \left(u_n \cdot cos(t) + v_n \cdot sin(t)\right)}$$

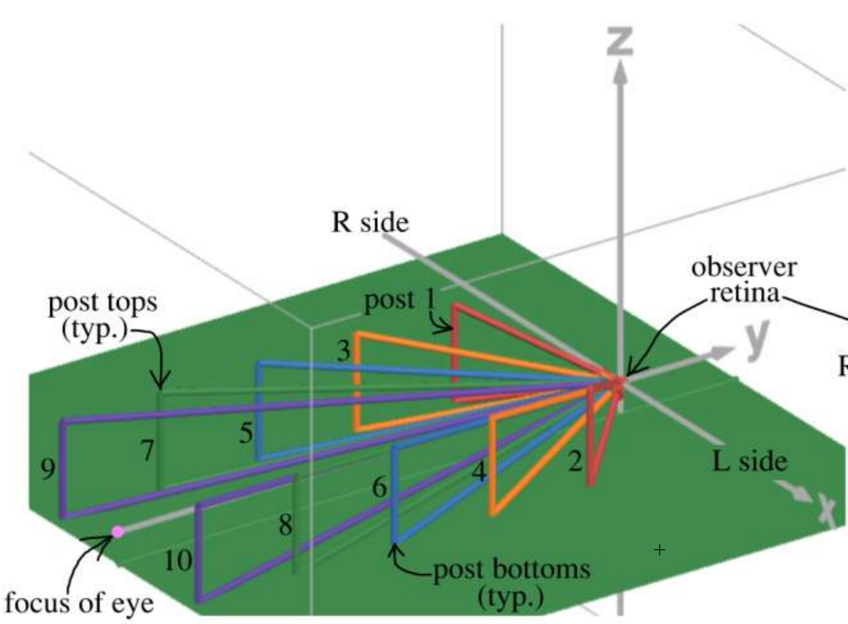


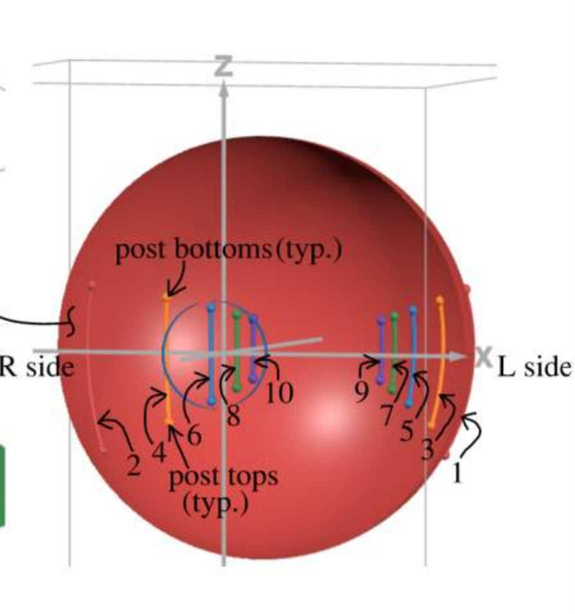
3D front view

3D front view retina pulled back and light rays trimmed for clarity



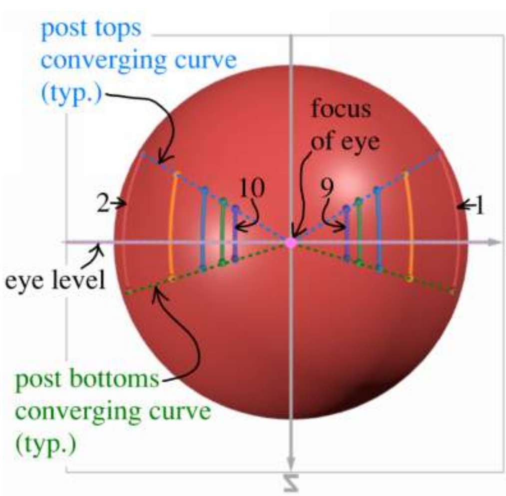
As can be seen in the diagrams below and on the previous page, light from the posts on the right side of the observer (posts 1,3,5,7,9) hit on the left side of the retina and light from the posts on the left side of the observer (2,4,6,8,10) hit on the right side of the retina. Also, the posts are inverted vertically meaning light from the post bottoms hits the top half of the retina and light from the post tops hits the bottom half of the retina. The brain apparently makes these corrections and provides a visual perception aligning with physical space.



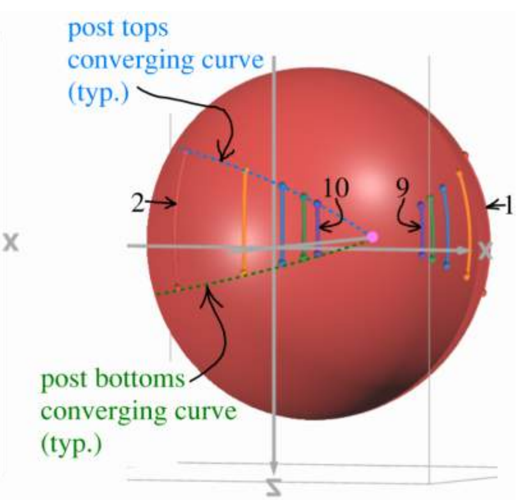


To represent the observer's visual field (what the observer sees), the retina diagram (as viewed from the front) can be flipped about the horizontal x-axis (as done in the below diagrams).

This diagram shows a 2D representation of the observer's vision if the observer is focusing straight out along eye level at the imaginary pink dot.



What the observer sees (2D representation)



This slightly rotated diagram is included to show the curve of the convergence lines in the visual field.

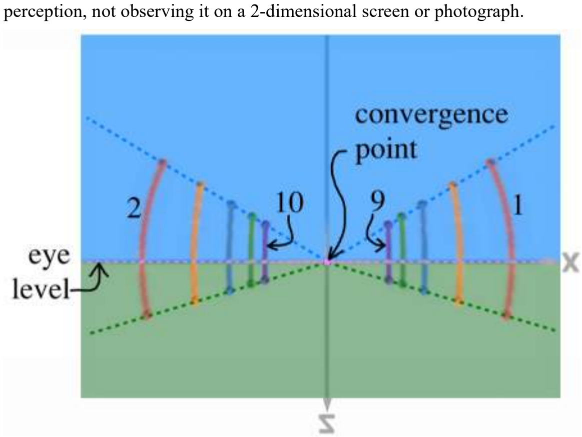
Slightly rotated view to show the curve of the converging post lines.

The diagram below is a 2D representation of the observer's visual perception (if the observer is focused straight out at the pink dot). The curves are a lot more subtle in your visual perception. A 2-dimensional representation (like a photograph) of visual perception cannot accurately display the perception we have. We are immersed in our perception, not observing it on a 2-dimensional screen or photograph.

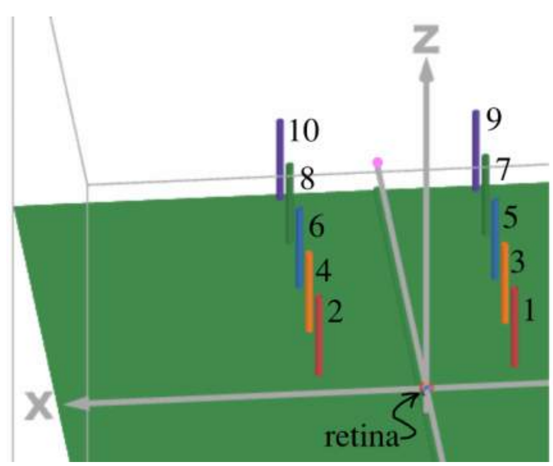
The curves are only apparent in the periphery of your vision.

If the observer was to change focus from the pink dot to post 1, then post 1 is now the focus point (and would now appear as a vertical planar object) and a new curved periphery would be formed around post 1.

Your vision is constantly creating these spheres of visual perception as you look around from focus point to focus point.

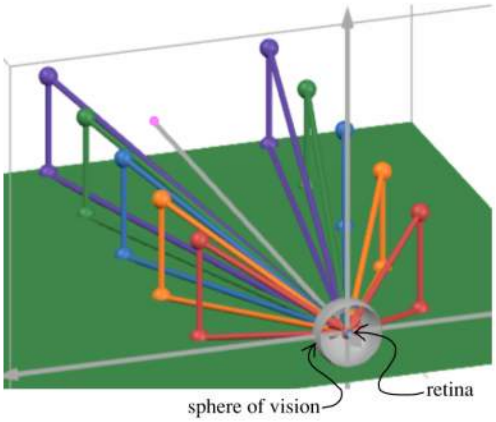


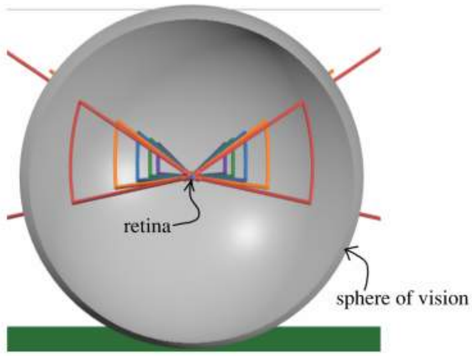
What the observer sees (2D representation, retina removed and ground and sky added)

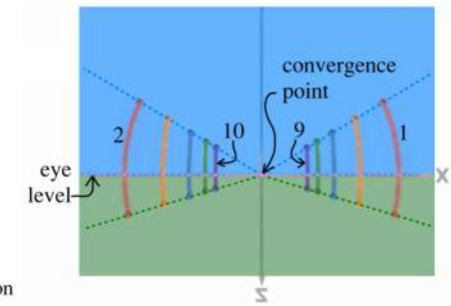


3D geometric layout.

Another way to conceptualize the field of vision is by adding a "sphere of vision" in the geometric layout which is just a larger sphere that is concentric with the retinal sphere.







This example demonstrates the convergence point that occurs in our vision straight out along eye level (at the pink dot). This is discussed on the following page after a brief discussion on the angular resolution limit of our eyes.

The human eye has an angular resolution limit of about 1 arcminute or 0.017° due to limitations of the human eye (quantity and configuration of the rods and cones on the retina) and limitations of the behavior of light (diffraction).

The example below involving a ladybug is used to illustrate the angular resolution limit of our eyes.

Height of ladybug $h_{ladybug} := 0.3\ \boldsymbol{in}$

Distance from observer to ladybug $d := 60\ \boldsymbol{ft}$

Angular resolution limit $\Theta_{lim} := 0.017\ \boldsymbol{deg}$

Coordinates of ladybug in physical space

$$p_1 := \begin{bmatrix} 0\ \boldsymbol{ft} \\ -d \\ h_{ladybug} \cdot 0.5 \end{bmatrix} = \begin{bmatrix} 0.0000 \\ -60.0000 \\ 0.0125 \end{bmatrix} \boldsymbol{ft}$$

$$p_2 := \begin{bmatrix} 0\ \boldsymbol{ft} \\ -d \\ h_{ladybug} \cdot -0.5 \end{bmatrix} = \begin{bmatrix} 0.0000 \\ -60.0000 \\ -0.0125 \end{bmatrix} \boldsymbol{ft}$$

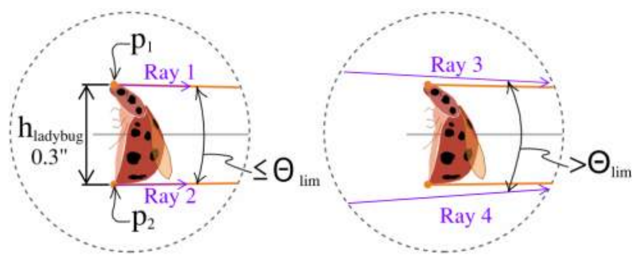
Coordinates on retina of light from ladybug

$$p_{1.ret} := \begin{bmatrix} x_{ret}\left(p_{1_1}, p_{1_2}, p_{1_3}\right) \\ y_{ret}\left(p_{1_1}, p_{1_2}, p_{1_3}\right) \\ z_{ret}\left(p_{1_1}, p_{1_2}, p_{1_3}\right) \end{bmatrix} = \begin{bmatrix} 0.00000 \\ 0.06437 \\ -0.00001 \end{bmatrix} \boldsymbol{ft}$$

$$p_{2.ret} := \begin{bmatrix} x_{ret}\left(p_{2_1}, p_{2_2}, p_{2_3}\right) \\ y_{ret}\left(p_{2_1}, p_{2_2}, p_{2_3}\right) \\ z_{ret}\left(p_{2_1}, p_{2_2}, p_{2_3}\right) \end{bmatrix} = \begin{bmatrix} 0.00000 \\ 0.06437 \\ 0.00001 \end{bmatrix} \boldsymbol{ft}$$

The minimum distance between the ladybug and the observer in which the ladybug is no longer visible to the observer $d_{lim} := \dfrac{h_{ladybug}}{\tan\left(\Theta_{lim}\right)} = 84\ \boldsymbol{ft}$

At this distance, the two light rays (1&2) are indistinguishable from one another at an angular size of $\le \Theta_{lim}$.

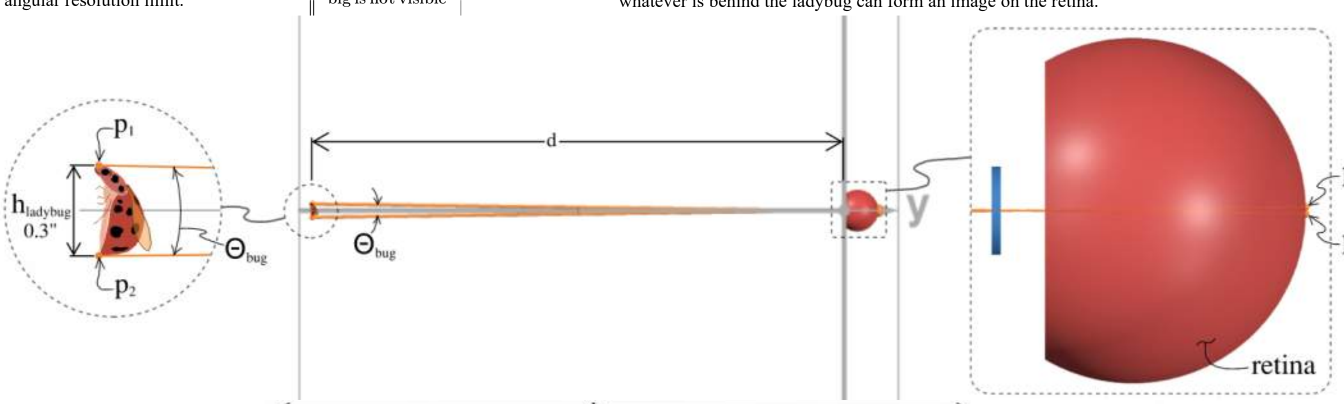


Angular size of ladybug $\Theta_{bug} := 2 \cdot \operatorname{atan}\left(\dfrac{0.5 \cdot h_{ladybug}}{d}\right) = 0.024\ \boldsymbol{deg}$

Is the ladybug visible at $d = 60\ \boldsymbol{ft}$ away from the observer? Yes, the angular size of the bug is above the angular resolution limit.

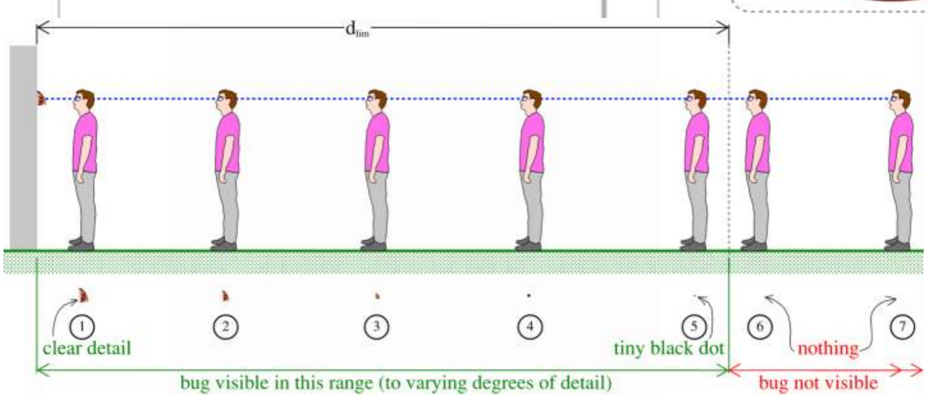
```
check_vis := if Θ_bug > Θ_lim        = "bug is visible"
                 "bug is visible"
             else
                 "big is not visible"
```

For an image to be formed on the retina, at least two distinct light rays are required. Since these two light rays are indistinguishable, an image cannot be formed on the retina. The bug is not visible to the observer. The angle subtended at the eye by light rays 3 & 4 > Θ_{lim} so two distinct rays coming from whatever is behind the ladybug can form an image on the retina.



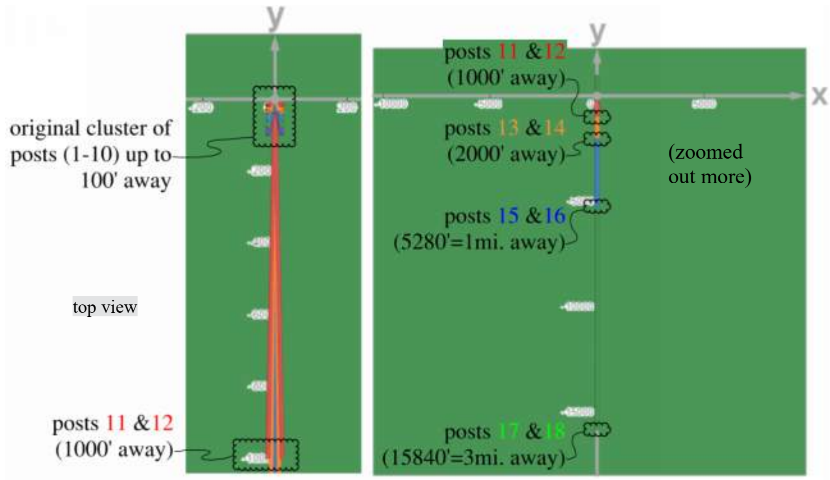
When the observer is very close to the ladybug on the wall at position 1, the details of the bug are easily perceptible.

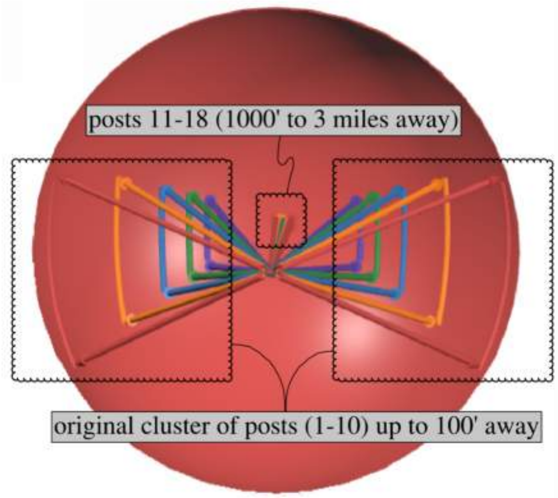
As the observer backs away from the bug, less and less detail can be perceived. In position 5 only a tiny black dot may be visible.



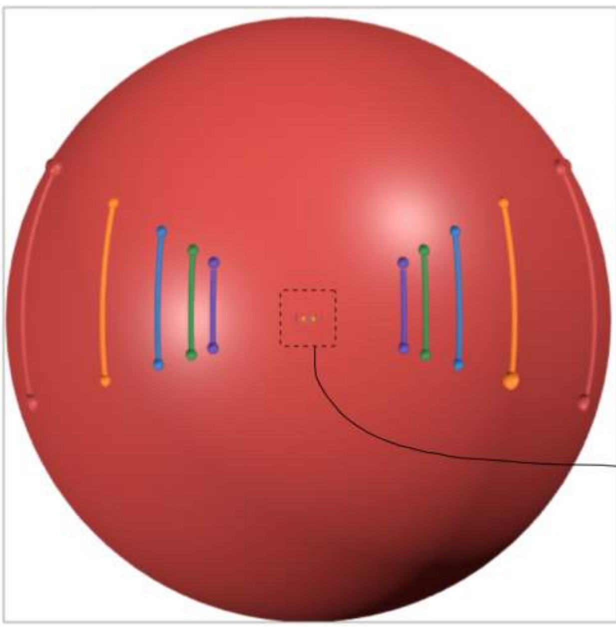
When the observer is standing beyond $d_{lim} = 84\ \boldsymbol{ft}$ at positions 6 and 7, the bug is no longer visible.

More lamp posts that are farther away (1000', 2000', 1 mile, and 3 miles) are added to the example from the previous pages.

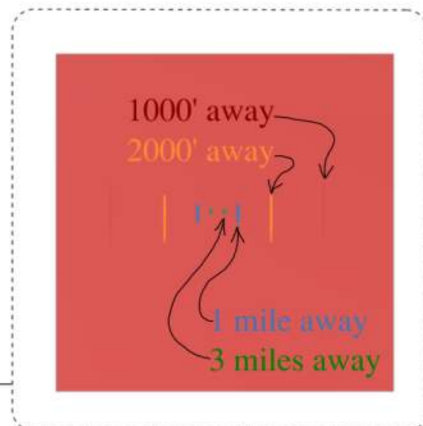




3D front view showing arcs of lamp posts formed on retina



What the observer sees (2D representation)



Zoomed in on central area of retina

As can be seen in these diagrams, these new posts that are 1000' to 3 miles away are forming very small images on the retina.

If more posts were to keep being added farther and farther away, eventually (similar to the diagrams with the ladybug) the posts will be beyond a distance corresponding to the angular resolution limit of the eye and the posts will no longer be visible to the observer.

However, in the ladybug example the observer's eye level was placed at mid-height of the ladybug. In this lamp post example, eye level is not at mid-height of the lamp posts. Also, the ground surface will play a role in the lamp post example.

The following is a discussion on horizon formation regarding the lower visual field and the ground surface.

The purpose of the first example is to get a practical sense of how much certain distances of the ground surface in front of an observer occupy the lower visual field of perception. This example uses a common surface (football field).

Length of one football field $l_{field} := 100\ yd = 0.057\ mi$ (not including end zones)

Length of multiple football fields $l_{fields} := l_{field} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 300 \\ 600 \\ 900 \\ 1200 \end{bmatrix} ft$ $\begin{bmatrix} \text{“field 1”} \\ \text{“fields 1+2”} \\ \text{“fields 1+2+3”} \\ \text{“fields 1+2+3+4”} \end{bmatrix}$

Observer eye height $h := 6\ ft$

Lower field of view angular size $\Theta_{lower} = 65\ deg$

Distance from observer to first point of ground in field of view $l_1 := h \cdot \tan\left(90\ deg - \Theta_{lower}\right) = 2.80\ ft$

Angular size of field 1 $\Theta_{field.1} := \operatorname{atan}\left(\dfrac{l_1 + l_{field}}{h}\right) - \left(90\ deg - \Theta_{lower}\right) = 63.86\ deg$

Cumulative angular size of fields $\Theta_{field.n} := \operatorname{atan}\left(\dfrac{l_1 + l_{fields}}{h}\right) - \left(90\ deg - \Theta_{lower}\right) = \begin{bmatrix} 63.86 \\ 64.43 \\ 64.62 \\ 64.71 \end{bmatrix} deg$ $\begin{bmatrix} \text{“field 1”} \\ \text{“fields 1+2”} \\ \text{“fields 1+2+3”} \\ \text{“fields 1+2+3+4”} \end{bmatrix}$

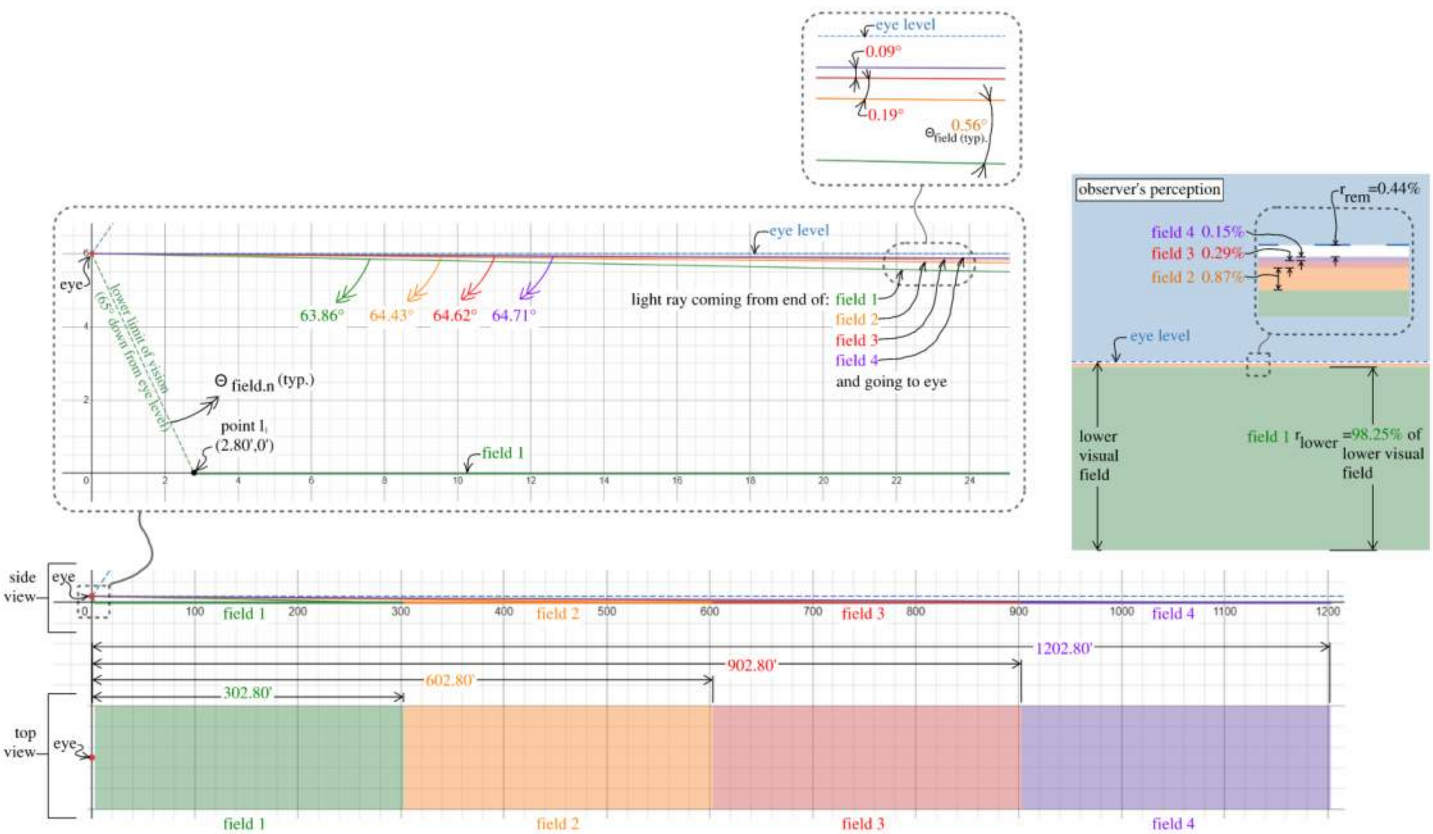
Matrix manipulation $n := 2 \,..\, \operatorname{rows}\left(\Theta_{field.n}\right)$ $B_n := \Theta_{field.n_n} - \Theta_{field.n_{n-1}}$

Angle subtended at eye by each field $\Theta_{field} := \operatorname{stack}\left(\operatorname{submatrix}\left(\Theta_{field.n}, 1, 1, 1, 1\right), \operatorname{submatrix}\left(B, 2, 4, 1, 1\right)\right) = \begin{bmatrix} 63.86 \\ 0.56 \\ 0.19 \\ 0.09 \end{bmatrix} deg$ $\begin{bmatrix} \text{“field 1”} \\ \text{“field 2”} \\ \text{“field 3”} \\ \text{“field 4”} \end{bmatrix}$

Percentage of lower field of view that each field takes up $r_{lower} := \dfrac{\Theta_{field}}{\Theta_{lower}} = \begin{bmatrix} 98.25\% \\ 0.87\% \\ 0.29\% \\ 0.15\% \end{bmatrix}$ $\begin{bmatrix} \text{“field 1”} \\ \text{“field 2”} \\ \text{“field 3”} \\ \text{“field 4”} \end{bmatrix}$

Remaining space in lower visual field $r_{rem} := 1 - \sum r_{lower} = 0.44\%$ (nothing is added in diagram beyond field 4, so it is shown as white space)

If you are standing $l_1 = 2.8\ ft$ behind 4 football fields (fields are in a line one after another) and are looking straight out at eye level, then the first football field in front of you takes up $r_{lower_1} = 98.25\%$ of your lower field of view (in the vertical direction). The remaining 3 football fields (2, 3, and 4) take up $r_{lower_2} + r_{lower_3} + r_{lower_4} = 1.31\%$ of your lower field of view. As can be seen, the lower field of view is occupied quickly after not too far of a distance (with the observer's eye height at 6' above ground).



The plot on this page shows the exponential decrease of the angular size in the lower visual field of 1' intervals of ground surface.

Theoretical distance to optical horizon $d_{lim} := \dfrac{h}{\tan(\Theta_{lim})} = 3.83\ mi$

Range for plotting $n := 1, 2 .. 500$

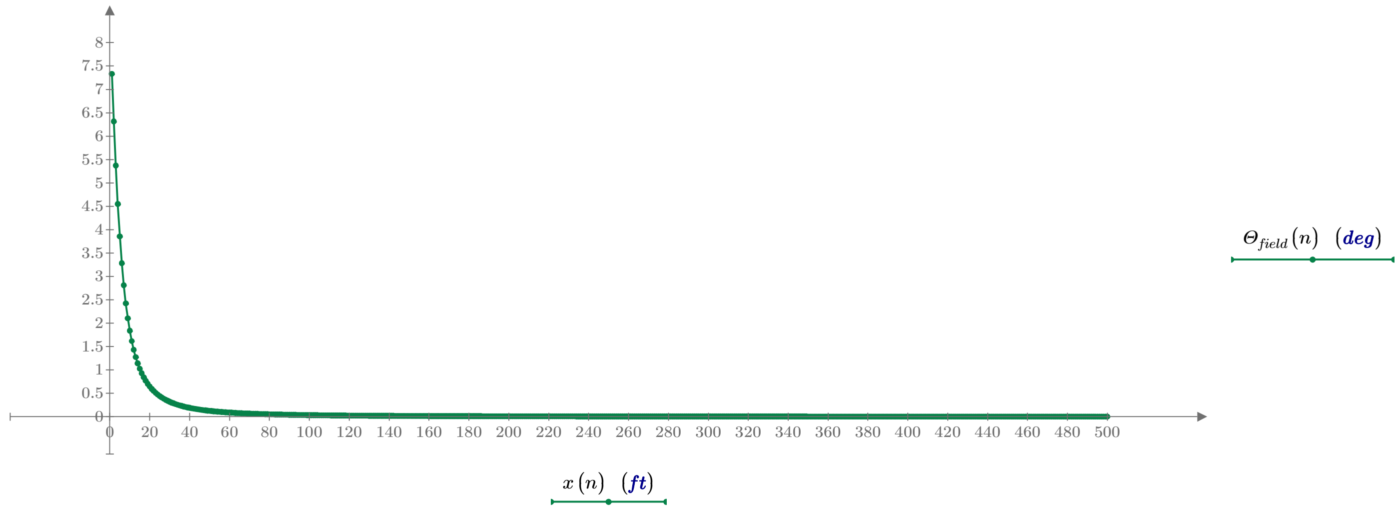
Distance matrix for plotting $x(n) := 1\ ft \cdot n$

Distance from observer to first point of ground in field of view $l_1 := h \cdot \tan(90\ deg - \Theta_{lower}) = 2.80\ ft$

Angular size first 1' interval $\Theta_{field.1} := \operatorname{atan}\left(\dfrac{l_1 + x(1)}{h}\right) - (90\ deg - \Theta_{lower}) = 7.33276\ deg$

Cumulative angular size of 1' intervals $\Theta_{field.n}(n) := \operatorname{atan}\left(\dfrac{l_1 + x(n)}{h}\right) - (90\ deg - \Theta_{lower})$

Angular size of each 1' interval $\Theta_{field}(n) := \Theta_{field.n}(n) - \Theta_{field.n}(n-1)$



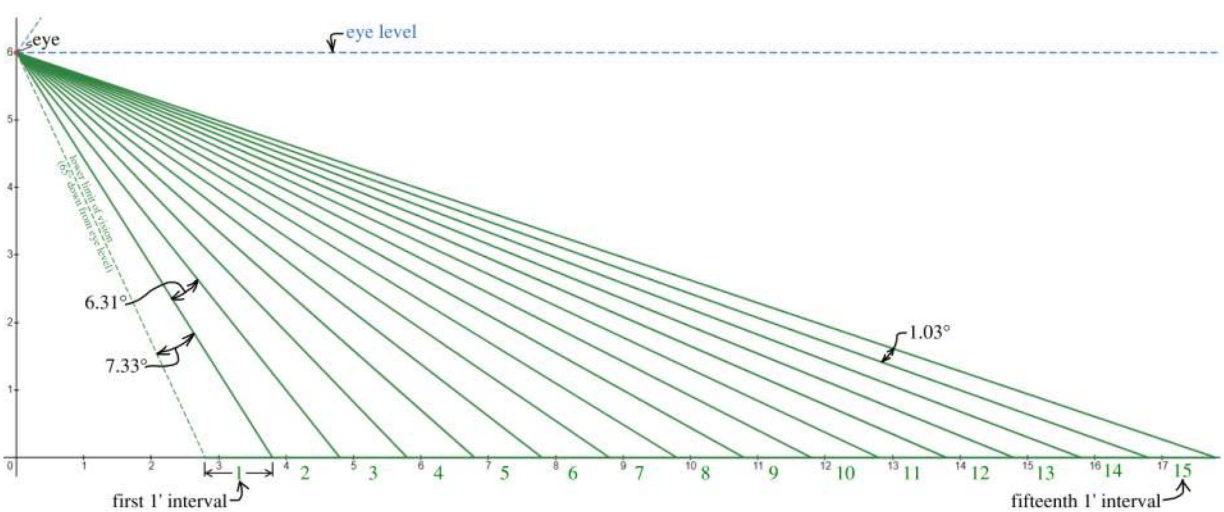
For example, the angular size of the first 1' interval of ground surface in the lower visual field is: $\Theta_{field.1} := \operatorname{atan}\left(\dfrac{l_1 + x(1)}{h}\right) - (90\ deg - \Theta_{lower}) = 7.33\ deg$

The angular size of the second 1' interval of ground surface is: $\Theta_{field}(n) := \Theta_{field.n}(2) - \Theta_{field.n}(2-1) = 6.31\ deg$

The angular size of a football field (three hundred 1' intervals) is: $\Theta_{field.n}(300) = 63.86\ deg$

The first fifteen 1' intervals of ground surface and their corresponding angular sizes are provided below.

$$x(n) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ \vdots \end{bmatrix} ft \qquad \Theta_{field.n}(n) - \Theta_{field.n}(n-1) = \begin{bmatrix} 7.33 \\ 6.31 \\ 5.37 \\ 4.55 \\ 3.86 \\ 3.28 \\ 2.81 \\ 2.42 \\ 2.10 \\ 1.84 \\ 1.62 \\ 1.43 \\ 1.27 \\ 1.14 \\ 1.03 \\ \vdots \end{bmatrix} deg$$



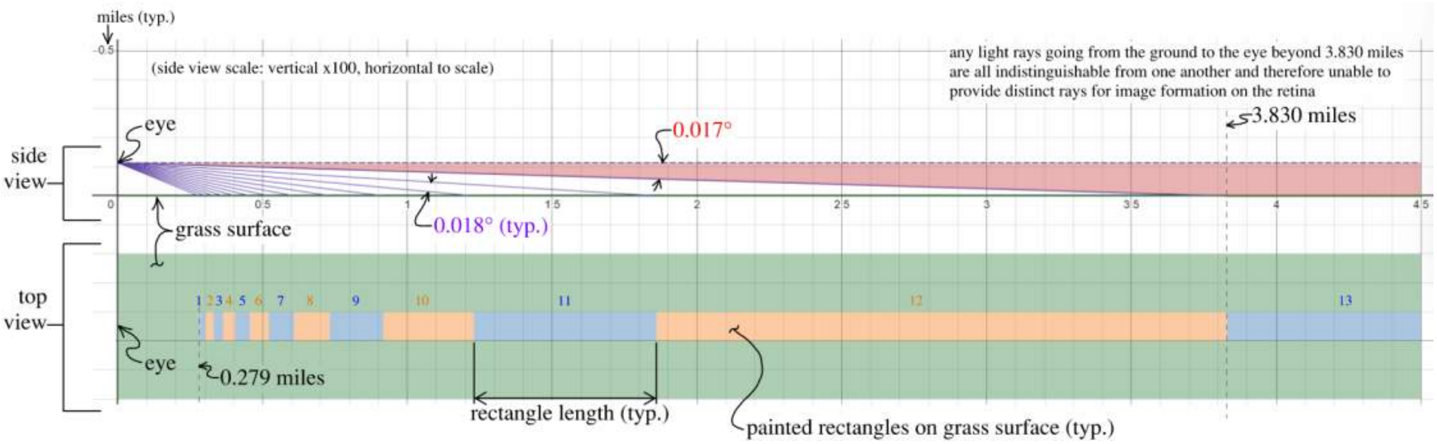
The last 1' interval that can be perceived by the observer is the 139th interval.

Angular size of 139th 1' interval: $\Theta_{field}(n) := \Theta_{field.n}(139) - \Theta_{field.n}(139-1) = 0.0172\ deg$ (> 0.017°)

Angular size of 140th 1' interval: $\Theta_{field}(n) := \Theta_{field.n}(140) - \Theta_{field.n}(140-1) = 0.0169\ deg$ (< 0.017°)

Beyond the 139th 1' interval, stretches of ground longer than 1' are only perceptible.

The diagram below illustrates how far the eye is able to perceive the ground. In the side view, the vertical direction is scaled up by a factor of 100 for clarity. The horizontal direction is to scale. The diagram represents an observer (whose eye is 6' above the surface) standing on a level grass field with blue and yellow painted rectangles on it.



The blue and yellow rectangles (1 through 12) all have an angular size of 0.018° and therefore are able to be perceived. Blue rectangle 13 is unable to be perceived since the angle subtended at the eye by any two light rays coming from blue rectangle 13 will always be less than 0.017°, even if this rectangle was stretched to the 100 mile marker. The observer can only theoretically perceive the ground up until 3.83 miles.

Angular resolution limit $\Theta_{lim} := 0.017\ deg$

Observer height $h := 6\ ft = 0.001\ mi$

Distance to last point of perceptible ground $d_{hor} := \dfrac{h}{\tan\left(\Theta_{lim}\right)} = 3.830\ mi$

Angular size of painted rectangles $\Theta_{rect} := 0.018\ deg$

Rectangle starting distance (rectangle)

$$d_{rect} := \text{stack}\left(h \cdot \tan\left(90\ deg - \Theta_{lim} - \begin{bmatrix} 12 \\ 11 \\ 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \cdot \Theta_{rect}\right), d_{hor}\right) = \begin{bmatrix} 0.279 \\ 0.303 \\ 0.331 \\ 0.364 \\ 0.404 \\ 0.455 \\ 0.521 \\ 0.608 \\ 0.732 \\ 0.917 \\ 1.228 \\ 1.860 \\ 3.830 \end{bmatrix} mi \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{bmatrix}$$

Rectangle length (rectangle)

$$n := 2 \,..\, \text{rows}\left(d_{rect}\right) \qquad l_{rect_n} := d_{rect_n} - d_{rect_{n-1}} = \begin{bmatrix} 0.023 \\ 0.028 \\ 0.033 \\ 0.041 \\ 0.051 \\ 0.066 \\ 0.088 \\ 0.123 \\ 0.185 \\ 0.311 \\ 0.632 \\ 1.970 \end{bmatrix} mi \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{bmatrix}$$

Each of these rectangle lengths correspond to an angular size of 0.018°.

It is clear from the rectangle lengths above the compression that occurs to the ground surface in your perception. Rectangle 1 is $l_{rect}(1) = 124\ ft$ and rectangle 12 is $l_{rect}(12) = 10400\ ft$. Each of these rectangles occupies the same amount of the visual field (same angular size), but rectangle 12 is $l_{rect}(12) - l_{rect}(1) = 10276\ ft$ longer.

Lower field of view angular size $\Theta_{lower} = 65\ deg$

Angular size of perceptible green grass before painted rectangles (up to 0.279 miles) $\Theta_{grass} := \text{atan}\left(\dfrac{d_{rect_1}}{h}\right) - \left(90\ deg - \Theta_{lower}\right) = 64.767\ deg$

Percentages of lower field of view

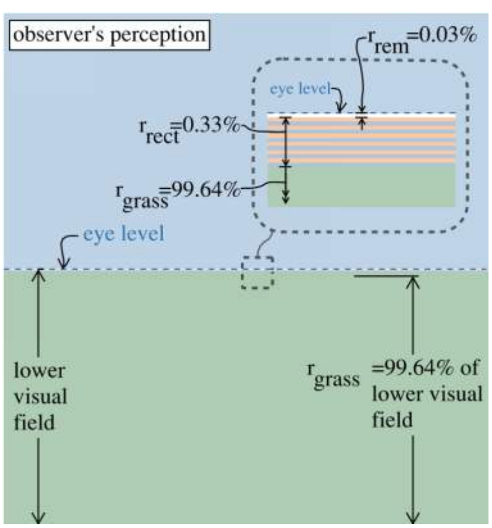
$r_{grass} := \dfrac{\Theta_{grass}}{\Theta_{lower}} = 99.64\%$

Angular size of rectangles 1 through 12 $\Theta_{rect} := \text{atan}\left(\dfrac{d_{rect_1} + \sum l_{rect}}{h}\right) - \left(90\ deg - \Theta_{lower}\right) - \Theta_{grass} = 0.216\ deg$

$r_{rect} := \dfrac{\Theta_{rect}}{\Theta_{lower}} = 0.33\%$

Theoretical angular size of imperceptible ground beyond rectangle 13 $\Theta_{rem} := \Theta_{lower} - \Theta_{grass} - \Theta_{rect} = 0.017\ deg$

$r_{rem} := \dfrac{\Theta_{rem}}{\Theta_{lower}} = 0.03\%$



The right side of rectangle 12 represents the last "point" of perceptible ground. This is the theoretical optical horizon of the ground surface.

Length of green grass in visual field $l_{grass} := d_{rect_1} - h \cdot \tan\left(90\ deg - \Theta_{lower}\right) = 1473\ ft$

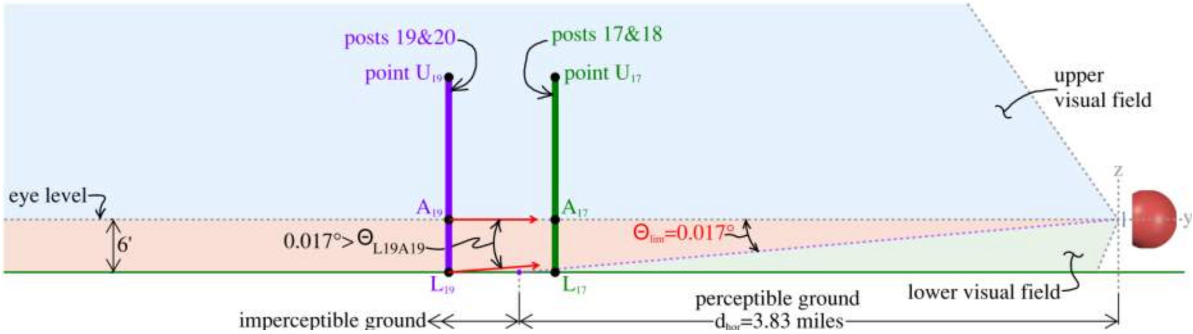
Length of painted rectangles (1-12) in visual field: $l_{rectangles} := \sum l_{rect} = 18747\ ft$

In the observer's perception, the green grass (with a length of $l_{grass} = 0.28\ mi$) occupies $r_{grass} = 99.64\%$ of the lower visual field while the painted rectangles (with a length of $l_{rectangles} = 3.55\ mi$) occupies only $r_{rect} = 0.33\%$ of the lower visual field, demonstrating the significant compression (fitting a lot of ground surface into a small angular size).

What is occurring in the lower visual field corresponding to the ground surface beyond rectangle 12 (the imperceptible ground - the white space in the diagram to the left) depends on various factors and will be discussed farther down in this document after the upper visual field is discussed.

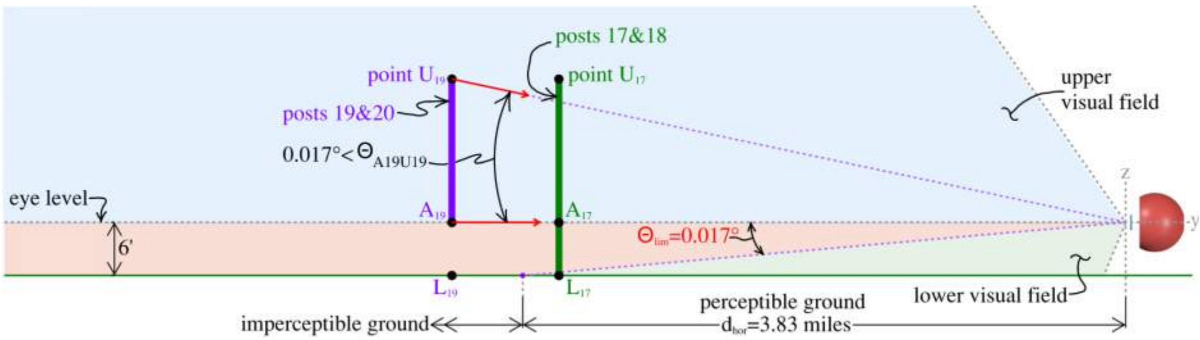
The following is a discussion on the upper visual field. This discussion refers back to the lamp post example from previous pages. Two new posts, 19 & 20, are added 4 miles away from the observer.

The upper visual field is not limited by a surface (like the ground), so objects in the upper visual field beyond $d_{hor} = 3.83\ mi$ have the potential to be partially visible.



Since posts 17 & 18 (3 miles away) are closer than $d_{hor} = 3.83\ mi$, the entirety of the posts are able to be perceived by the observer. The angle subtended at the eye by post segment L_{17} - $A_{17} > \Theta_{lim}$, therefore this segment is visible. The same goes for the upper portion of these posts.

Posts 19 & 20 (4 miles away) are beyond $d_{hor} = 3.83\ mi$, and therefore a portion of the bottom of the post will be imperceptible. The angle subtended at the eye by post segment L_{19} - $A_{19} < \Theta_{lim}$, therefore this segment is not visible. The diagram below shows this segment removed.



For posts 19 & 20, the remaining portion is segment A_{19} - U_{19}. Calculations are shown below to determine whether or not this remaining segment is visible.

Distance to posts 19 & 20 $\quad d_{19} := 4\ mi$

Overall post height $\quad h_{post} := 18\ ft$

Height of segment L_{19}-A_{19} $\quad h_{L19A19} := 6\ ft$

Height of segment A_{19}-U_{19} $\quad h_{A19U19} := h_{post} - h_{L19A19} = 12\ ft$

Angle subtended at the eye by post segment L_{19} - A_{19} $\quad \Theta_{L19A19} := \operatorname{atan}\left(\dfrac{h_{L19A19}}{d_{19}}\right) = 0.016\ deg \quad$ ($\Theta_{L19A19} < \Theta_{lim}$, therefore segment L_{19} - A_{19} not visible) $\quad$ ($\Theta_{lim} = 0.017\ deg$)

Angle subtended at the eye by post segment A_{19} - U_{19} $\quad \Theta_{A19U19} := \operatorname{atan}\left(\dfrac{h_{A19U19}}{d_{19}}\right) = 0.033\ deg \quad$ ($\Theta_{A19U19} > \Theta_{lim}$, therefore segment A_{19} - U_{19} is visible)

In summary, posts 1-16 (not shown in the diagrams on these pages) and posts 17 & 18 can be fully perceived by the observer as well as the upper segments, A_{19}-U_{19}, of posts 19 & 20.

Lamp post 19 & 20 coordinates:

$x_{post.1} := 20\ ft \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \end{bmatrix} ft \quad y_{post.1} := \begin{bmatrix} -4 \\ -4 \end{bmatrix} mi = \begin{bmatrix} -21120 \\ -21120 \end{bmatrix} ft \quad z_{post.1} := 0\ ft$

$x_{post.2} := 20\ ft \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \end{bmatrix} ft \quad y_{post.2} := \begin{bmatrix} -4 \\ -4 \end{bmatrix} mi = \begin{bmatrix} -21120 \\ -21120 \end{bmatrix} ft \quad z_{post.2} := 12\ ft$

$p_{x.1} := x_{ret}\left(x_{post.1}, y_{post.1}, z_{post.1}\right) \quad p_{y.1} := y_{ret}\left(x_{post.1}, y_{post.1}, z_{post.1}\right) \quad p_{z.1} := z_{ret}\left(x_{post.1}, y_{post.1}, z_{post.1}\right)$

$p_{x.2} := x_{ret}\left(x_{post.2}, y_{post.2}, z_{post.2}\right) \quad p_{y.2} := y_{ret}\left(x_{post.2}, y_{post.2}, z_{post.2}\right) \quad p_{z.2} := z_{ret}\left(x_{post.2}, y_{post.2}, z_{post.2}\right)$

$p_{x.f.1} := x_f\left(x_{post.1}, y_{post.1}, z_{post.1}\right) \quad p_{y.f.1} := y_f\left(x_{post.1}, y_{post.1}, z_{post.1}\right) \quad p_{z.f.1} := z_f\left(x_{post.1}, y_{post.1}, z_{post.1}\right)$

$p_{x.f.2} := x_f\left(x_{post.2}, y_{post.2}, z_{post.2}\right) \quad p_{y.f.2} := y_f\left(x_{post.2}, y_{post.2}, z_{post.2}\right) \quad p_{z.f.2} := z_f\left(x_{post.2}, y_{post.2}, z_{post.2}\right)$

$x_{avg.}\left(p_{x.1}, p_{x.2}\right) := \overrightarrow{\operatorname{mean}\left(p_{x.1}, p_{x.2}\right)} \quad y_{avg.}\left(p_{y.1}, p_{y.2}\right) := \overrightarrow{\operatorname{mean}\left(p_{y.1}, p_{y.2}\right)} \quad z_{avg.}\left(p_{z.1}, p_{z.2}\right) := \overrightarrow{\operatorname{mean}\left(p_{z.1}, p_{z.2}\right)}$

$x_{avg} := \overrightarrow{x_{avg.}\left(p_{x.1}, p_{x.2}\right)} \quad y_{avg} := \overrightarrow{y_{avg.}\left(p_{y.1}, p_{y.2}\right)} \quad z_{avg} := \overrightarrow{z_{avg.}\left(p_{z.1}, p_{z.2}\right)}$

$p_{x.bi.1} := x_{ret}\left(x_{avg}, y_{avg}, z_{avg}\right) \quad p_{y.bi.1} := y_{ret}\left(x_{avg}, y_{avg}, z_{avg}\right) \quad p_{z.bi.1} := z_{ret}\left(x_{avg}, y_{avg}, z_{avg}\right)$

$p_{x.bi.2} := x_{ret}\left(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}\right) \quad p_{y.bi.2} := y_{ret}\left(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}\right) \quad p_{z.bi.2} := z_{ret}\left(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}\right)$

$$r_{cir} := \frac{\sqrt{{p_{x.bi.1}}^2 + {p_{y.bi.1}}^2 + {p_{z.bi.1}}^2} + \sqrt{{p_{x.bi.2}}^2 + {p_{y.bi.2}}^2 + {p_{z.bi.2}}^2}}{2}$$

$x_{cir.}\left(p_{x.bi.1}, p_{x.bi.2}\right) := \overrightarrow{\text{mean}\left(p_{x.bi.1}, p_{x.bi.2}\right)}$ $\quad y_{cir.}\left(p_{y.bi.1}, p_{y.bi.2}\right) := \overrightarrow{\text{mean}\left(p_{y.bi.1}, p_{y.bi.2}\right)}$ $\quad z_{cir.}\left(p_{z.bi.1}, p_{z.bi.2}\right) := \overrightarrow{\text{mean}\left(p_{z.bi.1}, p_{z.bi.2}\right)}$

$x_{cir} := \overrightarrow{x_{cir.}\left(p_{x.bi.1}, p_{x.bi.2}\right)}$ $\quad y_{cir} := \overrightarrow{y_{cir.}\left(p_{y.bi.1}, p_{y.bi.2}\right)}$ $\quad z_{cir} := \overrightarrow{z_{cir.}\left(p_{z.bi.1}, p_{z.bi.2}\right)}$

Parametrizing the circles that the arcs of posts 19 & 20 are on (used for plotting arcs):

$n := 1$

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft \qquad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft \qquad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2$$

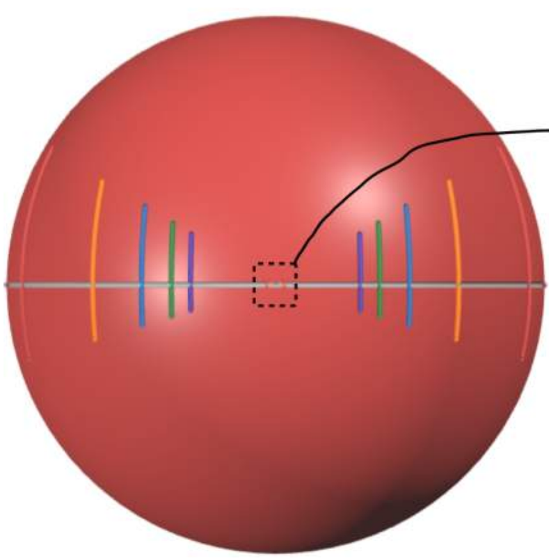
$$\check{n} := \frac{n}{\|n\|} = \begin{bmatrix} 1.000 \\ -0.001 \\ 0.000 \end{bmatrix} \qquad u_1 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} 0.000 \\ -0.001 \\ -1.000 \end{bmatrix} \qquad v_1 := \check{n} \times u_1 = \begin{bmatrix} 0.001 \\ 1.000 \\ -0.001 \end{bmatrix}$$

$n := 2$

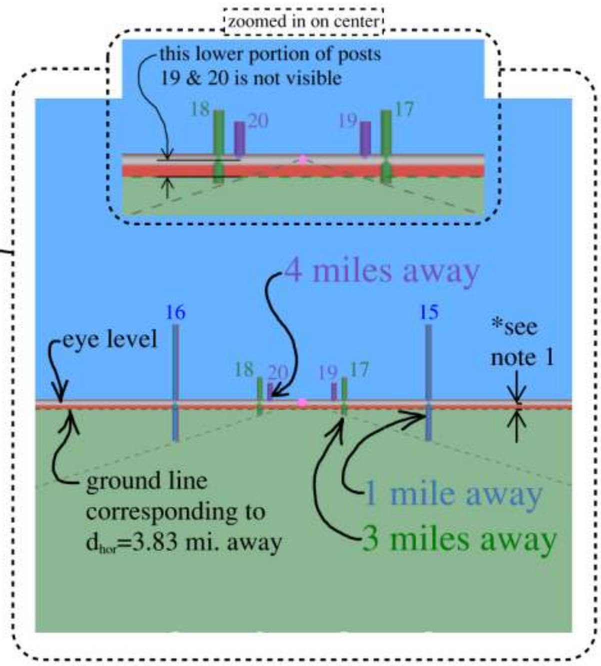
$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft \qquad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft \qquad n := e_1 \times e_2 = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} ft^2$$

$$\check{n} := \frac{n}{\|n\|} = \begin{bmatrix} 1.000 \\ 0.001 \\ 0.000 \end{bmatrix} \qquad u_2 := \frac{e_1}{\|e_1\|} = \begin{bmatrix} 0.000 \\ -0.001 \\ -1.000 \end{bmatrix} \qquad v_2 := \check{n} \times u_2 = \begin{bmatrix} -0.001 \\ 1.000 \\ -0.001 \end{bmatrix}$$

$$x_{arc} = \overrightarrow{x_{cir} + r_{cir} \cdot \left(u_n \cdot \cos(t) + v_n \cdot \sin(t)\right)} \quad y_{arc} = \overrightarrow{y_{cir} + r_{cir} \cdot \left(u_n \cdot \cos(t) + v_n \cdot \sin(t)\right)} \quad z_{arc} = \overrightarrow{z_{cir} \cdot 0 + r_{cir} \cdot \left(u_n \cdot \cos(t) + v_n \cdot \sin(t)\right)}$$



What the observer sees (2D representation)

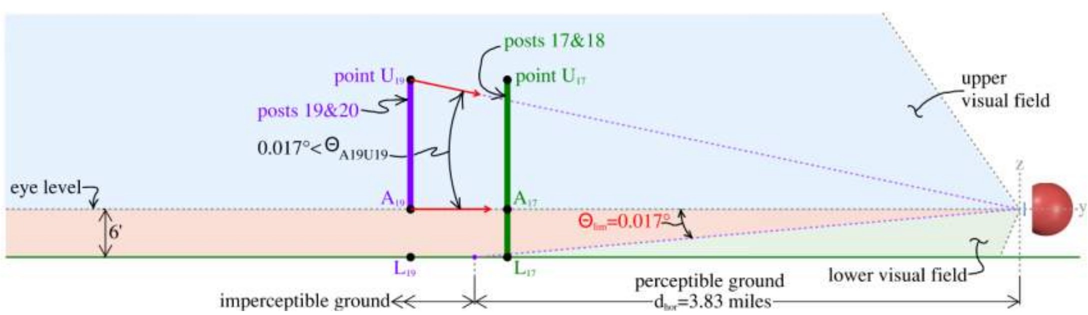


Zoomed in on central area of retina (sky and ground added)

This demonstrates how objects (like posts 19 & 20) can appear to disappear from bottom up (if they are beyond the theoretical distance to the optical ground horizon).

Note 1: What occurs in the red region depends on various factors which will be discussed on the following page.

The same diagram from the bottom of the previous page is included here for reference.



In some of the examples previously shown a few diagrams (of the observer's perception) include a white or red shaded area which has been left empty. In these areas in the lower visual field the ground is no longer perceptible. Light rays coming from objects or surfaces in this region are not able to contribute to image formation on the retina. It could be that the angular size of this region is less than the angular resolution limit of 0.017 degrees. It could also be that the light rays coming from this region have to travel though such a large distance of atmosphere that the light rays get affected by the air molecules in such a way (perhaps being scattered) as to prevent the light rays from reaching the retina.

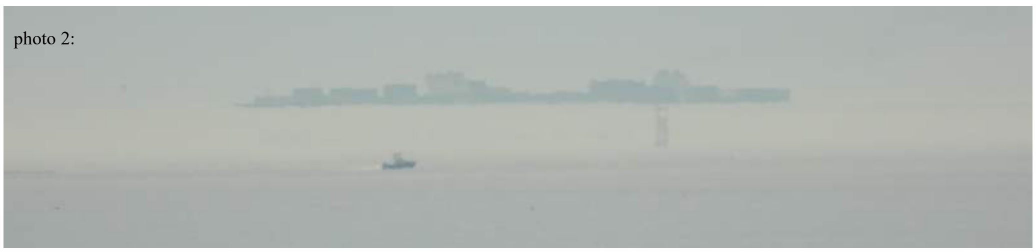
This region in your visual perception can be thought of as empty space momentarily. In this empty space, it appears that various things can occur. One thing that can occur is called an inferior mirage. This is demonstrated in the photographs below and diagrams on the following page.

Below is a photograph (photo 1) of an Evergreen F-class container ship (specifically the Ever Focus ship) coming into port in Boston on 6/6/2026. The front of the $1096\ ft$ long ship is on the left side of the photo. All photos were taken from about 20' above sea level.

In physical reality, in region A, there exists the lower portion of the ship down to the ocean surface and more ocean surface leading to & going beyond the ship.

However, it appears like there is air in this region. Whatever is occurring in this region, the light from the remainder of the ship and the surrounding ocean surface could not form an image in the camera.

In this particular case it is certain that this is not a superior mirage because I watched the entire ship slowly and fully become perceptible. Shown below are additional photographs of the ship as it came closer and closer to me.



This photo is pretty similar to photo 1 in terms of the amount of visible ship. A smaller boat is sailing by and appears to be under the floating ship. Obviously it is just in front of the ship much closer to the camera as the surface of the water can be seen around the small boat.

Another small boat is in front of the container ship. Also, there is a change occurring in region A.

A dark irregular shape is perceptible.

More of the ship is visible. The lettering on the ship, "EVERGREEN", is mostly perceptible.

In physical reality, the ocean surface meets the ship not too far below the lettering of the ship, but this is still an imperceptible region.

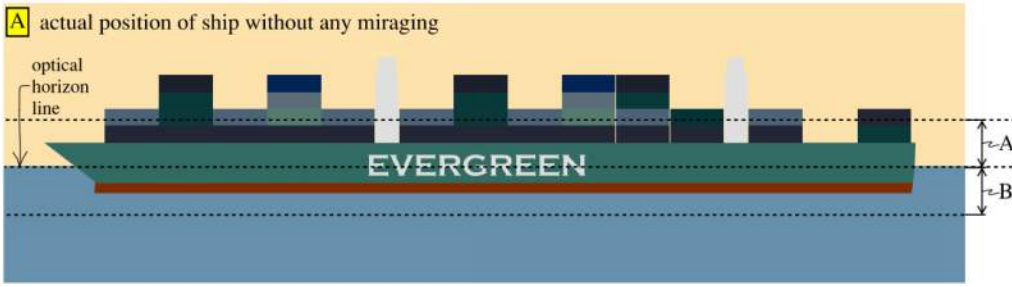
More pronounced inferior miraging is occurring in region A. Photo 5 is the last photo of the ship I took. Eventually the entire ship could be seen sitting on the surface of the water similar to the smaller boats in these photos.

Photo 7 provides some perspective of the container ship, compared to waves hitting the shore.

What is visibly occurring in these photographs does not always occur. The temperature & pressure of the air, humidity, position of the sun (time of day), clarity or fogginess of the air, etc. are a few factors which influence how light behaves and which light rays make it to the camera / your eye in this highly compressed region of your visual perception. Refer to the previous example involving the blue and yellow painted rectangles, vast distances of surface are contained within a very small angular size of your perception.

Diagrams illustrating the formation of an inferior mirage.

Diagram A shows the position of a ship in the ocean and the actual location of the optical horizon (effectively at eye level for this example of an observer eye very close to sea level). Diagram B shows the side view of this ship.



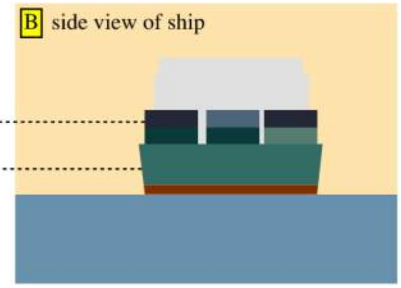


Diagram C is a side view showing the ship in the ocean and the observer on the shore. Most explanations of an inferior mirage online say that the cause of this optical phenomenon is due to a temperature gradient in the air that the light is traveling through. There is a warmer layer of air (say below eye level to the ocean surface) and a colder layer of air in region A. This gradient causes the reflected light coming from the ship and heading down toward the ocean to be bent upwards and towards the observer's eye. The mirage occurs in the uppermost portion of the lower visual field due to where the bent light appears to now be originating from.

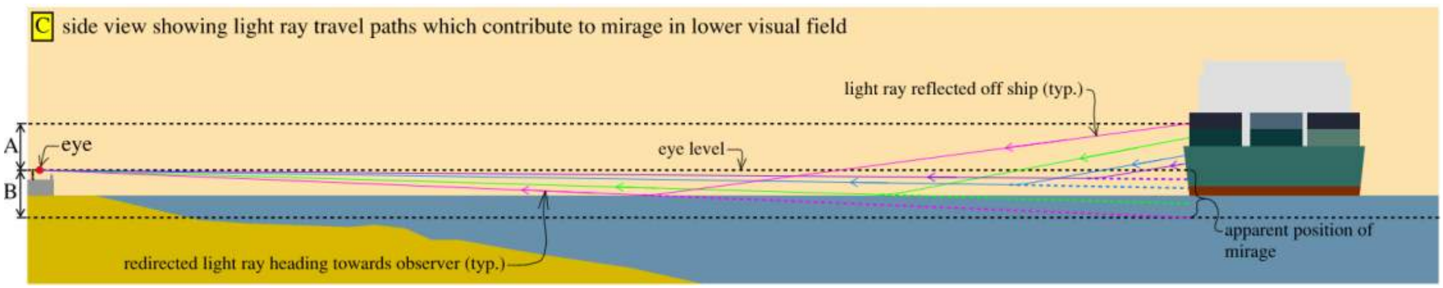
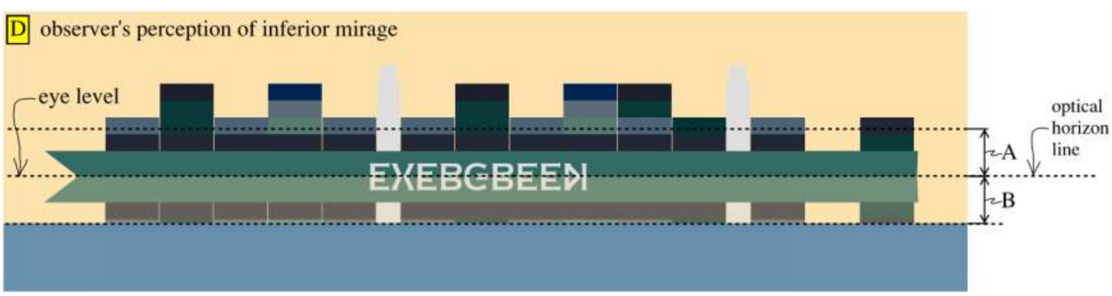


Diagram D shows the observer's perception. In region B, the observer will perceive the inferior mirage (an inverted image of region A of the ship). Although the actual location of the optical horizon cannot be perceived due to the miraging and it appears as if the ocean horizon is at the bottom of region B, the true optical horizon is still in the same location (effectively at eye level). I suppose mainstream explanations would argue that both images of the ship (in region A and the inverted region B) are mirages and the occluding edge horizon is at the bottom of region B. In this example, that cannot be the case as I watched the ship come closer and closer until the entire ship could be seen normally sitting in the ocean and I would argue that it never is the case.



This example is included to demonstrate one of the effects that can occur in the upper portion of the lower visual field when observing objects at great distances.

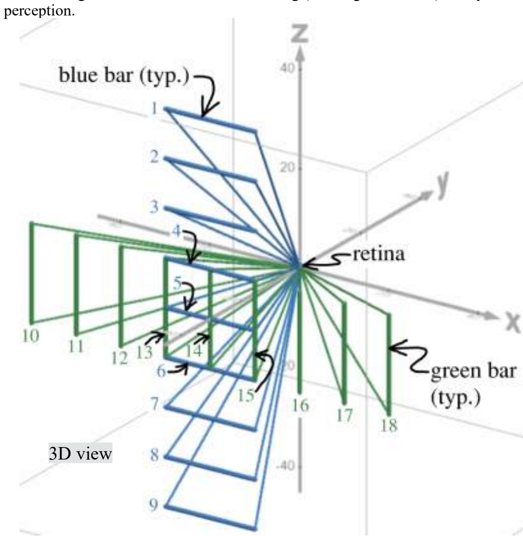
This example includes 9 horizontal blue bars (each 20' in length) & 9 vertical green bars (each 20' in length). The plane in which the bars are in (parallel to the x-z plane) is 30' away from the observer.

Up until now the convergence point straight out along eye level has been discussed. Depending on the orientation of planar or linear objects in physical space, there will also be convergence points on either side of the object (to the left, right, above, and below). For example, if bars 4, 6, 13, and 15 (in the diagrams below) represent the front facade of a square building. The floor and ceiling of the building are parallel to the ground and the walls of the building are perpendicular to the ground. Given this orientation, there will be convergence points in your visual perception to the left, right, above, and below the building (see diagrams below). If say the building was rotated 45 degrees, then the convergence points would be in the top left, top right, bottom right, and bottom left of your visual perception.

Coordinates of bars in physical space:

$$x_1 := \begin{bmatrix} -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -40 \\ -30 \\ -20 \\ -10 \\ 0 \\ 10 \\ 20 \\ 30 \\ 40 \end{bmatrix} ft \qquad y_1 := -30\,ft \qquad z_1 := \begin{bmatrix} 40 \\ 30 \\ 20 \\ 10 \\ 0 \\ -10 \\ -20 \\ -30 \\ -40 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} ft$$

$$x_2 := \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ -40 \\ -30 \\ -20 \\ -10 \\ 0 \\ 10 \\ 20 \\ 30 \\ 40 \end{bmatrix} ft \qquad y_2 := -30\,ft \qquad z_2 := \begin{bmatrix} 40 \\ 30 \\ 20 \\ 10 \\ 0 \\ -10 \\ -20 \\ -30 \\ -40 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \end{bmatrix} ft$$



Calculations for plotting images (arcs) of bars formed on retina. The results are not shown here, just the equations used for plotting.

$p_{x.1} := x_{ret}(x_1, y_1, z_1)$ $\quad p_{y.1} := y_{ret}(x_1, y_1, z_1)$ $\quad p_{z.1} := z_{ret}(x_1, y_1, z_1)$

$p_{x.2} := x_{ret}(x_2, y_2, z_2)$ $\quad p_{y.2} := y_{ret}(x_2, y_2, z_2)$ $\quad p_{z.2} := z_{ret}(x_2, y_2, z_2)$

$p_{x.f.1} := x_f(x_1, y_1, z_1)$ $\quad p_{y.f.1} := y_f(x_1, y_1, z_1)$ $\quad p_{z.f.1} := z_f(x_1, y_1, z_1)$

$p_{x.f.2} := x_f(x_2, y_2, z_2)$ $\quad p_{y.f.2} := y_f(x_2, y_2, z_2)$ $\quad p_{z.f.2} := z_f(x_2, y_2, z_2)$

$x_{avg.}(p_{x.1}, p_{x.2}) := \overrightarrow{\mathrm{mean}(p_{x.1}, p_{x.2})}$ $\quad y_{avg.}(p_{y.1}, p_{y.2}) := \overrightarrow{\mathrm{mean}(p_{y.1}, p_{y.2})}$ $\quad z_{avg.}(p_{z.1}, p_{z.2}) := \overrightarrow{\mathrm{mean}(p_{z.1}, p_{z.2})}$

$x_{avg} := \overrightarrow{x_{avg.}(p_{x.1}, p_{x.2})}$ $\quad y_{avg} := \overrightarrow{y_{avg.}(p_{y.1}, p_{y.2})}$ $\quad z_{avg} := \overrightarrow{z_{avg.}(p_{z.1}, p_{z.2})}$

$p_{x.bi.1} := x_{ret}(x_{avg}, y_{avg}, z_{avg})$ $\quad p_{y.bi.1} := y_{ret}(x_{avg}, y_{avg}, z_{avg})$ $\quad p_{z.bi.1} := z_{ret}(x_{avg}, y_{avg}, z_{avg})$

$p_{x.bi.2} := x_{ret}(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1})$ $\quad p_{y.bi.2} := y_{ret}(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1})$ $\quad p_{z.bi.2} := z_{ret}(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1})$

$$r_{cir} := \frac{\sqrt{p_{x.bi.1}^{\;2} + p_{y.bi.1}^{\;2} + p_{z.bi.1}^{\;2}} + \sqrt{p_{x.bi.2}^{\;2} + p_{y.bi.2}^{\;2} + p_{z.bi.2}^{\;2}}}{2}$$

$x_{cir.}(p_{x.bi.1}, p_{x.bi.2}) := \overrightarrow{\mathrm{mean}(p_{x.bi.1}, p_{x.bi.2})}$ $\quad y_{cir.}(p_{y.bi.1}, p_{y.bi.2}) := \overrightarrow{\mathrm{mean}(p_{y.bi.1}, p_{y.bi.2})}$ $\quad z_{cir.}(p_{z.bi.1}, p_{z.bi.2}) := \overrightarrow{\mathrm{mean}(p_{z.bi.1}, p_{z.bi.2})}$

$n := 1$ $\quad \overrightarrow{x_{cir.}(p_{x.bi.1}, p_{x.bi.2})}$ $\quad y_{cir} := \overrightarrow{y_{cir.}(p_{y.bi.1}, p_{y.bi.2})}$ $\quad z_{cir} := \overrightarrow{z_{cir.}(p_{z.bi.1}, p_{z.bi.2})}$

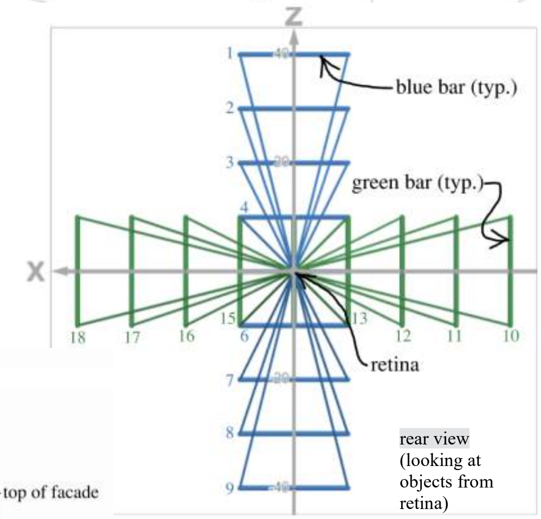
(n: 1-18)

$$e_1 := \begin{bmatrix} p_{x.2_n} - p_{x.1_n} \\ p_{y.2_n} - p_{y.1_n} \\ p_{z.2_n} - p_{z.1_n} \end{bmatrix} \qquad e_2 := \begin{bmatrix} p_{x.bi.2_n} - p_{x.1_n} \\ p_{y.bi.2_n} - p_{y.1_n} \\ p_{z.bi.2_n} - p_{z.1_n} \end{bmatrix} \qquad n := e_1 \times e_2 \quad \check{n} := \frac{n}{\|n\|} \quad u_1 := \frac{e_1}{\|e_1\|} \quad v_1 := \check{n} \times u_1$$

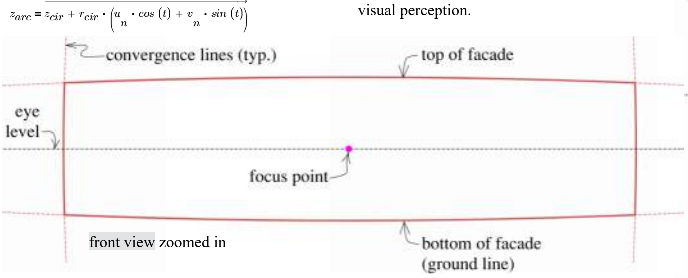
$$x_{arc} = \overrightarrow{x_{cir} + r_{cir} \cdot \left(u_n \cdot \cos(t) + v_n \cdot \sin(t)\right)}$$

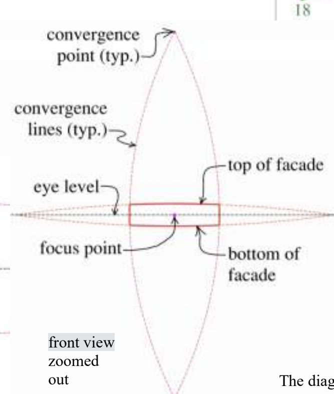
$$y_{arc} = \overrightarrow{y_{cir} + r_{cir} \cdot \left(u_n \cdot \cos(t) + v_n \cdot \sin(t)\right)}$$

$$z_{arc} = \overrightarrow{z_{cir} + r_{cir} \cdot \left(u_n \cdot \cos(t) + v_n \cdot \sin(t)\right)}$$

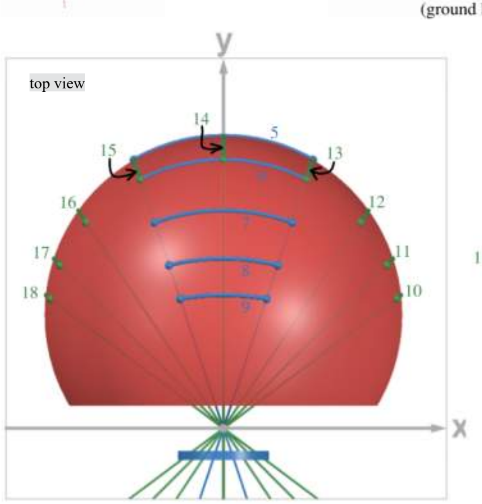


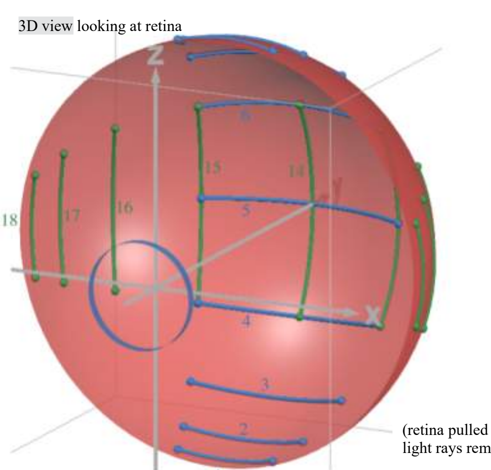
The diagrams below illustrate the convergence lines of the edges of a building facade in your visual perception.

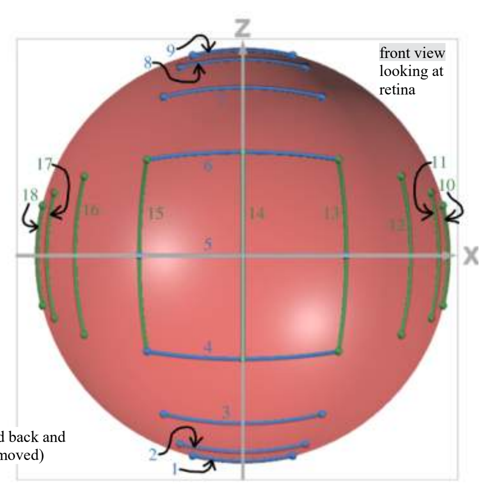




The diagrams below show the images formed on the retina of the bars.

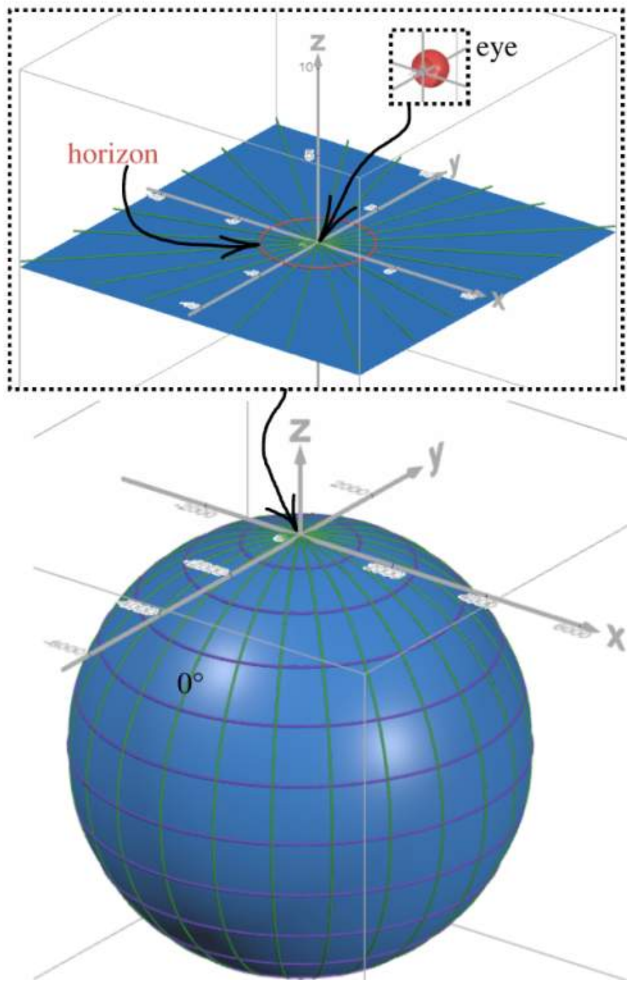






As the bars get more and more in the periphery, the images that are formed on the retina of these bars decrease in size (demonstrating the convergence that occurs of planar objects as they move farther away from you in the up, down, left, and right directions). In this specific example since the bars are so close, as more bars are added lower (below bar 9) and higher (above bar 1) they would eventually be outside of the visual field of the eye.

Globe earth model example to see how the occluding edge horizon would form on the retina.



Observer (eye) height $h := 6\ ft = 0.00114\ mi$

Radius of earth $r = 3958.8\ mi$

Equation for globe earth $x^2 + y^2 + (z + r + h)^2 = (r)^2$

Distance to horizon $d := \sqrt{(r+h)^2 - r^2} = 2.9995456354\ mi$

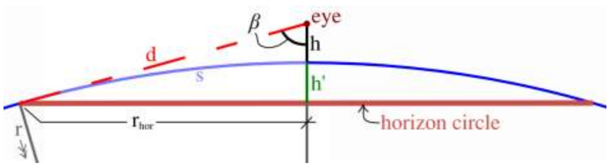
Angle subtended at the eye by arc segment of earth surface (from observer to horizon) $\beta := \operatorname{atan}\left(\frac{r}{d}\right) = 89.96\ deg$ (this is also the latitude of the horizon small circle)

Distance to horizon (arc length along globe surface) $s := r \cdot (90\ deg - \beta) = 2.9995450614\ mi$

Radius of small circle (along horizon) $r_{hor} := d \cdot \sin(\beta) = 2.9995447744\ mi$

Distance from globe surface to plane of horizon small circle $h' := d \cdot \cos(\beta) - h = 0.00114\ mi$

Equations for horizon circle $x = r_{hor} \cdot \sin(t)$ $y = r_{hor} \cdot \cos(t)$ $z = -1 \cdot (h' + h)$



The occluding edge horizon (red circle) is shown in the diagram to the top right. For this example, the observer is at the north pole and is looking south (towards -y direction) along the 0° longitude line. The horizon is $s = 3\ mi$ away or $90\ deg - \beta = 0.043\ deg$ away in terms of change in latitude. At any observer location on the globe, the occluding edge horizon will be $\pm 0.043\ deg$ away from your latitude (if looking N-S) and longitude (if looking W-E), for an eye height of 6 feet.

You can test this in reality by knowing your latitude and longitude (when taking a photograph with a camera) and aiming the camera to the N, S, W, or E (preferably looking out over the ocean towards other land masses or objects for some reference points). Based on your coordinates and direction you are aiming the camera, add $\pm 0.043\ deg$ to the longitude or latitude of your coordinates, determine where that is using a map, and see if that new coordinate is somewhat in alignment with where the globe earth occluding edge horizon is supposed to be based on the photograph. The key factor of this experiment is the camera height above sea level. At a 6 foot camera height, the difference between the distance to the globe earth occluding edge horizon and the planar earth optical horizon may not be great enough to determine which model's horizon is in alignment with the photograph. As shown in the calculations below, as the observer height increases the difference between the two increases rapidly and it should be easier to determine which model aligns better with your observation. The amount of degrees you need to add to your coordinates varies with height above sea level. If the camera is 50 feet above sea level, the amount is $\pm 0.125\ deg$. It is important to note that visual observations of an optical phenomena are not the "end all be all" in terms of determining the shape of the surface you stand upon. Comprehension of the behavior of the substances in our reality provides a deeper knowing, specifically true motion vs. apparent motion and how applied pressure (substances pushing other substances) is related to these motions[2, 3, 4, Appendix B].

Observer height

$$h := \begin{bmatrix} 10^{-8} \\ 6 \\ 10 \\ 20 \\ 50 \\ 100 \\ 250 \\ 500 \\ 1000 \\ 5000 \\ 20000 \\ 33000 \\ 50000 \\ 75000 \\ 100000 \\ 120000 \end{bmatrix} ft$$

Distance to horizon (line of sight)

$$d := \sqrt{(r+h)^2 - r^2} = \begin{bmatrix} 0.00 \\ 3.00 \\ 3.87 \\ 5.48 \\ 8.66 \\ 12.25 \\ 19.36 \\ 27.38 \\ 38.72 \\ 86.59 \\ 173.22 \\ 222.54 \\ 273.98 \\ 335.66 \\ 387.70 \\ 424.81 \end{bmatrix} mi$$

Angle subtended at the eye by arc segment of earth surface to horizon

$$\beta := \operatorname{atan}\left(\frac{r}{d}\right) = \begin{bmatrix} 90.00 \\ 89.96 \\ 89.94 \\ 89.92 \\ 89.87 \\ 89.82 \\ 89.72 \\ 89.60 \\ 89.44 \\ 88.75 \\ 87.49 \\ 86.78 \\ 86.04 \\ 85.15 \\ 84.41 \\ 83.88 \end{bmatrix} deg$$

Distance to occluding edge horizon (arc length along globe surface)

$$d_{hor.globe} := r \cdot (90\ deg - \beta) = \begin{bmatrix} 0.00 \\ 3.00 \\ 3.87 \\ 5.48 \\ 8.66 \\ 12.25 \\ 19.36 \\ 27.38 \\ 38.72 \\ 86.58 \\ 173.11 \\ 222.31 \\ 273.55 \\ 334.86 \\ 386.47 \\ 423.19 \end{bmatrix} mi$$

Note: The first row of these matrices can be ignored. It exists to avoid an error when calculating β.

Angular resolution limit $\Theta_{lim} = 0.017\ deg$

Theoretical distance to optical horizon (last piece of perceptible ground) on planar earth

$$d_{hor.opt} := \frac{h}{\tan(\Theta_{lim})} = \begin{bmatrix} 0.00 \\ 3.83 \\ 6.38 \\ 12.77 \\ 31.92 \\ 63.83 \\ 159.58 \\ 319.16 \\ 638.32 \\ 3191.61 \\ 12766.44 \\ 21064.62 \\ 31916.10 \\ 47874.15 \\ 63832.19 \\ 76598.63 \end{bmatrix} mi$$

Difference in distance to horizon between globe earth occluding edge horizon and planar earth optical horizon

Observer heights

$$\Delta_{horizon} := d_{hor.opt} - d_{hor.globe} = \begin{bmatrix} 0.00 \\ 0.83 \\ 2.51 \\ 7.29 \\ 23.26 \\ 51.59 \\ 140.22 \\ 291.78 \\ 599.60 \\ 3105.03 \\ 12593.33 \\ 20842.32 \\ 31642.55 \\ 47539.29 \\ 63445.72 \\ 76175.44 \end{bmatrix} mi \qquad h = \begin{bmatrix} 0 \\ 6 \\ 10 \\ 20 \\ 50 \\ 100 \\ 250 \\ 500 \\ 1000 \\ 5000 \\ 20000 \\ 33000 \\ 50000 \\ 75000 \\ 100000 \\ 120000 \end{bmatrix} ft$$

The theoretical distance to the last "piece" of perceptible ground $(d_{hor.optical})$ is by no means indicating that a person can clearly see the ground to this point. It is just the mathematical distance when only considering the angular resolution limit. A camera on a high altitude balloon (at 120,000 feet) cannot practically resolve the surface of the ground up to 76175.44 miles away. Refer to the example on previous pages regarding the yellow and blue painted rectangles. At an observer (camera) height of 120,000 feet, rectangle 12 (the last theoretical perceptible "section" of ground surface) would be beginning at a distance of 37,205 miles away from the camera and would be 39,394 miles in length, representing the last 0.018° of your visible angular perception before the ground falls into the imperceptible limit. In other words, mathematically only two individual rays of light coming from a ground surface having a length of 39,394 miles are distinguishable by your eyes.

The point is that the theoretical distance is very large, but that does not mean you can make out any detail in these distant regions. Besides this, the scattering and absorption of the light rays reflecting off the ground from these regions and going to your eyes by the vast distances of atmosphere the rays are traveling through would hinder these light rays from ever making it to your eye. When you look at high altitude camera videos, a great stretch of ground can be perceived but not nearly as great as the calculated theoretical distances to the last piece of perceptible ground as shown on this page. At these distances (and well before it) due to the reasons mentioned above, all that is formed in your visual perception (towards the horizon) is typically a blurry whitish blue haze (if the sun is still relatively close to the camera and overhead) and a blurry reddish orange haze (if the sun is at a relatively large distance away from the camera and in the direction the camera is facing, aka "sunset" or "sunrise" positions).

A decay function representing the decrease in ground surface perceptibility (due to high compression of ground surface at vast distances and atmospheric light scattering and absorption) is introduced on this page.

For this first calculation a decay constant of $\lambda := 0.23$ is used. This represents optimal viewing conditions for long distance perception (for example, viewing towards the sun's direction when it is just before "sunrise"). Long distance photographers utilize this time of day because the sky will be reddish orange, providing a contrast with the distant faint silhouettes of land features which allows these land features to be perceived.

This decay function is entirely made up. I cannot calculate the amount that the atmosphere hinders perception. It is included to illustrate that the practical distance to the last perceptible ground surface is less than the mathematical distance considering only the angular resolution limit.

Note: when describing this distance, $d_{hor.opt}$, the descriptors mathematical and theoretical are used interchangeably, but the same value is being referred to.

$n := 1 \,..\, \text{rows}(h)$

$decay_n := e^{-\lambda \cdot n}$

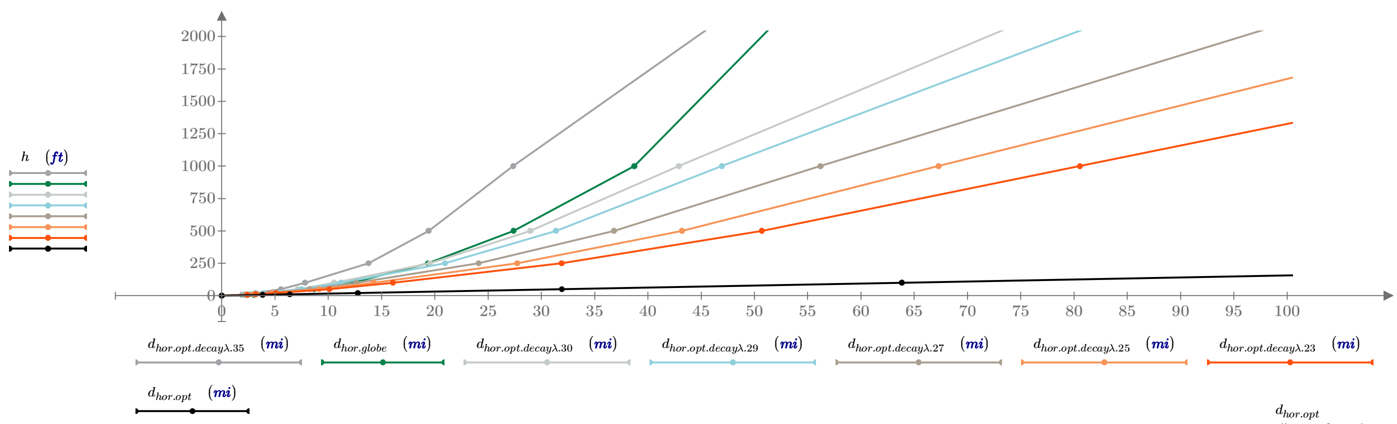
$d_{hor.opt.decay\lambda.23} := \overrightarrow{d_{hor.opt} \cdot decay}$

Decay function $decay_n =$	Distance to last practically perceptible ground surface $d_{hor.opt.decay\lambda.23} =$ (mi)	Percentage of theoretical distance $\dfrac{d_{hor.opt.decay\lambda.23}}{d_{hor.opt}} =$	Observer height $h =$ (ft)
0.794533603	0.00	79.5%	0
0.631283646	2.42	63.1%	6
0.501576069	3.20	50.2%	10
0.398519041	5.09	39.9%	20
0.316636769	10.11	31.7%	50
0.251578553	16.06	25.2%	100
0.199887614	31.90	20.0%	250
0.158817426	50.69	15.9%	500
0.126185782	80.55	12.6%	1000
0.100258844	319.99	10.0%	5000
0.07965902	1016.96	8.0%	20000
0.063291768	1333.22	6.3%	33000
0.050287437	1604.98	5.0%	50000
0.039955058	1912.81	4.0%	75000
0.031745636	2026.39	3.2%	100000
0.025222975	1932.05	2.5%	120000

Additional decay rates are added for plotting to represent different conditions (like sky clarity and time of day).

$\lambda_{0.25} := 0.25 \quad decay_n := e^{-\lambda_{0.25} \cdot n} \quad d_{hor.opt.decay\lambda.25} := \overrightarrow{d_{hor.opt} \cdot decay}$ | $\lambda_{0.27} := 0.27 \quad decay_n := e^{-\lambda_{0.27} \cdot n} \quad d_{hor.opt.decay\lambda.27} := \overrightarrow{d_{hor.opt} \cdot decay}$ | $\lambda_{0.29} := 0.29 \quad decay_n := e^{-\lambda_{0.29} \cdot n} \quad d_{hor.opt.decay\lambda.29} := \overrightarrow{d_{hor.opt} \cdot decay}$

$\lambda_{0.30} := 0.30 \quad decay_n := e^{-\lambda_{0.30} \cdot n} \quad d_{hor.opt.decay\lambda.30} := \overrightarrow{d_{hor.opt} \cdot decay}$ | $\lambda_{0.35} := 0.35 \quad decay_n := e^{-\lambda_{0.35} \cdot n} \quad d_{hor.opt.decay\lambda.35} := \overrightarrow{d_{hor.opt} \cdot decay}$



$d_{hor.opt.decay\lambda.35}$ distance from observer to last perceptible ground surface (considering decay) - this represents a foggy day when visibility is poor

$d_{hor.globe}$ distance from observer to globe earth occluding edge horizon

$d_{hor.opt.decay\lambda.30}$ distance from observer to last perceptible ground surface (considering decay) - this represents when sun is overhead and horizon is a whitish haze

$d_{hor.opt.decay\lambda.29}$ distance from observer to last perceptible ground surface (considering decay) - this represents a few hours or so after sunrise and horizon is a whitish blue

$d_{hor.opt.decay\lambda.27}$ distance from observer to last perceptible ground surface (considering decay) - this represents an hour and a half or so after sunrise and horizon is a bluish orange

$d_{hor.opt.decay\lambda.25}$ distance from observer to last perceptible ground surface (considering decay) - this represents a half hour or so after sunrise and horizon is more orange than red

$d_{hor.opt.decay\lambda.23}$ distance from observer to last perceptible ground surface (considering decay) - this represents just before or right at sunrise and horizon is more red than orange

$d_{hor.opt}$ distance from observer to last theoretical perceptible ground surface (considering angular resolution limit only) - distances shown on previous page

observer height

$d_{hor.opt.decay\lambda.35} =$ (mi)	$d_{hor.globe} =$ (mi)	$d_{hor.opt.decay\lambda.30} =$ (mi)	$d_{hor.opt.decay\lambda.29} =$ (mi)	$d_{hor.opt.decay\lambda.27} =$ (mi)	$d_{hor.opt.decay\lambda.25} =$ (mi)	$d_{hor.opt.decay\lambda.23} =$ (mi)	$h =$ (ft)
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0
1.9	3.0	2.1	2.1	2.2	2.3	2.4	6
2.2	3.9	2.6	2.7	2.8	3.0	3.2	10
3.1	5.5	3.8	4.0	4.3	4.7	5.1	20
5.5	8.7	7.1	7.5	8.3	9.1	10.1	50
7.8	12.2	10.6	11.2	12.6	14.2	16.1	100
13.8	19.4	19.5	21.0	24.1	27.7	31.9	250
19.4	27.4	29.0	31.4	36.8	43.2	50.7	500
27.4	38.7	42.9	46.9	56.2	67.3	80.5	1000
96.4	86.6	158.9	175.6	214.5	262.0	320.0	5000
271.7	173.1	470.9	525.6	655.0	816.1	1017.0	20000
315.9	222.3	575.6	648.9	825.0	1048.7	1333.2	33000
337.3	273.5	646.0	735.7	954.2	1237.5	1605.0	50000
356.5	334.9	717.9	825.8	1092.6	1445.7	1912.8	75000
335.0	386.5	709.1	823.9	1112.1	1501.2	2026.4	100000
283.3	423.2	630.4	739.8	1018.8	1403.0	1932.0	120000

These calculations are included to help illustrate some of the variables (sky clarity and time of day) which decrease or increase the ability to perceive ground surfaces at vast distances. It is a mathematical model with the main point being that we cannot see even close to the theoretical perceptible ground surface considering angular resolution limit only. The intensity of the light coming from objects is another factor affecting perceptibility. A very bright object will remain perceptible (as it recedes from your location) at a farther distance than a duller object of the same size would.

Retinal image formed from the globe earth occluding edge horizon.

Observer height $h := 6\ ft$ $h_1 := 6\ ft$ (top of earth sphere is set at -6ft)

Typical values of visual field limits (laterally) $\Theta_{left} = 75\ deg$ $\Theta_{right} = 75\ deg$

Radius of small circle (along horizon) $r_{hor} := \sqrt{(r+h)^2 - r^2} \cdot \sin\left(\operatorname{atan}\left(\frac{r}{\sqrt{(r+h)^2 - r^2}}\right)\right) = 2.99954\ mi$

Distance from globe surface to plane of horizon small circle $h' := \sqrt{(r+h)^2 - r^2} \cdot \cos\left(\operatorname{atan}\left(\frac{r}{\sqrt{(r+h)^2 - r^2}}\right)\right) - h = 0.00114\ mi$

Angles within visual field for which geometric coordinates will be calculated

$$t := \frac{\begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \end{bmatrix} \cdot \pi}{12} = \begin{bmatrix} 105 \\ 120 \\ 135 \\ 150 \\ 165 \\ 180 \\ 195 \\ 210 \\ 225 \\ 240 \\ 255 \end{bmatrix} deg$$

Geometric coordinates of points along the horizon

$$x := r_{hor} \cdot \sin(t) = \begin{bmatrix} 2.89734 \\ 2.59768 \\ 2.12100 \\ 1.49977 \\ 0.77634 \\ 0.00000 \\ -0.77634 \\ -1.49977 \\ -2.12100 \\ -2.59768 \\ -2.89734 \end{bmatrix} mi \quad y := r_{hor} \cdot \cos(t) = \begin{bmatrix} -0.77634 \\ -1.49977 \\ -2.12100 \\ -2.59768 \\ -2.89734 \\ -2.99954 \\ -2.89734 \\ -2.59768 \\ -2.12100 \\ -1.49977 \\ -0.77634 \end{bmatrix} mi$$

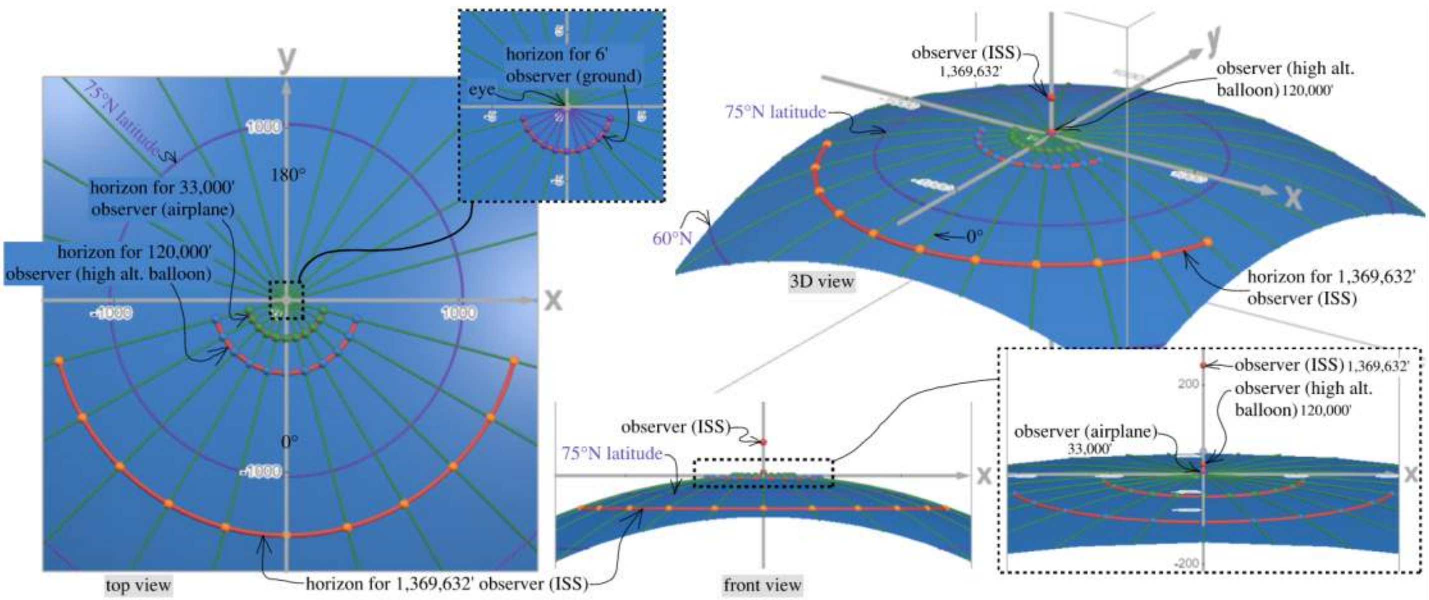
$$z := -1 \cdot (h' + h) - (h - h_1) = -0.00227\ mi$$

Output coordinates on retina

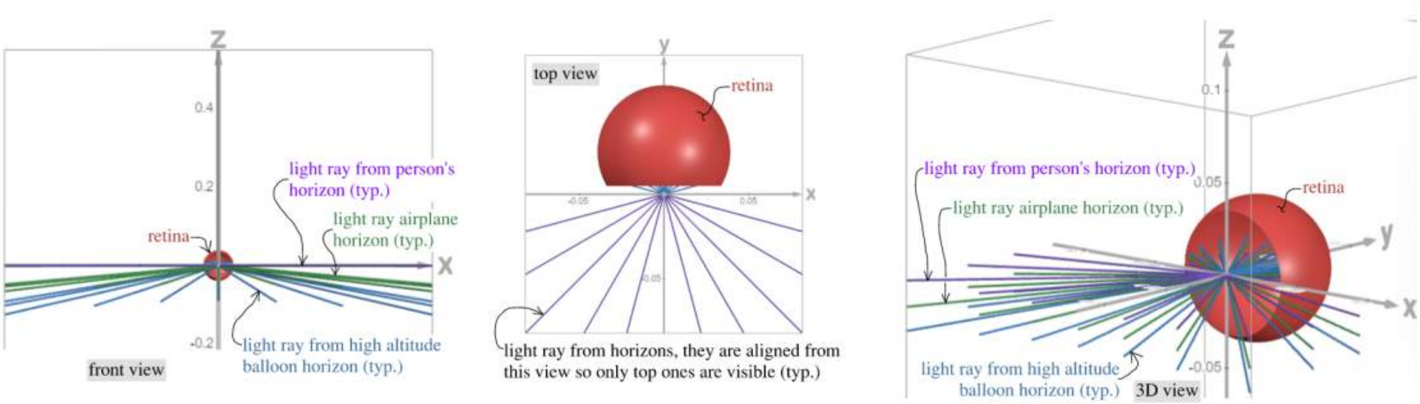
$$x_{ret}(x, y, z) = \begin{bmatrix} -0.000006872183194 \\ -0.000007443698192 \\ -0.000007078597103 \\ -0.000005585548063 \\ -0.000003087318212 \\ 0 \\ 0.000003087372379 \\ 0.000005585602489 \\ 0.000007078433733 \\ 0.000007443805217 \\ 0.000006872183193 \end{bmatrix} mi \quad y_{ret}(x, y, z) = \begin{bmatrix} 0.000001841395937 \\ 0.000004297621155 \\ 0.000007078597103 \\ 0.000009674453033 \\ 0.000011522028426 \\ 0.000012191146924 \\ 0.000011522230579 \\ 0.000009674547302 \\ 0.000007078433733 \\ 0.000004297682946 \\ 0.000001841395937 \end{bmatrix} mi \quad z_{ret}(x, y, z) = \begin{bmatrix} 0.000000005390671 \\ 0.000000006512534 \\ 0.000000007584974 \\ 0.000000008464235 \\ 0.000000009038099 \\ 0.000000009237118 \\ 0.000000009038257 \\ 0.000000008464317 \\ 0.000000007584799 \\ 0.000000006512628 \\ 0.000000005390671 \end{bmatrix} mi$$

(Note: The calculations for the geometric coordinates of points along the horizon result in irrational numbers due to the trigonometric functions, sin and cos, of the angles (t). This software obviously only stores and uses finite numbers so for the calculations of x & y there is error and for the calculations of the retina functions (x_{ret},y_{ret},z_{ret}), there is an accumulation of error, which is why the coordinates are not perfectly symmetric about the y-z plane. They are very close and the difference cannot even be noticed in the plots.)

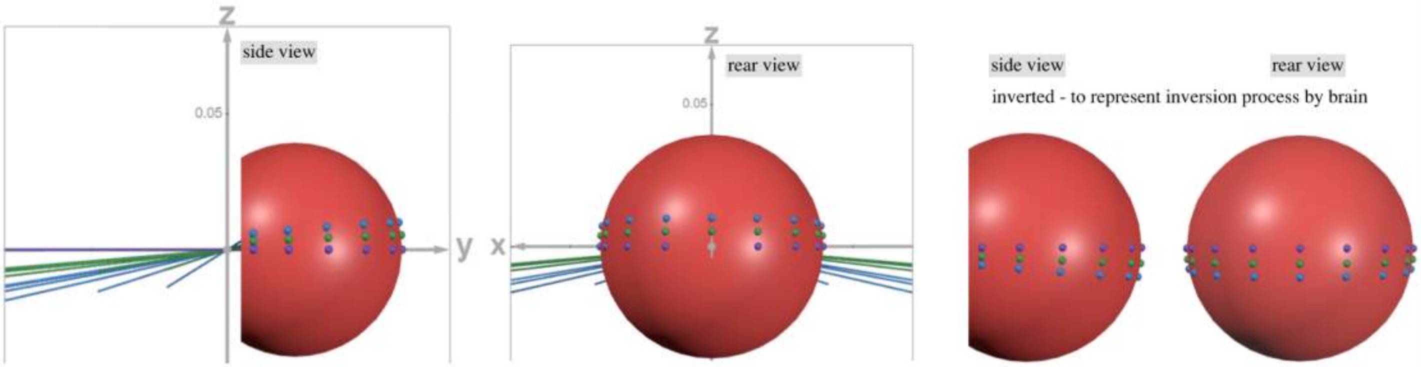
The calculations are shown for an observer height of 6 feet. Calculations were also performed (but not included here) for observer heights of 33,000 feet (airplane), 120,000 feet (high altitude balloon) and 1,369,632' (alleged ISS). Shown below are diagrams of a model of globe earth and a few different horizons formed from a few different observer heights. Units of axes are in miles.



The diagrams below are showing the light rays (coming from the occluding edge horizons shown above) going to the retina. The eye is looking straight out at eye level (so the occluding edge horizons are below this level). The diagrams show the retina in one position, but they are a combination of the three scenarios described in the diagrams above (not including ISS position).



The diagrams below are of the light rays hitting the retina. Points are included at these intersection points along the retina to distinguish the shape of the horizon formed.



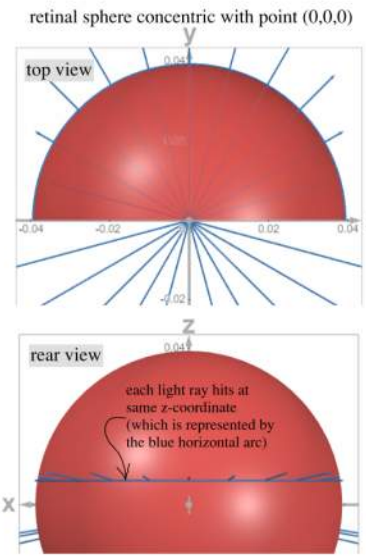
As can be seen in the images above, the light from the occluding edge horizons forms curved arcs on the retina. The point on the horizon below where you are focused (straight out along eye level) would appear to be the lowest point and as you pay attention to your periphery (while keeping focus at same spot) on either side, the horizon would appear to curve upwards slightly. As you move your focus horizontally along eye level, the curves in your periphery would move accordingly. From the observation (with an eye) of the occluding edge horizon on a globe earth, when the horizon is in your lower field of view one might conclude the surface of the earth is concave when paying close attention to the periphery.

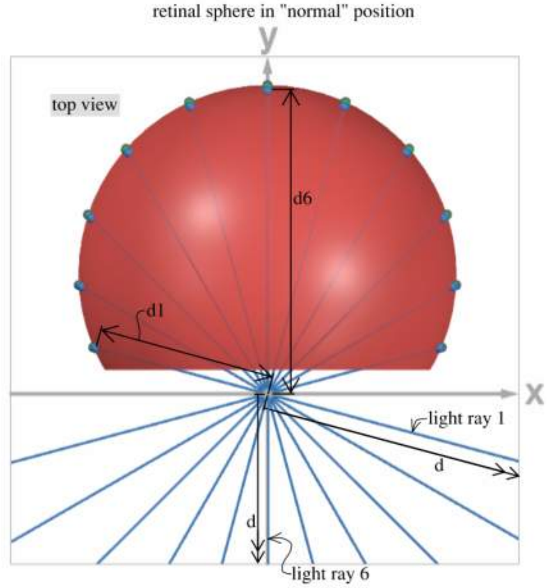
The upward apparent curvature of the horizon in the periphery can be explained using the diagrams to the right. In this example, only the light rays from the high altitude balloon are shown.

Refer to the diagrams titled "retinal sphere concentric with point (0,0,0)". If the retinal sphere was centered at the light ray intersection point (point 0,0,0 in the diagram, the intersection of the x,y,z axes), then each ray would travel the same distance beyond point (0,0,0) to hit the retina and therefore they would hit the retina at the same z-coordinate and the horizon would appear level (from periphery to focus point to the other periphery in your perception). In the diagram titled "retinal sphere in "normal" position", due to the offset of the retina's center from point (0,0,0), the light rays will hit the retina at slightly different z-coordinates and thus the horizon will appear slightly curved upwards in the periphery (as shown in the diagrams at the top of the page).

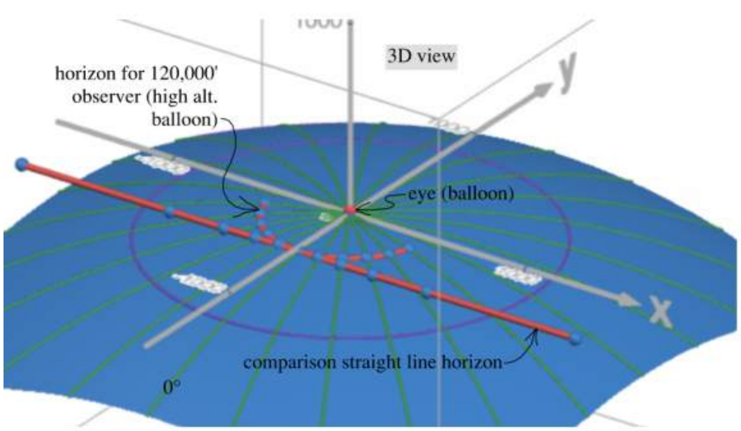
The higher up the eye goes above the surface, then the more apparent this curvature would become. Moving your eye level would affect this apparent curvature as well. If you focused on a point well before the horizon and the horizon was in the periphery of your upper field of view, a convex horizon would be formed in your upper periphery.

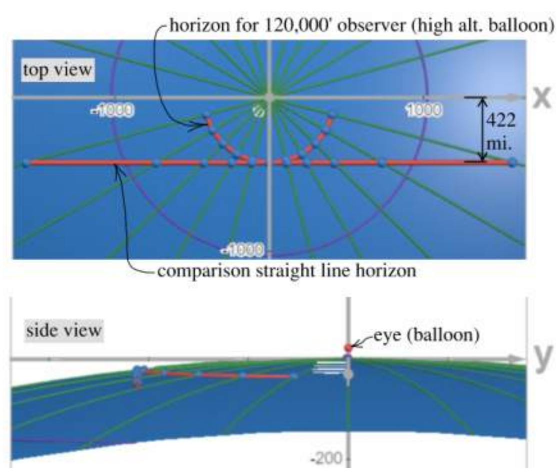
This curvature is going to be impossible to invalidate visually if you are standing on the ground and I imagine it would be difficult to invalidate on a plane due to field of view restrictions and the plane's motion. A high altitude balloon camera height is a good height probably to notice this non-existent upward curvature in the periphery, however I do not think any eyeballs are making it up there. Cameras have a different image forming configuration than the eye. Some are designed to remove what would be equivalent to the eye's peripheral distortion and create rectilinear photographs and others are designed to include distortion like a fisheye lens. A fisheye lens with minor distortion is actually more realistic than a rectilinear lens (comparing to our visual perception). If a radial blur is added to the photo (to mimic our unfocused periphery) it would be even closer.

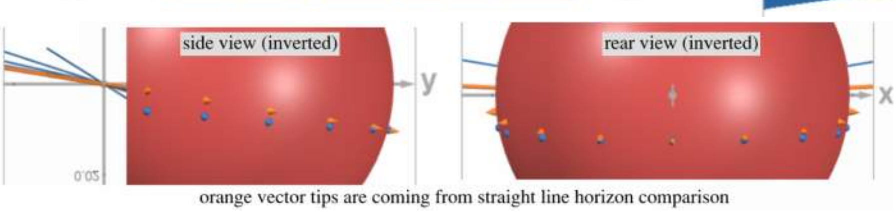




For comparison purposes, a straight line occluding edge horizon is included. This would represent the horizon on a rectangular earth and the observer is above the top surface at the high altitude balloon elevation of 120,000 ft and is $\sqrt{(r + 120000\,ft)^2 - r^2} \cdot \sin\left(\operatorname{atan}\left(\frac{r}{\sqrt{(r + 120000\,ft)^2 - r^2}}\right)\right) = 422\,mi$ away from the edge. The value of 422 miles is the radius of the small circle formed by the occluding edge horizon on the globe earth (see top view below).



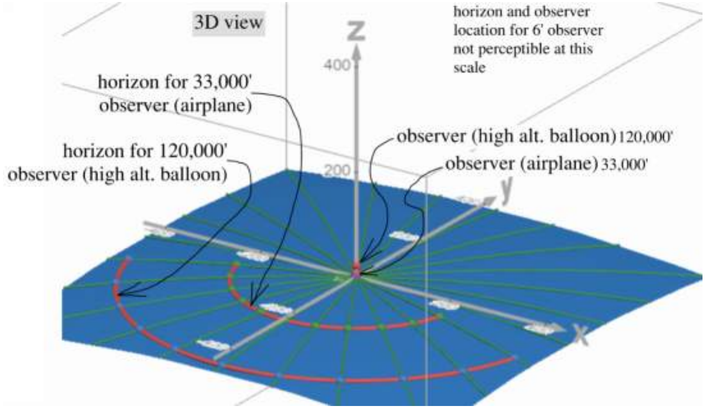


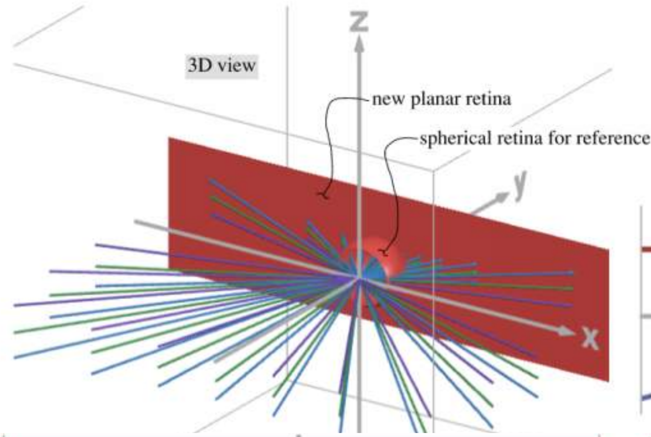


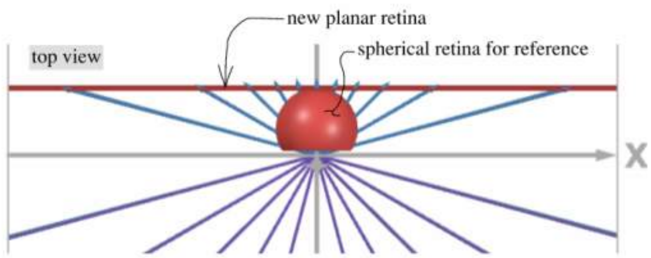
The retina diagrams to the left show the horizon formed for the 120,000 ft observer - blue points (same as shown at top of page) and the horizon formed from the straight line comparison (orange vector tips). There is a more drastic curvature in the periphery occurring for the straight line comparison horizon due to the points along the straight line horizon being farther and farther away from the eye (as you move away from the 0° longitude line), which makes the angles of these light rays shallower and shallower.

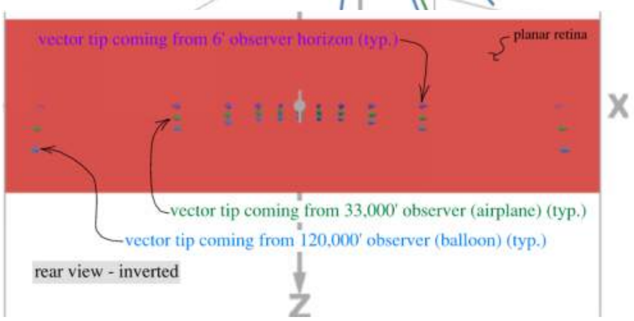
Shown on this page are the same horizon examples from the previous page, but this time the retina will be a vertical plane. This retinal configuration represents an image sensor that preserves linearity in the image formation (if light is coming from linear objects), similar to a rectilinear lens. This preservation of linearity is not very applicable to the curved horizons shown here in this example, but it is included for reference because typical everyday cameras utilize rectilinear lenses to minimize or eliminate distortion and preserve physical straight edges of buildings, roads, and other linear/planar features. Although most cameras and artists create photographs and art utilizing a form of linear perspective, our spherical retinas create a hyperbolic visual perception.

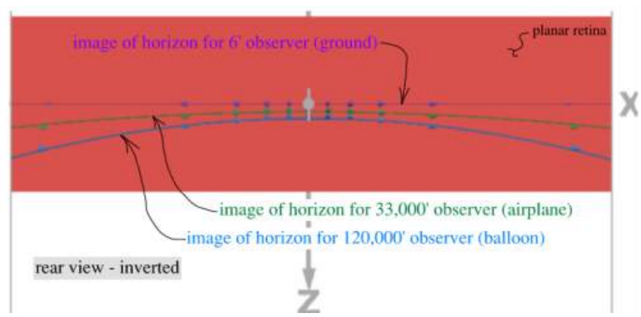
In the 3D view to the right, the distance between each point on the horizon (either one) is equal (15° increments). So in physical reality, the distance between points is identical. Refer to the 3D and top views below, it is clear that the corresponding points on the planar retina of these equidistant horizon points (in reality) are no longer equidistant. The image formed on the planar retina is stretched more and more (in the x-axis direction) as you move farther from the intersection of the axes. This is a common result of rectilinear lenses - the periphery is stretched.





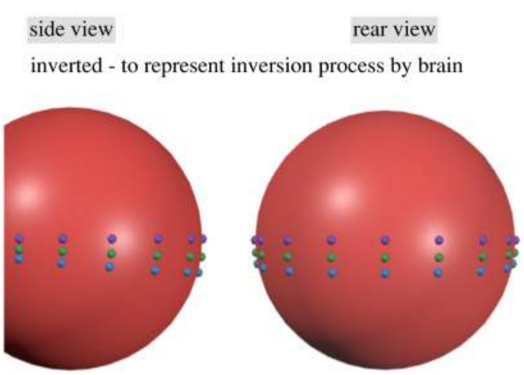




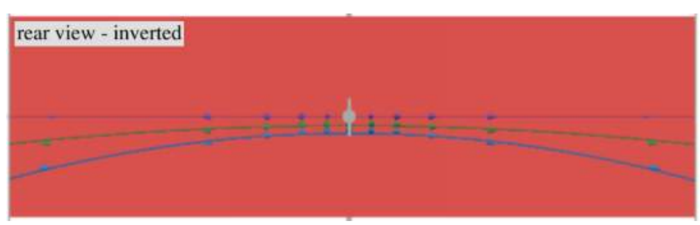


Summary of globe earth occluding edge horizon image formation.

Eyeball with spherical retina



Some type of camera with a planar image sensor



purple - ground observer (6')
green - airplane observer (33,000')
blue - high altitude balloon observer (120,000')

Using an eye, the shape of the horizon is a concave curve. Using a device with a planar image sensor, the shape of the horizon is a convex curve. These examples are very basic and most cameras have multiple differently shaped lenses in which the light travels and gets redirected through. The point of these examples is to demonstrate that in order to make a claim about the shape of a horizon, it is important to understand how that shape is being formed. As seen in this example, the same horizons are forming different shapes based on the shape of the image sensor.

Retinal image formed from the level earth optical horizon.

As discussed previously, the distance that the ground can be perceived up until the optical horizon varies according to many factors. For this example, some of the decay functions from farther up in this document will be used to provide example values of distances to perceptible ground at varying observer elevations. ($d_{hor.opt.decay\lambda.30}$) will be used for an observer height of 33,000 ft. & ($d_{hor.opt.decay\lambda.27}$) for an observer height of 120,000 ft. For an observer height of 6 ft., the angular resolution limit will govern the distance of perceptible ground.

Distance from observer to last perceptible ground surface

$$r_{hor} := \begin{bmatrix} \dfrac{6\ ft}{\tan\left(\Theta_{lim}\right)} \\ d_{hor.opt.decay\lambda.30_{12}} \\ d_{hor.opt.decay\lambda.27_{16}} \end{bmatrix} = \begin{bmatrix} 3.82993 \\ 575.56395 \\ 1018.75290 \end{bmatrix} mi \quad \text{for observer heights} \quad h := \begin{bmatrix} 6 \\ 33000 \\ 120000 \end{bmatrix} ft \quad \begin{bmatrix} \text{“ground”} \\ \text{“airplane”} \\ \text{“high altitude balloon”} \end{bmatrix}$$

The calculations are shown for an observer height of 6 feet. Calculations were also performed (but not included here) for observer heights of 33,000 feet (airplane), and 120,000 feet (high altitude balloon)

Angles within visual field for which geometric coordinates will be calculated

$$t := \frac{\begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \end{bmatrix} \cdot \pi}{12} = \begin{bmatrix} 105 \\ 120 \\ 135 \\ 150 \\ 165 \\ 180 \\ 195 \\ 210 \\ 225 \\ 240 \\ 255 \end{bmatrix} deg$$

$n := 1$

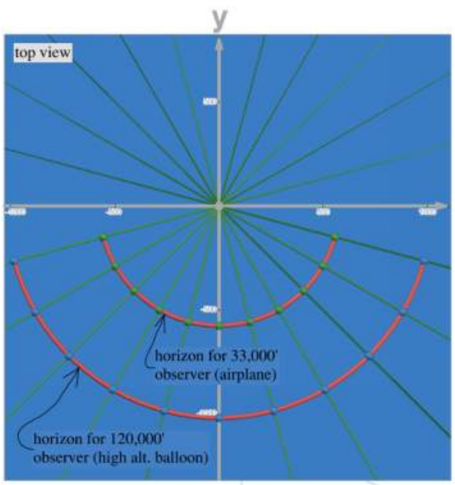
"Geometric coordinates" of points along the optical horizon (with respect to the eye of the observer)

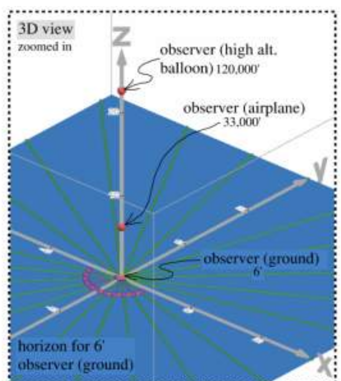
$$x := r_{hor_n} \cdot \sin(t) = \begin{bmatrix} 3.69943 \\ 3.31682 \\ 2.70817 \\ 1.91497 \\ 0.99126 \\ 0.00000 \\ -0.99126 \\ -1.91497 \\ -2.70817 \\ -3.31682 \\ -3.69943 \end{bmatrix} mi \qquad y := r_{hor_n} \cdot \cos(t) = \begin{bmatrix} -0.99126 \\ -1.91497 \\ -2.70817 \\ -3.31682 \\ -3.69943 \\ -3.82993 \\ -3.69943 \\ -3.31682 \\ -2.70817 \\ -1.91497 \\ -0.99126 \end{bmatrix} mi \qquad z := -h_n = -0.00114\ mi$$

Output coordinates on retina

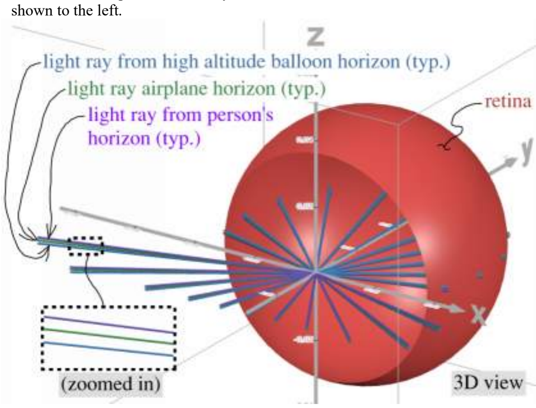
$$x_{ret}(x, y, z) = \begin{bmatrix} -0.03628571 \\ -0.03930169 \\ -0.03737633 \\ -0.02949182 \\ -0.01630104 \\ 0 \\ 0.01630139 \\ 0.02949218 \\ 0.03737633 \\ 0.03930377 \\ 0.03628489 \end{bmatrix} ft \qquad y_{ret}(x, y, z) = \begin{bmatrix} 0.00972273 \\ 0.02269084 \\ 0.03737633 \\ 0.05108134 \\ 0.06083629 \\ 0.06437006 \\ 0.0608376 \\ 0.05108195 \\ 0.03737633 \\ 0.02269204 \\ 0.00972251 \end{bmatrix} ft \qquad z_{ret}(x, y, z) = \begin{bmatrix} 0.00001115 \\ 0.00001347 \\ 0.00001568 \\ 0.00001750 \\ 0.00001869 \\ 0.00001910 \\ 0.00001869 \\ 0.00001750 \\ 0.00001568 \\ 0.00001347 \\ 0.00001115 \end{bmatrix} ft$$

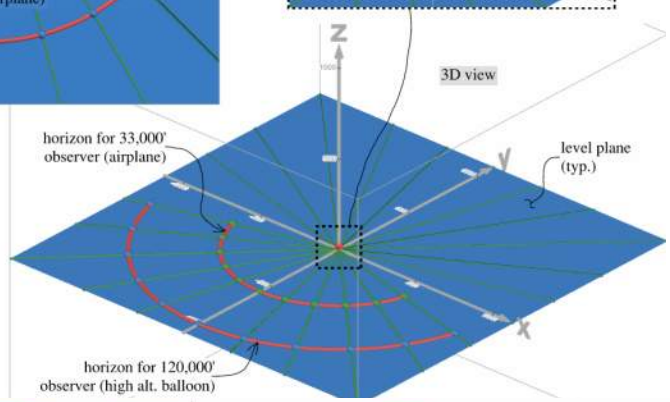
Shown below are diagrams of a model of a level planar earth and a few different horizons formed from a few different observer heights. Units of axes are in miles.



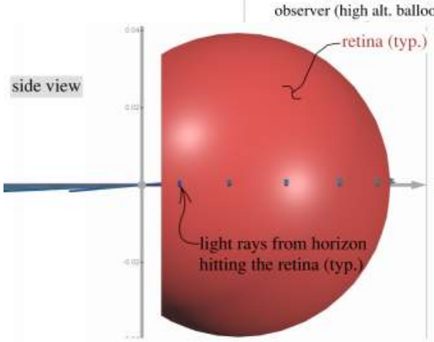


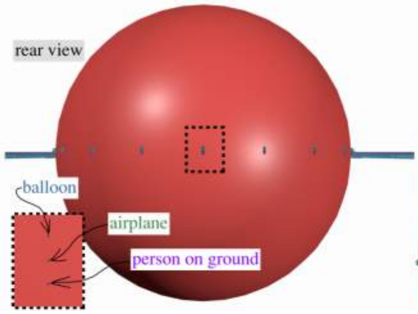
The diagram below is showing the light rays (coming from the optical horizons shown to the left) going to the retina. The eye is looking straight out at eye level (so the optical horizons are slightly below this level). The diagrams show the retina in one position, but they are a combination of the three scenarios shown to the left.

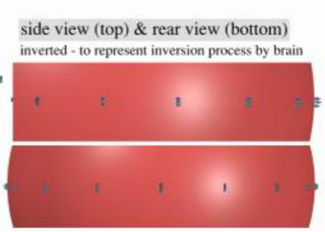




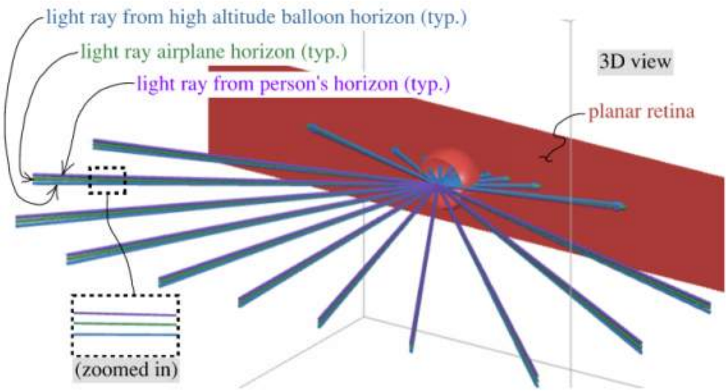
The diagrams along the bottom show the points where the light rays from the optical horizons hit the retina. As can be seen in the inverted diagrams, there is very slight curvature upwards as you move into the periphery (basically imperceptible), not nearly as perceptible as the globe earth occluding edge horizons. Essentially, from each observer height the horizon should appear effectively level using an eyeball as the image sensor and if you are looking straight out along eye level.

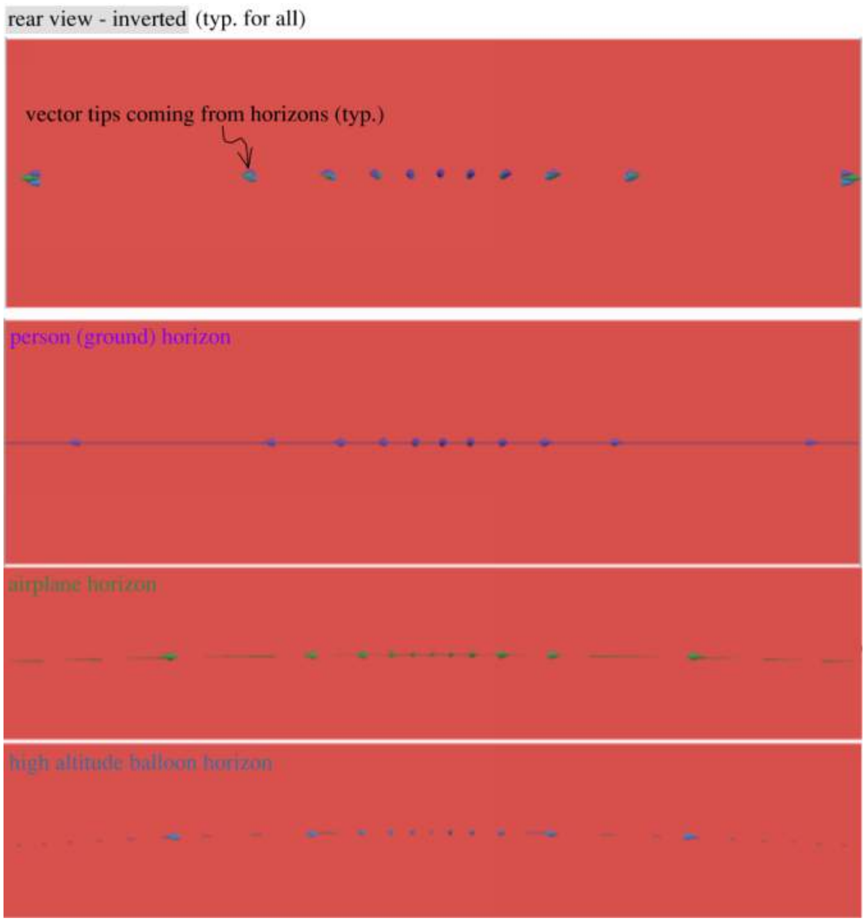






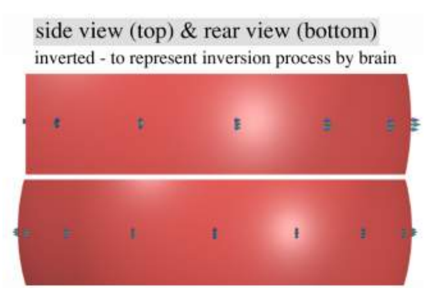
Shown on this page are the same horizon examples from the previous page, but this time the retina will be a vertical plane.





<u>Summary of level plane earth optical horizon image formation.</u>

Eyeball with spherical retina



Some type of camera with a planar image sensor

(see diagrams above)

Using an eye, the shape of the horizon is a slight concave curve. Using a device with a planar image sensor, the shape of the horizon is a slight convex curve.

Calculations to determine the amount that ocean swells could obstruct perceptibility of the bottom portion of a ship.

Observer eye height $h_{eye} := 6\ ft$

Height of swell underneath ship $h_{swell.ship} := 6\ ft$

Height of swell (height from top of swell to sea level) $h_{swell} := 8\ ft$

Distance from observer to ship $d_{ship} := \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} mi$

Distance from observer to swell $d_{swell} := 1.5\ mi$

Height of ship from ocean surface to tallest point on ship $h_{ship} = 60\ ft$

Slope of line of sight to top of swell $m := \dfrac{h_{swell} - h_{eye}}{d_{swell}} = 0.00025\ \dfrac{ft}{ft}$

Height above sea level where corresponding line of sight "hits" ship $y := h_{eye} + m \cdot d_{ship} = \begin{bmatrix} 8.67 \\ 11.33 \\ 19.33 \end{bmatrix} ft$ $\begin{bmatrix} \text{“ship 2 miles away”} \\ \text{“ship 4 miles away”} \\ \text{“ship 10 miles away”} \end{bmatrix}$

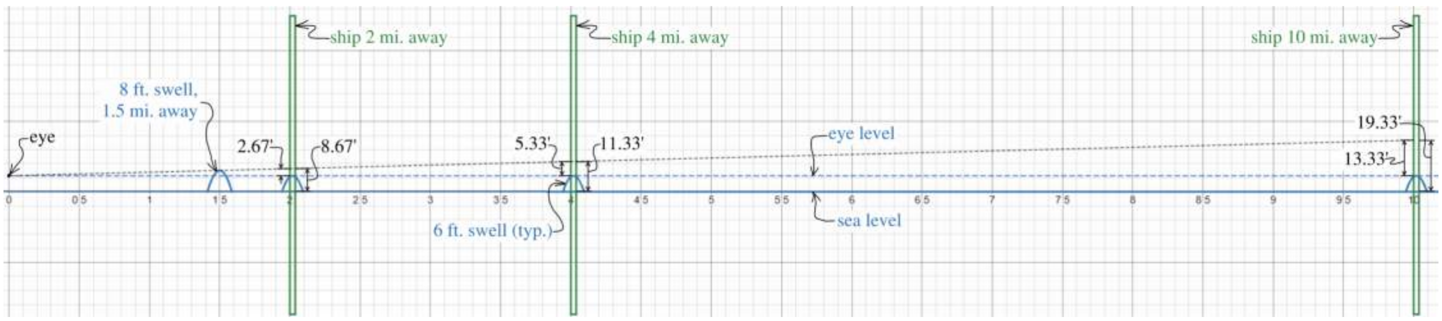
Percentage of ship height (above surface) obstructed by swell

$r := \dfrac{y}{h_{ship}} = \begin{bmatrix} 14.4\% \\ 18.9\% \\ 32.2\% \end{bmatrix}$ (for ships at sea level)

Height of ship obstructed by swell $y_2 := h_{eye} + m \cdot d_{ship} - h_{swell.ship} = \begin{bmatrix} 2.67 \\ 5.33 \\ 13.33 \end{bmatrix} ft$ $\begin{bmatrix} \text{“ship 2 miles away”} \\ \text{“ship 4 miles away”} \\ \text{“ship 10 miles away”} \end{bmatrix}$

$r_2 := \dfrac{y_2}{h_{ship}} = \begin{bmatrix} 4.4\% \\ 8.9\% \\ 22.2\% \end{bmatrix}$ (for ships on 6' swell)

Note: In the diagram below, x-axis units are in miles. The y-axis is scaled up by a factor of 100 for clarity.



If a swell is $d_{swell} = 1.5\ mi$ away from an observer and is $h_{swell} = 8\ ft$ tall (above normal sea level), and a ship is $d_{ship} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} mi$ away sitting atop a $h_{swell.ship} = 6\ ft$ tall swell, then the bottom $y_2 = \begin{bmatrix} 2.67 \\ 5.33 \\ 13.33 \end{bmatrix} ft$ of the ship will be obstructed by the swell and will be imperceptible to the observer. If the ship is sitting atop normal sea level (not on a swell), then the bottom $y = \begin{bmatrix} 8.67 \\ 11.33 \\ 19.33 \end{bmatrix} ft$ of the ship will be obstructed by the swell.

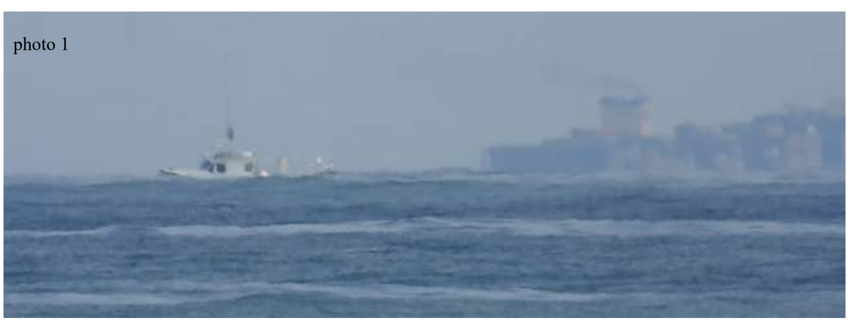


photo 1

In photos 1 and 2 (which are stills from a video), the surface of the ocean had many visible swells. These can be seen in the photo (the lighter shade of blue is the top of the swells and the darker shade of blue is the downslope of the swells) in photo 1.

In photo 2, two swells are prominent (each a different shade of blue, the one farther away is lighter).

An occluding edge obstruction is observable. However, this edge has nothing to do with a physical horizon on a globular earth with water magically sticking to the exterior of it, it just has to do with surface swells. The actual horizon is an optical phenomenon, not a physical thing in three-dimensional reality.


photo 2

This example is included to show the relationship between the height of the images of objects formed on the retina (in this model) and distance from the observer to the objects.

Coordinates of bars in physical space

$x_1 := 0\,ft$ $\quad y_1 := \begin{bmatrix} -5 \\ -10 \\ -20 \\ -40 \\ -80 \\ -160 \end{bmatrix} ft$ $\quad z_1 := -5\,ft$ $\qquad x_2 := 0\,ft$ $\quad y_2 := \begin{bmatrix} -5 \\ -10 \\ -20 \\ -40 \\ -80 \\ -160 \end{bmatrix} ft$ $\quad z_2 := 5\,ft$

Corresponding coordinates on retina

$p_{x.1} := x_{ret}(x_1, y_1, z_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ft$ $\quad p_{y.1} := y_{ret}(x_1, y_1, z_1) = \begin{bmatrix} 0.037374717083959 \\ 0.053763922757978 \\ 0.061268309084301 \\ 0.063560081851196 \\ 0.064165295155456 \\ 0.064318737987184 \end{bmatrix} ft$ $\quad p_{z.1} := z_{ret}(x_1, y_1, z_1) = \begin{bmatrix} 0.037374717083959 \\ 0.026881961378989 \\ 0.015317077271075 \\ 0.007945010231399 \\ 0.004010330947216 \\ 0.002009960562099 \end{bmatrix} ft$

$p_{x.2} := x_{ret}(x_2, y_2, z_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ft$ $\quad p_{y.2} := y_{ret}(x_2, y_2, z_2) = \begin{bmatrix} 0.037374717083959 \\ 0.053763922757978 \\ 0.061268309084301 \\ 0.063560081851196 \\ 0.064165295155456 \\ 0.064318737987184 \end{bmatrix} ft$ $\quad p_{z.2} := z_{ret}(x_2, y_2, z_2) = \begin{bmatrix} -0.037374717083959 \\ -0.026881961378989 \\ -0.015317077271075 \\ -0.007945010231399 \\ -0.004010330947216 \\ -0.002009960562099 \end{bmatrix} ft$

$p_{x.f.1} := x_f(x_1, y_1, z_1) \quad p_{y.f.1} := y_f(x_1, y_1, z_1) \quad p_{z.f.1} := z_f(x_1, y_1, z_1) \quad p_{x.f.2} := x_f(x_2, y_2, z_2) \quad p_{y.f.2} := y_f(x_2, y_2, z_2) \quad p_{z.f.2} := z_f(x_2, y_2, z_2)$

$x_{avg.}(p_{x.1}, p_{x.2}) := \overrightarrow{\text{mean}(p_{x.1}, p_{x.2})} \quad y_{avg.}(p_{y.1}, p_{y.2}) := \overrightarrow{\text{mean}(p_{y.1}, p_{y.2})} \quad z_{avg.}(p_{z.1}, p_{z.2}) := \overrightarrow{\text{mean}(p_{z.1}, p_{z.2})} \quad x_{avg} := \overrightarrow{x_{avg.}(p_{x.1}, p_{x.2})} \quad y_{avg} := \overrightarrow{y_{avg.}(p_{y.1}, p_{y.2})} \quad z_{avg} := \overrightarrow{z_{avg.}(p_{z.1}, p_{z.2})}$

$p_{x.bi.1} := x_{ret}(x_{avg}, y_{avg}, z_{avg}) \quad p_{y.bi.1} := y_{ret}(x_{avg}, y_{avg}, z_{avg}) \quad p_{z.bi.1} := z_{ret}(x_{avg}, y_{avg}, z_{avg}) \quad p_{x.bi.2} := x_{ret}(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}) \quad p_{y.bi.2} := y_{ret}(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1}) \quad p_{z.bi.2} := z_{ret}(p_{x.bi.1}, p_{y.bi.1}, p_{z.bi.1})$

$r_{cir} := \dfrac{\sqrt{{p_{x.bi.1}}^2 + {p_{y.bi.1}}^2 + {p_{z.bi.1}}^2} + \sqrt{{p_{x.bi.2}}^2 + {p_{y.bi.2}}^2 + {p_{z.bi.2}}^2}}{2} \quad x_{cir.}(p_{x.bi.1}, p_{x.bi.2}) := \overrightarrow{\text{mean}(p_{x.bi.1}, p_{x.bi.2})} \quad y_{cir.}(p_{y.bi.1}, p_{y.bi.2}) := \overrightarrow{\text{mean}(p_{y.bi.1}, p_{y.bi.2})} \quad z_{cir.}(p_{z.bi.1}, p_{z.bi.2}) := \overrightarrow{\text{mean}(p_{z.bi.1}, p_{z.bi.2})}$

$x_{cir} := \overrightarrow{x_{cir.}(p_{x.bi.1}, p_{x.bi.2})} \quad y_{cir} := \overrightarrow{y_{cir.}(p_{y.bi.1}, p_{y.bi.2})} \quad z_{cir} := \overrightarrow{z_{cir.}(p_{z.bi.1}, p_{z.bi.2})} \quad l_{xyz.1} := \overrightarrow{\sqrt{(x_{avg} - x_{cir})^2 + (y_{avg} - y_{cir})^2 + (z_{avg} - z_{cir})^2}} \quad l_{xyz.2} := \overrightarrow{\sqrt{(p_{x.2} - x_{avg})^2 + (p_{y.2} - y_{avg})^2 + (p_{z.2} - z_{avg})^2}}$

$\Theta_{cir} := 2 \cdot \text{atan}\left(\dfrac{l_{xyz.2}}{l_{xyz.1}}\right)$

Distance from observer to object $\qquad$ Arc length of object on retina

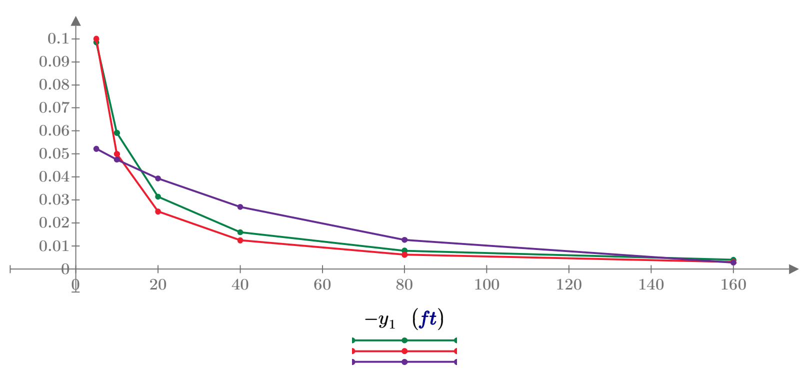
$y_1 = \begin{bmatrix} -5 \\ -10 \\ -20 \\ -40 \\ -80 \\ -160 \end{bmatrix} ft \quad \begin{bmatrix} \text{“obj 1”} \\ \text{“obj 2”} \\ \text{“obj 3”} \\ \text{“obj 4”} \\ \text{“obj 5”} \\ \text{“obj 6”} \end{bmatrix} \quad s_{arc} := \overrightarrow{r_{cir} \cdot \Theta_{cir}} = \begin{bmatrix} 0.0985085 \\ 0.0591804 \\ 0.0314649 \\ 0.0159999 \\ 0.0080346 \\ 0.0040217 \end{bmatrix} ft$

Approximate equations to fit data

Eq1 = inverse proportionality, k=1/2 $\qquad$ Eq2 = exponential regression best fit curve

$eq1 := \dfrac{\frac{1}{2} \cdot ft}{-\dfrac{y_1}{ft}} = \begin{bmatrix} 0.100 \\ 0.050 \\ 0.025 \\ 0.013 \\ 0.006 \\ 0.003 \end{bmatrix} ft \qquad eq2 := \left(0.0574091228182745 \cdot 0.981339575191041^{-\frac{y_1}{ft}}\right) \cdot ft = \begin{bmatrix} 0.052 \\ 0.048 \\ 0.039 \\ 0.027 \\ 0.013 \\ 0.003 \end{bmatrix} ft$

Note: Calculations not shown here for numbers in above equation.



The green plot represents the arc length of the object on the retina (the apparent size of the object in your perception) with respect to the distance to the object from the observer.

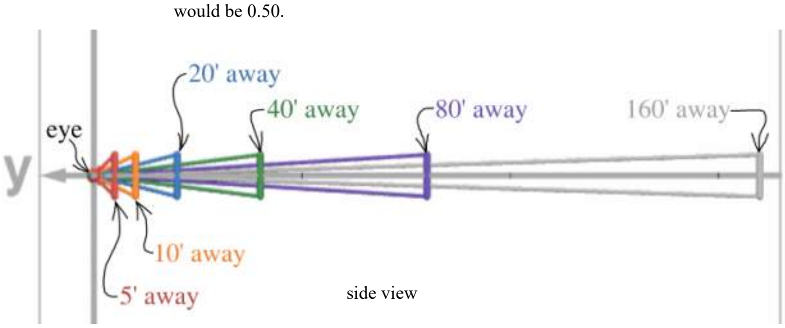
The red and purple plots are some common best fit relationships to fit the set of data points (shown above).

Ratio of image height of object to image height of adjacent (closer in) object

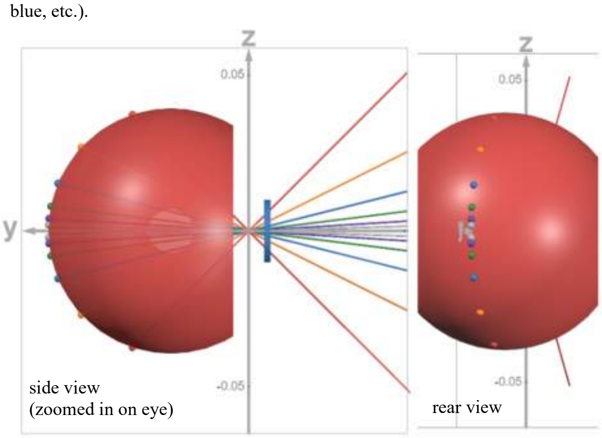
$n := 2\,..\,\text{rows}(y_1)$

$\dfrac{s_{arc_n}}{s_{arc_{n-1}}} = \begin{bmatrix} 0.601 \\ 0.532 \\ 0.508 \\ 0.502 \\ 0.501 \end{bmatrix}$ For example, $\dfrac{s_{arc_2}}{s_{arc_1}} = 0.601$, $\dfrac{s_{arc_3}}{s_{arc_2}} = 0.532$ $\qquad \dfrac{y_{1_n}}{y_{1_{n-1}}} = \begin{bmatrix} 2.00 \\ 2.00 \\ 2.00 \\ 2.00 \\ 2.00 \end{bmatrix}$

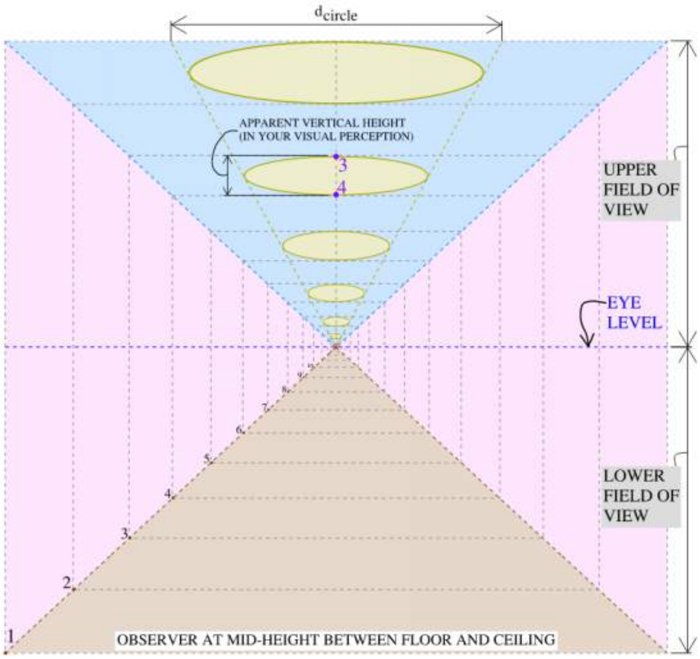
Not quite a halving of the retinal image height as distance to the object doubles. If that were the case, all values in the matrix to the left would be 0.50.

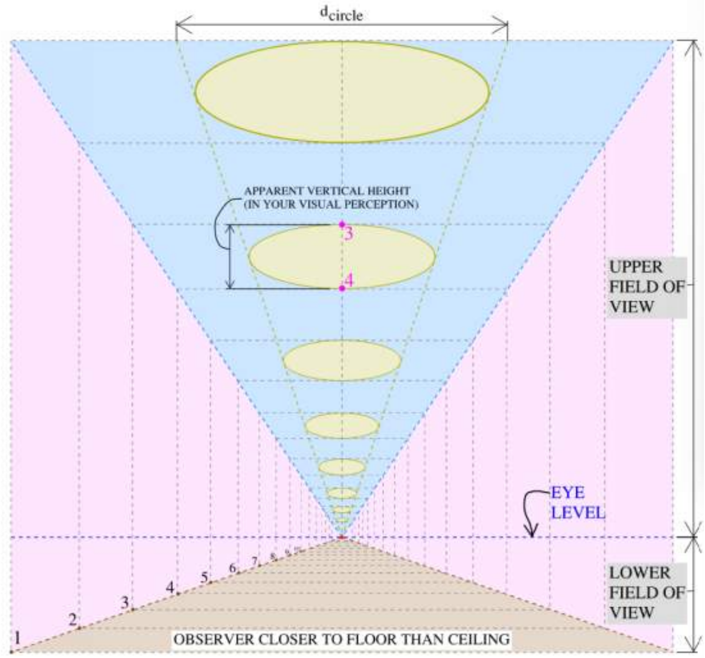


Points are shown on the retina which represent where the light rays from the objects are landing on the retina. Arcs of images formed are not shown, but you can imagine arcs between the same colored points (red to red, blue to blue, etc.).

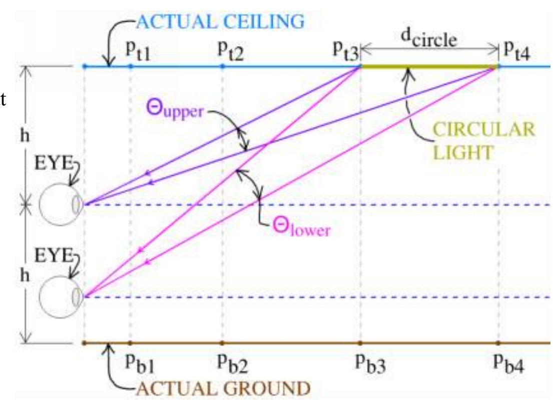


Below are two diagrams of the same tunnel showing the observer's eye level at different heights. The diagram on the left shows an observer at mid-height and the diagram on the right shows an observer closer to the floor. Circular lights are on the ceiling having a diameter of d_{circle}. In the diagram on the left the ceiling is less steep than the diagram on the right and therefore the apparent vertical height (or angular size) of the light is less than the same light in the diagram on the right. This is due to the angles (Θ_{upper}, Θ_{lower}) subtended at the eye by the imaginary line from point 3 to point 4 (see diagram below).





The light going from points 3 and 4 to the eye form an angle (Θ_{upper} for the mid-height observer, Θ_{lower} for the lower observer). These angles (which are a result of the circular light's dimensions and the observer's height) result in the angular size (apparent vertical height) of objects in your visual perception. The light is physically a circle (it can be measured to be a circle with a tape measure), but in our visual perception the circular lights are ellipses. This demonstrates the difference between actual physical shape and apparent visual shape. The only position the observer can be in when the actual physical shape and apparent visual shape are the same is by standing directly underneath it.



The steeper the surface appears in your visual perception then the less compressed the apparent size of the circular light becomes (the apparent shape of the lights for the observer closer to the ground are closer to circles than are the apparent shapes of the lights for the observer at mid-height). So if you are the observer closer to the floor and you start raising the ceiling higher and higher (but still remaining parallel to the floor), then as the ceiling is raised the apparent elliptical lights will get closer and closer to resembling the actual physical circular shape of the light.

Appendix A: Long range photography example [5, 6, 7].

Photographer: Richard Jezik

Year of photo: 2024

Photo taken from: Atop a mountain (Karagöl) in Turkey at an elevation of 1.93 miles (above sea level)

Distance to objects visible in photo: 261 - 306 miles (other mountains)

Elevation of camera $el_A := 3107\ m = 1.93\ mi$ (Karagöl)

Radius of earth $r := 3958.80\ mi$

Height of camera above sea level $h := el_A = 1.93\ mi$

Distance to geometrical horizon (from mainstream teaching) $d_{hor.A} := \sqrt{(r+h)^2 - r^2} = 124\ mi$

Distance from camera to observed mountains

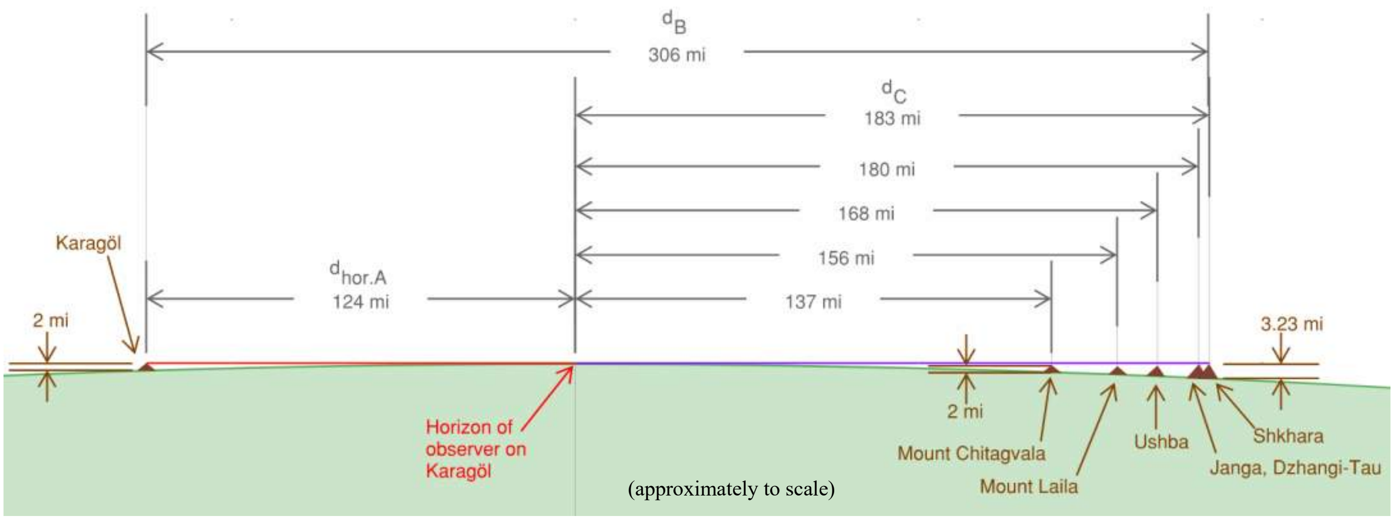
$$d_B := \begin{bmatrix} 420 \\ 450 \\ 470 \\ 489 \\ 493 \end{bmatrix} km = \begin{bmatrix} 261 \\ 280 \\ 292 \\ 304 \\ 306 \end{bmatrix} mi \quad \begin{bmatrix} \text{“Mount Chitagvala”} \\ \text{“Mount Laila”} \\ \text{“Ushba”} \\ \text{“Janga, Dzhangi–Tau”} \\ \text{“Shkhara”} \end{bmatrix}$$

Elevation of observed mountain peaks

$$el_B := \begin{bmatrix} 3226 \\ 4009 \\ 4710 \\ 5085 \\ 5203 \end{bmatrix} m = \begin{bmatrix} 2.00 \\ 2.49 \\ 2.93 \\ 3.16 \\ 3.23 \end{bmatrix} mi \quad \begin{bmatrix} \text{“Mount Chitagvala”} \\ \text{“Mount Laila”} \\ \text{“Ushba”} \\ \text{“Janga, Dzhangi–Tau”} \\ \text{“Shkhara”} \end{bmatrix}$$

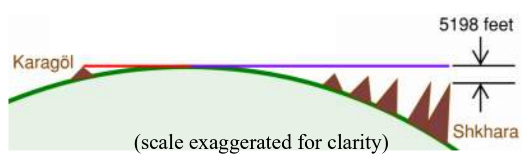
Distance from camera to mountain minus distance from camera to horizon

$$d_C := d_B - d_{hor.A} = \begin{bmatrix} 137 \\ 156 \\ 168 \\ 180 \\ 183 \end{bmatrix} mi \quad \begin{bmatrix} \text{“Mount Chitagvala”} \\ \text{“Mount Laila”} \\ \text{“Ushba”} \\ \text{“Janga, Dzhangi–Tau”} \\ \text{“Shkhara”} \end{bmatrix}$$



Vertical height by which the observed mountains (on the right of the horizon) are obstructed by earth's curvature, from the viewpoint at Karagöl

$$y := r - \sqrt{r^2 - {d_C}^2} - el_B = \begin{bmatrix} 1996 \\ 3075 \\ 3466 \\ 4983 \\ 5198 \end{bmatrix} ft \quad \begin{bmatrix} \text{“Mount Chitagvala”} \\ \text{“Mount Laila”} \\ \text{“Ushba”} \\ \text{“Janga, Dzhangi–Tau”} \\ \text{“Shkhara”} \end{bmatrix}$$



On a globe earth, these mountains are thousands of feet below the horizon. It is optically impossible to see these mountains from Karagöl on a globe. The observed mountains should be physically obstructed by the curvature of the earth by thousands of feet. It is not just the mountain peaks which are observable, it is all or almost all of the mountains.

Appendix B: Density gradients resulting from motion.

This example begins in diagram 1 with examining a portion of "space" with "nothing" in it. "Nothing" is a mainstream concept used to describe the infinite vacuum that objects (like planets) are located within. In reality, "nothing" does not exist because in order for something to exist in this reality it must have shape (or form) and location. "Nothing" does not have shape or location and you cannot point to "nothing" and say "there is nothing, right over there". It is more probable that there is something that exists in what mainstream calls the infinite vacuum. I refer to this as the "substrate" of our physical reality in which all objects are located within. Some people refer to this substrate as "the ether".

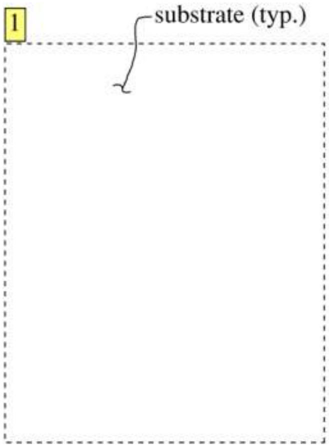


Diagram 1: This depicts a portion of the substrate with no objects in it.

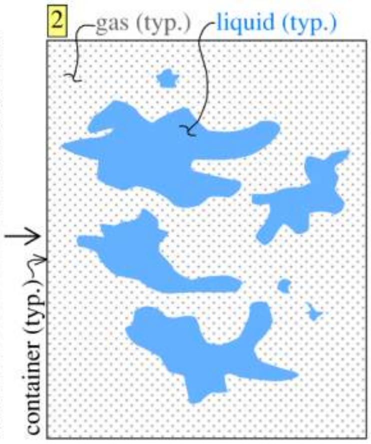


Diagram 2: Objects (liquid matter and gas matter) are added in this portion of the substrate. Since gas has been added, a container has to be added to contain the gas, otherwise the gas would move into the lower (0 psi - no gas) pressure regions surrounding the diagram.

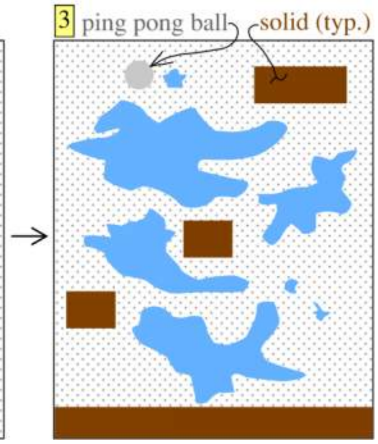


Diagram 3: More objects (solid matter) and an air-filled ping pong ball are added.

Note: The substrate is not "nothing". It is a substance and a medium in which all matter is placed into. Using a video game analogy, the environment is required first - the possibility of placement of matter or that which harbors matter. Whether it has been truly detected or is able to be manipulated, I don't know yet. But it is clear that if you claim outer space is devoid of gas (a mainstream vacuum), and your rocket propulsion is apparently provided by throwing gas down, then you are not going to move because the substrate does not provide the necessary reaction, only an atmosphere full of gas would. Unless the substrate is manipulated, no one is flying around in the space beyond the reaches of our gaseous atmosphere, crafts will begin decelerating with respect to the stationary substrate.

Newton's full third law (not the mainstream bastardized version): *"To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

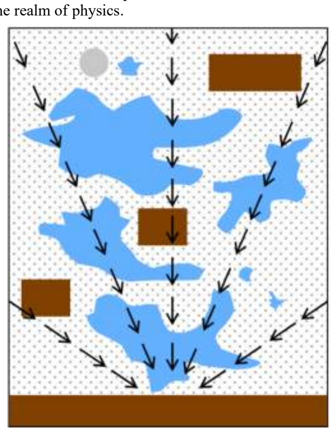
Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone: for the distended rope, by the same endeavour to relax or unbend itself, will draw the horse as much towards the stone, as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other.[Pg 84] If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are reciprocally proportional to the bodies. This law takes place also in attractions, as will be proved in the next scholium."

As of diagram 3, there are three different types of matter (solid, liquid, and gas). The ping pong ball is a solid ball filled with gas (air). Let's just say this matter is magically (similar to the magic of the big bang) placed into existence in the locations shown in the diagrams. There is no motion at all - the substrate, solids, liquid, and gas are stationary. There is also no directional reference - no up, down, left, or right with respect to anything.

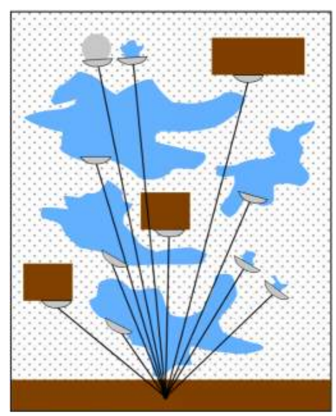
The solids, liquid, and gas have intrinsic properties, one of them being density. The solids are the most dense objects, followed by the liquid, followed by the ping pong ball, and the gas is the least dense. The fact that these objects have different densities does not result in anything moving. The solids are not somehow aware that they are surrounded by less dense matter and begin moving in a particular direction. Which direction would they move anyway? There is no directional reference for anything and they are surrounded on all sides equally be a lesser dense gas (besides the bottom solid). Since the gas exists within this portion of "space" and is contained, there is an associated gas pressure that accompanies the existence of the gas. Let's say the gas pressure is 1 psi. The gas is exerting this pressure on each surface of the solids. Equal pressure on each side means the solids are in equilibrium with the surrounding gas, meaning there isn't a higher pressure on the right side of the solids than the left side. If that were the case, the solids would move to the left due to the unequal pressures.

If nothing else happens in this example (diagram 3 is the final state), then the matter will stay exactly in the position that it was in when it was placed into existence. There is no explanation for why these objects would start moving, unless you **believe** in gravitational or 4-dimensional spacetime models. These models are theories. They exist on paper written in the language of mathematics. These theories cannot be demonstrated in our physical reality and therefore do not belong in the realm of physics.

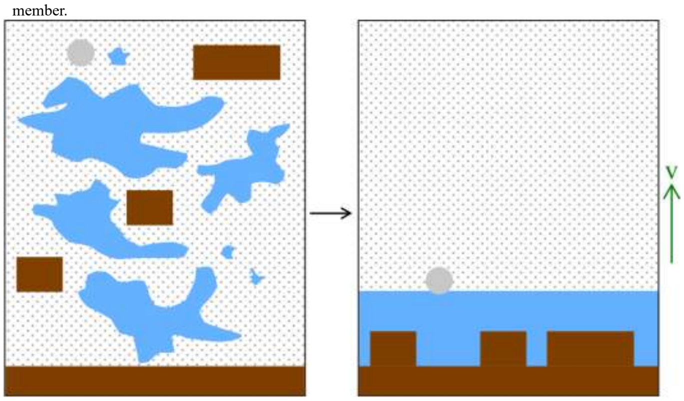
Under Newton's imagined concept of gravity, the large solid (largest mass in this example) along the lower edge of the container somehow pulls all of the other smaller masses towards it. In this theory all of the objects are pulling on each other, but the net force is towards the largest mass (see diagram to the right). Newton at least understood that his own theory was magical - masses exerting forces on other masses at a distance with no identifiable physical mechanism of doing so and performing this through "nothingness" (like planets "pulling" other objects towards them in the vacuum of space). But people later on started fantasizing about hypothetical "fields", particles called "gravitons", and other conceptual inventions. They cannot be demonstrated or shown to exist, and if this is the case, it is not physics.



I can imagine that the solid (along the bottom) extends invisible suction cups out and attaches to all of the other objects and then pulls them in. Of course I cannot demonstrate this, but this theory is as valid as Newton's. Neither can be demonstrated using the substances in our reality so they are equally nonsensical and require belief.



Relativity theories are equally as nonsensical and do not belong in the realm of physics. These are more theories imagined by some of the priests of "science", similar to the big bang and the heliocentric-globe model (the largest religion in existence currently that its members do not even recognize they are in). Challenge someone about their heliocentric-globe beliefs and see what happens. In the majority of cases people will defend this belief with incomprehension, heightened emotions, disbelief, bewilderment, belief in composite images and hearsay, and assumptions and guesses about the apparent motion of intangible lights in the sky rather than with rational logical physical explanations of why they believe it. They will hold onto this belief with dear life when encountering a challenge to their unrecognized belief masquerading as science in a similar manner to a cult member.



So how do we get any motion or any sort of density gradient being produced (similar to what we witness in our physical reality)? Is there any physical demonstrations that could support an explanation, which is what physics actually is?

What if the container is in motion? If the container is in motion then a density gradient will be formed. For example, if the motion is in the upwards direction (shown to the left) then the least dense matter will move to the top of the container and the most dense matter will be at the bottom of the container, and now we have a directional reference. This is exactly how we see matter behave on this earth. It is a simple explanation that can be practically demonstrated by anyone. It is actual physics. No complex nonsense or religious beliefs required. On one hand we have theoretical nonsense that cannot be physically demonstrated and on the other hand we have an explanation that can be backed by practical demonstration. It is a pretty logical choice once the psychological barrier is overcome.

See references 8 & 9 for physical demonstrations.

Appendix C - Calculations to plot the apparent positions of the sun at different times of the year (solstices and equinoxes).

Angular speed of earth $\alpha := \frac{360\ deg}{24\ hr} = 15\frac{deg}{hr}$ Radius of earth $r := 3958.8\ mi$

Time (taken from time & date website)

Length of day

$$t_{june} := \begin{bmatrix} 20\ hr + 26\ min + 19\ s \\ 12\ hr + 46\ min + 06\ s \\ 05\ hr + 05\ min + 54\ s \end{bmatrix} \quad \begin{bmatrix} \text{"sunset"} \\ \text{"solar noon"} \\ \text{"sunrise"} \end{bmatrix}$$

(summer solstice - June 21, 2026) $l_{day.june} := t_{june_1} - t_{june_3} = 15.34\ hr$

$$t_{sept} := \begin{bmatrix} 18\ hr + 43\ min + 15\ s \\ 12\ hr + 36\ min + 52\ s \\ 06\ hr + 30\ min + 29\ s \end{bmatrix} \quad \begin{bmatrix} \text{"sunset"} \\ \text{"solar noon"} \\ \text{"sunrise"} \end{bmatrix}$$

(autumnal equinox - Sept. 22, 2026) $l_{day.sept} := t_{sept_1} - t_{sept_3} = 12.21\ hr$

$$t_{dec} := \begin{bmatrix} 17\ hr + 16\ min + 19\ s \\ 12\ hr + 42\ min + 24\ s \\ 08\ hr + 08\ min + 28\ s \end{bmatrix} \quad \begin{bmatrix} \text{"sunset"} \\ \text{"solar noon"} \\ \text{"sunrise"} \end{bmatrix}$$

(winter solstice - Dec. 21, 2026)
(ignoring daylight savings time end) $l_{day.dec} := t_{dec_1} - t_{dec_3} = 9.13\ hr$

$$t_{march} := \begin{bmatrix} 18\ hr + 57\ min + 43\ s \\ 12\ hr + 51\ min + 38\ s \\ 06\ hr + 45\ min + 32\ s \end{bmatrix} \quad \begin{bmatrix} \text{"sunset"} \\ \text{"solar noon"} \\ \text{"sunrise"} \end{bmatrix}$$

(vernal equinox - March 20, 2026) $l_{day.march} := t_{march_1} - t_{march_3} = 12.20\ hr$

Coordinates of Boston, MA $latlong_{bos} := \begin{bmatrix} 42.361145 \\ -71.057083 \end{bmatrix} deg$

Longitude of sun at sunset

$$long_{sunset.} := latlong_{bos_2} - \alpha \cdot \left(\begin{bmatrix} t_{june_1} \\ t_{sept_1} \\ t_{dec_1} \\ t_{march_1} \end{bmatrix} - \begin{bmatrix} t_{june_2} \\ t_{sept_2} \\ t_{dec_2} \\ t_{march_2} \end{bmatrix} \right) = \begin{bmatrix} -186.11 \\ -162.65 \\ -139.54 \\ -162.58 \end{bmatrix} deg \quad \text{(west)} \quad \begin{bmatrix} \text{"summer solstice"} \\ \text{"atumnal equinox"} \\ \text{"winter solstice"} \\ \text{"vernal equinox"} \end{bmatrix}$$

$$f\left(long_{sunset.}\right) := \overrightarrow{\begin{array}{l} \text{if } long_{sunset.} < -180\ deg \\ \quad \| \ 180\ deg - \left|long_{sunset.} + 180\ deg\right| \\ \text{else} \\ \quad \| \ long_{sunset.} \end{array}}$$

(correcting)

$$long_{sunset} := \overrightarrow{f\left(long_{sunset.}\right)} = \begin{bmatrix} 173.89 \\ -162.65 \\ -139.54 \\ -162.58 \end{bmatrix} deg \quad \begin{bmatrix} \text{"east"} \\ \text{"west"} \\ \text{"west"} \\ \text{"west"} \end{bmatrix} \quad \begin{bmatrix} \text{"summer solstice"} \\ \text{"atumnal equinox"} \\ \text{"winter solstice"} \\ \text{"vernal equinox"} \end{bmatrix}$$

Longitude of sun at sunrise

$$long_{sunrise} := latlong_{bos_2} + \alpha \cdot \left(\begin{bmatrix} t_{june_2} \\ t_{sept_2} \\ t_{dec_2} \\ t_{march_2} \end{bmatrix} - \begin{bmatrix} t_{june_3} \\ t_{sept_3} \\ t_{dec_3} \\ t_{march_3} \end{bmatrix} \right) = \begin{bmatrix} 43.99 \\ 20.54 \\ -2.57 \\ 20.47 \end{bmatrix} deg \quad \begin{bmatrix} \text{"east"} \\ \text{"east"} \\ \text{"west"} \\ \text{"east"} \end{bmatrix} \quad \begin{bmatrix} \text{"summer solstice"} \\ \text{"atumnal equinox"} \\ \text{"winter solstice"} \\ \text{"vernal equinox"} \end{bmatrix}$$

Notes:

1. Assumed locations of land masses (based on an azimuthal equidistant projection) are included in the diagram below. I am not claiming this is an accurate layout of the lands on this plane.
2. According to Merriam-Webster's dictionary, the word "tropic" comes from the Latin word "tropicus" meaning "of a solstice or equinox" and the Greek word "tropikós" meaning "pertaining to a turn or change". This could possibly be referring to the change in direction of the sun's "looping spiral" that occurs at the tropics. For example, the sun reaches it's northernmost "loop" during the summer solstice over the tropic of cancer and then changes direction and starts heading south where it will reach it's southernmost "loop" during the winter solstice over the tropic of capricorn 6 months later.

Globe earth radii at certain latitudes

$$r_{circle.globe} := r \cdot \cos\left(\left(\begin{bmatrix} latlong_{bos_1} \\ 23.43586\ deg \\ 0\ deg \\ -23.43586\ deg \end{bmatrix}\right)\right) = \begin{bmatrix} 2925.21 \\ 3632.22 \\ 3958.80 \\ 3632.22 \end{bmatrix} mi \quad \begin{bmatrix} \text{"boston, ma"} \\ \text{"tropic of cancer"} \end{bmatrix}$$

Globe earth circumference at certain latitudes

$$c_{circle.globe} := 2 \cdot \pi \cdot r_{circle.globe} = \begin{bmatrix} 18380 \\ 22822 \\ 24874 \\ 22822 \end{bmatrix} mi \quad \begin{bmatrix} \text{"boston, ma"} \\ \text{"tropic of cancer"} \\ \text{"equator"} \\ \text{"tropic of capricorn"} \end{bmatrix}$$

Distance from north pole to latitudes on level plane earth

$$r_{circle} := \left(90\ deg - \begin{bmatrix} latlong_{bos_1} \\ 23.43586\ deg \\ 0\ deg \\ -23.43586\ deg \end{bmatrix}\right) \cdot r_{circle.globe_3} = \begin{bmatrix} 3291.56 \\ 4599.19 \\ 6218.47 \\ 7837.75 \end{bmatrix} mi \quad \begin{bmatrix} \text{"boston, ma"} \\ \text{"tropic of cancer"} \\ \text{"equator"} \\ \text{"tropic of capricorn"} \end{bmatrix}$$

Circumference of latitude circles on level plane earth

$$c_{circle} := 2 \cdot \pi \cdot r_{circle} = \begin{bmatrix} 20681.50 \\ 28897.56 \\ 39071.79 \\ 49246.02 \end{bmatrix} mi \quad \begin{bmatrix} \text{"boston, ma"} \\ \text{"tropic of cancer"} \\ \text{"equator"} \\ \text{"tropic of capricorn"} \end{bmatrix}$$

Angle subtended at the north pole by sunrise and sunset sun positions

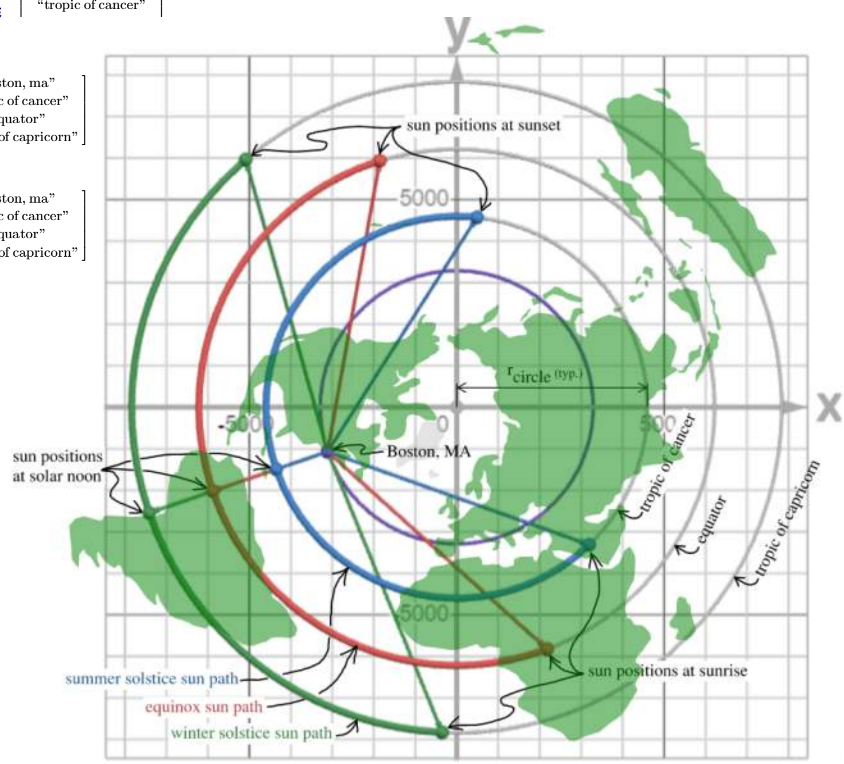
$$\beta := \overrightarrow{\left|long_{sunset.}\right| + long_{sunrise}} = \begin{bmatrix} 230.10 \\ 183.19 \\ 136.96 \\ 183.05 \end{bmatrix} deg \quad \begin{bmatrix} \text{"summer solstice"} \\ \text{"atumnal equinox"} \\ \text{"winter solstice"} \\ \text{"vernal equinox"} \end{bmatrix}$$

Distance traveled (arc length) of sun during the day

$$s := \overrightarrow{\beta \cdot \begin{bmatrix} r_{circle_2} \\ r_{circle_3} \\ r_{circle_4} \\ r_{circle_3} \end{bmatrix}} = \begin{bmatrix} 18470.69 \\ 19882.30 \\ 18735.72 \\ 19866.47 \end{bmatrix} mi \quad \begin{bmatrix} \text{"summer solstice"} \\ \text{"atumnal equinox"} \\ \text{"winter solstice"} \\ \text{"vernal equinox"} \end{bmatrix}$$

Velocity of sun at different latitudes

$$v := \frac{s}{\begin{bmatrix} l_{day.june} \\ l_{day.sept} \\ l_{day.dec} \\ l_{day.march} \end{bmatrix}} = \begin{bmatrix} 1204.06 \\ 1627.99 \\ 2051.92 \\ 1627.99 \end{bmatrix} mph \quad \begin{bmatrix} \text{"summer solstice"} \\ \text{"atumnal equinox"} \\ \text{"winter solstice"} \\ \text{"vernal equinox"} \end{bmatrix}$$



Model - North Pole at (0,0,0)

Angular speed of sun $\alpha = 15 \frac{deg}{hr}$

Lat/long of Boston $lat_{bos} := latlong_{bos_1} = 42.36\ deg$ $long_{bos} := latlong_{bos_2} = -71.06\ deg$

Coordinates of Boston $x_{bos} := \overrightarrow{r_{circle_1} \cdot \sin\left(long_{bos}\right) \cdot 1} = -3113.3\ mi$ $y_{bos} := \overrightarrow{r_{circle_1} \cdot \cos\left(long_{bos}\right) \cdot -1} = -1068.53\ mi$

Sun lat/long during summer solstice

$lat_{june} := 23.43586\ deg$ $long_{june} := \begin{bmatrix} long_{sunrise_1} \\ latlong_{bos_2} \\ long_{sunset_1} \end{bmatrix} = \begin{bmatrix} 43.99 \\ -71.06 \\ 173.89 \end{bmatrix} deg$ $\begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$

Sun lat/long during autumnal equinox

$lat_{sept} := 0\ deg$ $long_{sept} := \begin{bmatrix} long_{sunrise_2} \\ latlong_{bos_2} \\ long_{sunset_2} \end{bmatrix} = \begin{bmatrix} 20.54 \\ -71.06 \\ -162.65 \end{bmatrix} deg$ $\begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$

Sun lat/long during winter solstice

$lat_{dec} := -23.43586\ deg$ $long_{dec} := \begin{bmatrix} long_{sunrise_3} \\ latlong_{bos_2} \\ long_{sunset_3} \end{bmatrix} = \begin{bmatrix} -2.57 \\ -71.06 \\ -139.54 \end{bmatrix} deg$ $\begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$

Sun lat/long during vernal equinox

$lat_{mar} := 0\ deg$ $long_{mar} := \begin{bmatrix} long_{sunrise_4} \\ latlong_{bos_2} \\ long_{sunset_4} \end{bmatrix} = \begin{bmatrix} 20.47 \\ -71.06 \\ -162.58 \end{bmatrix} deg$ $\begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$

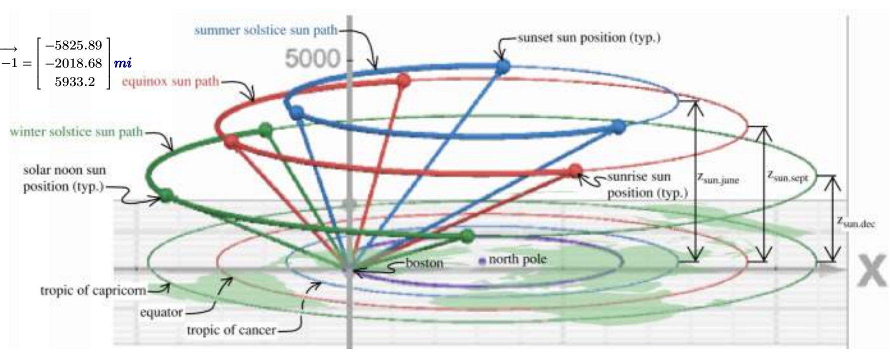
Sun coordinates (during solstices and equinoxes)

$x_{june} := \overrightarrow{r_{circle_2} \cdot \sin\left(long_{june}\right) \cdot 1} = \begin{bmatrix} 3194.46 \\ -4350.11 \\ 489.63 \end{bmatrix} mi$ $y_{june} := \overrightarrow{r_{circle_2} \cdot \cos\left(long_{june}\right) \cdot -1} = \begin{bmatrix} -3308.77 \\ -1493.02 \\ 4573.05 \end{bmatrix} mi$

$x_{sept} := \overrightarrow{r_{circle_3} \cdot \sin\left(long_{sept}\right) \cdot 1} = \begin{bmatrix} 2181.69 \\ -5881.69 \\ -1854.09 \end{bmatrix} mi$ $y_{sept} := \overrightarrow{r_{circle_3} \cdot \cos\left(long_{sept}\right) \cdot -1} = \begin{bmatrix} -5823.19 \\ -2018.68 \\ 5935.63 \end{bmatrix} mi$

$x_{dec} := \overrightarrow{r_{circle_4} \cdot \sin\left(long_{dec}\right) \cdot 1} = \begin{bmatrix} -351.96 \\ -7413.27 \\ -5086.44 \end{bmatrix} mi$ $y_{dec} := \overrightarrow{r_{circle_4} \cdot \cos\left(long_{dec}\right) \cdot -1} = \begin{bmatrix} -7829.84 \\ -2544.34 \\ 5963.09 \end{bmatrix} mi$

$x_{mar} := \overrightarrow{r_{circle_3} \cdot \sin\left(long_{mar}\right) \cdot 1} = \begin{bmatrix} 2174.49 \\ -5881.69 \\ -1861.86 \end{bmatrix} mi$ $y_{mar} := \overrightarrow{r_{circle_3} \cdot \cos\left(long_{mar}\right) \cdot -1} = \begin{bmatrix} -5825.89 \\ -2018.68 \\ 5933.2 \end{bmatrix} mi$



Angular altitude

Solar noon $\Theta_{alt} := \begin{bmatrix} 71.1 \\ 47.8 \\ 24.2 \\ 47.7 \end{bmatrix} deg$ $\begin{bmatrix} \text{“summer solstice”} \\ \text{“atumnal equinox”} \\ \text{“winter solstice”} \\ \text{“vernal equinox”} \end{bmatrix}$

Given these angular altitudes, the corresponding sun height would be:

$z_{sun.june} := \tan\left(\Theta_{alt_1}\right) \cdot \sqrt{\left(\left|x_{june_2}\right| - \left|x_{bos}\right|\right)^2 + \left(\left|y_{june_2}\right| - \left|y_{bos}\right|\right)^2} = 3819.26134124664\ mi$

$z_{sun.sept} := \tan\left(\Theta_{alt_2}\right) \cdot \sqrt{\left(\left|x_{sept_2}\right| - \left|x_{bos}\right|\right)^2 + \left(\left|y_{sept_2}\right| - \left|y_{bos}\right|\right)^2} = 3227.92642703494\ mi$

$z_{sun.dec} := \tan\left(\Theta_{alt_3}\right) \cdot \sqrt{\left(\left|x_{dec_2}\right| - \left|x_{bos}\right|\right)^2 + \left(\left|y_{dec_2}\right| - \left|y_{bos}\right|\right)^2} = 2043.13631503922\ mi$

$z_{sun.mar} := \tan\left(\Theta_{alt_4}\right) \cdot \sqrt{\left(\left|x_{mar_2}\right| - \left|x_{bos}\right|\right)^2 + \left(\left|y_{mar_2}\right| - \left|y_{bos}\right|\right)^2} = 3216.62654350769\ mi$

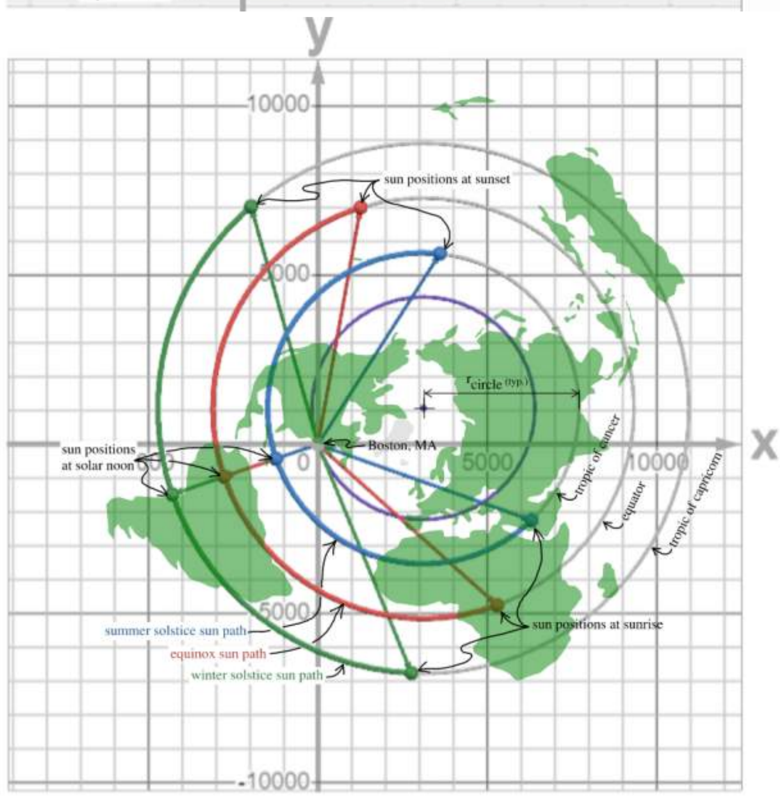
For modeling, the location of the observer (Boston) is set to coordinate (0,0,0) which requires previously calculated coordinates to be shifted by $\left|x_{bos}\right| = 3113.3\ mi$ and $\left|y_{bos}\right| = 1068.53\ mi$.

$x_{june.shift} := x_{june} + \left|x_{bos}\right| = \begin{bmatrix} 6307.76 \\ -1236.81 \\ 3602.93 \end{bmatrix} mi$ $y_{june.shift} := y_{june} + \left|y_{bos}\right| = \begin{bmatrix} -2240.25 \\ -424.49 \\ 5641.58 \end{bmatrix} mi$

$x_{sept.shift} := x_{sept} + \left|x_{bos}\right| = \begin{bmatrix} 5294.99 \\ -2768.39 \\ 1259.21 \end{bmatrix} mi$ $y_{sept.shift} := y_{sept} + \left|y_{bos}\right| = \begin{bmatrix} -4754.67 \\ -950.15 \\ 7004.16 \end{bmatrix} mi$

$x_{dec.shift} := x_{dec} + \left|x_{bos}\right| = \begin{bmatrix} 2761.34 \\ -4299.97 \\ -1973.14 \end{bmatrix} mi$ $y_{dec.shift} := y_{dec} + \left|y_{bos}\right| = \begin{bmatrix} -6761.31 \\ -1475.81 \\ 7031.62 \end{bmatrix} mi$

$x_{mar.shift} := x_{mar} + \left|x_{bos}\right| = \begin{bmatrix} 5287.79 \\ -2768.39 \\ 1251.44 \end{bmatrix} mi$ $y_{mar.shift} := y_{mar} + \left|y_{bos}\right| = \begin{bmatrix} -4757.36 \\ -950.15 \\ 7001.72 \end{bmatrix} mi$



Shifted model - Boston at (0,0,0)

Calculations for the size of the sun given the sun coordinates previously calculated and using an angular sun size of 0.5°.

June - summer solstice

$$x := \left|\overrightarrow{x_{june.shift}}\right| = \begin{bmatrix} 6307.76 \\ 1236.81 \\ 3602.93 \end{bmatrix} \boldsymbol{mi} \qquad y := \left|\overrightarrow{y_{june.shift}}\right| = \begin{bmatrix} 2240.25 \\ 424.49 \\ 5641.58 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$z := \begin{bmatrix} z_{sun.june} \\ z_{sun.sept} \\ z_{sun.dec} \\ z_{sun.mar} \end{bmatrix} = \begin{bmatrix} 3819.26 \\ 3227.93 \\ 2043.14 \\ 3216.63 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“summer solstice”} \\ \text{“autumnal equinox”} \\ \text{“winter solstice”} \\ \text{“vernal equinox”} \end{bmatrix}$$

$$l_{june} := \sqrt{x^2 + y^2 + {z_1}^2} = \begin{bmatrix} 7706.7 \\ 4036.91 \\ 7706.83 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$b_{sun.june} := 2 \cdot l_{june} \cdot \tan\left(0.5 \frac{\boldsymbol{deg}}{2}\right) = \begin{bmatrix} 67.25 \\ 35.23 \\ 67.26 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

September- autumnal equinox

$$x := \left|\overrightarrow{x_{sept.shift}}\right| = \begin{bmatrix} 5294.99 \\ 2768.39 \\ 1259.21 \end{bmatrix} \boldsymbol{mi} \qquad y := \left|\overrightarrow{y_{sept.shift}}\right| = \begin{bmatrix} 4754.67 \\ 950.15 \\ 7004.16 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$l_{sept} := \sqrt{x^2 + y^2 + {z_2}^2} = \begin{bmatrix} 7814.3 \\ 4357.33 \\ 7814.3 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$b_{sun.sept} := 2 \cdot l_{sept} \cdot \tan\left(\frac{0.5\ \boldsymbol{deg}}{2}\right) = \begin{bmatrix} 68.19 \\ 38.03 \\ 68.19 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

December - winter solstice

$$x := \left|\overrightarrow{x_{dec.shift}}\right| = \begin{bmatrix} 2761.34 \\ 4299.97 \\ 1973.14 \end{bmatrix} \boldsymbol{mi} \qquad y := \left|\overrightarrow{y_{dec.shift}}\right| = \begin{bmatrix} 6761.31 \\ 1475.81 \\ 7031.62 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$l_{dec} := \sqrt{x^2 + y^2 + {z_3}^2} = \begin{bmatrix} 7583.85 \\ 4984.19 \\ 7583.62 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$b_{sun.dec} := 2 \cdot l_{dec} \cdot \tan\left(\frac{0.5\ \boldsymbol{deg}}{2}\right) = \begin{bmatrix} 66.18 \\ 43.5 \\ 66.18 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

March - vernal equinox

$$x := \left|\overrightarrow{x_{mar.shift}}\right| = \begin{bmatrix} 5287.79 \\ 2768.39 \\ 1251.44 \end{bmatrix} \boldsymbol{mi} \qquad y := \left|\overrightarrow{y_{mar.shift}}\right| = \begin{bmatrix} 4757.36 \\ 950.15 \\ 7001.72 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$l_{mar} := \sqrt{x^2 + y^2 + {z_4}^2} = \begin{bmatrix} 7806.4 \\ 4348.96 \\ 7806.21 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$b_{sun.mar} := 2 \cdot l_{mar} \cdot \tan\left(\frac{0.5\ \boldsymbol{deg}}{2}\right) = \begin{bmatrix} 68.12 \\ 37.95 \\ 68.12 \end{bmatrix} \boldsymbol{mi} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

Using average sun widths to determine how the angular sizes of the sun would appear.

$$b_{sun.june.avg} := \text{mean}\left(b_{sun.june_1}, b_{sun.june_2}\right) = 51.24\ \boldsymbol{mi}$$

$$b_{sun.sept.avg} := \text{mean}\left(b_{sun.sept_1}, b_{sun.sept_2}\right) = 53.11\ \boldsymbol{mi}$$

$$b_{sun.dec.avg} := \text{mean}\left(b_{sun.dec_1}, b_{sun.dec_2}\right) = 54.84\ \boldsymbol{mi}$$

$$b_{sun.mar.avg} := \text{mean}\left(b_{sun.mar_1}, b_{sun.mar_2}\right) = 53.04\ \boldsymbol{mi}$$

deviation from 0.5° angular size (degrees)

deviation from 1800 arcsecond (0.5°) angular size

$$\Theta_{sun.size.june} := 2 \cdot \text{atan}\left(\frac{b_{sun.june.avg}}{2 \cdot l_{june}}\right) = \begin{bmatrix} 0.381 \\ 0.727 \\ 0.381 \end{bmatrix} \boldsymbol{deg} \qquad \delta_{june.deg} := \Theta_{sun.size.june} - 0.5\ \boldsymbol{deg} = \begin{bmatrix} -0.12 \\ 0.23 \\ -0.12 \end{bmatrix} \boldsymbol{deg} \qquad \delta_{june.arcsec} := \frac{\delta_{june.deg}}{\boldsymbol{deg}} \cdot 60 \cdot 60 = \begin{bmatrix} -429 \\ 818 \\ -429 \end{bmatrix} \text{arcseconds} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$\Theta_{sun.size.sept} := 2 \cdot \text{atan}\left(\frac{b_{sun.sept.avg}}{2 \cdot l_{sept}}\right) = \begin{bmatrix} 0.389 \\ 0.698 \\ 0.389 \end{bmatrix} \boldsymbol{deg} \qquad \delta_{sept.deg} := \Theta_{sun.size.sept} - 0.5\ \boldsymbol{deg} = \begin{bmatrix} -0.11 \\ 0.20 \\ -0.11 \end{bmatrix} \boldsymbol{deg} \qquad \delta_{sept.arcsec} := \frac{\delta_{sept.deg}}{\boldsymbol{deg}} \cdot 60 \cdot 60 = \begin{bmatrix} -398 \\ 714 \\ -398 \end{bmatrix} \text{arcseconds} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$\Theta_{sun.size.dec} := 2 \cdot \text{atan}\left(\frac{b_{sun.dec.avg}}{2 \cdot l_{dec}}\right) = \begin{bmatrix} 0.414 \\ 0.630 \\ 0.414 \end{bmatrix} \boldsymbol{deg} \qquad \delta_{dec.deg} := \Theta_{sun.size.dec} - 0.5\ \boldsymbol{deg} = \begin{bmatrix} -0.09 \\ 0.13 \\ -0.09 \end{bmatrix} \boldsymbol{deg} \qquad \delta_{dec.arcsec} := \frac{\delta_{dec.deg}}{\boldsymbol{deg}} \cdot 60 \cdot 60 = \begin{bmatrix} -309 \\ 469 \\ -308 \end{bmatrix} \text{arcseconds} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

$$\Theta_{sun.size.mar} := 2 \cdot \text{atan}\left(\frac{b_{sun.mar.avg}}{2 \cdot l_{mar}}\right) = \begin{bmatrix} 0.389 \\ 0.699 \\ 0.389 \end{bmatrix} \boldsymbol{deg} \qquad \delta_{mar.deg} := \Theta_{sun.size.mar} - 0.5\ \boldsymbol{deg} = \begin{bmatrix} -0.11 \\ 0.20 \\ -0.11 \end{bmatrix} \boldsymbol{deg} \qquad \delta_{mar.arcsec} := \frac{\delta_{mar.deg}}{\boldsymbol{deg}} \cdot 60 \cdot 60 = \begin{bmatrix} -399 \\ 715 \\ -399 \end{bmatrix} \text{arcseconds} \qquad \begin{bmatrix} \text{“sunrise”} \\ \text{“solar noon”} \\ \text{“sunset”} \end{bmatrix}$$

Calculations for plotting angular sizes to visually see difference.

0.5° sun size

$$x := 1\ \boldsymbol{ft}$$

$$\beta := 0.25\ \boldsymbol{deg}$$

$$\frac{x \cdot \tan(\beta)}{x} = 0.004363$$

biggest deviation (>0.5°)

vernal equinox solar noon

$$\frac{\delta_{mar.deg_2}}{2} = 0.09937\ \boldsymbol{deg}$$

$$\gamma := 0.25\ \boldsymbol{deg} + \frac{\delta_{mar.deg_2}}{2} = 0.35\ \boldsymbol{deg}$$

$$\frac{x \cdot \tan(\gamma)}{x} = 0.006098$$

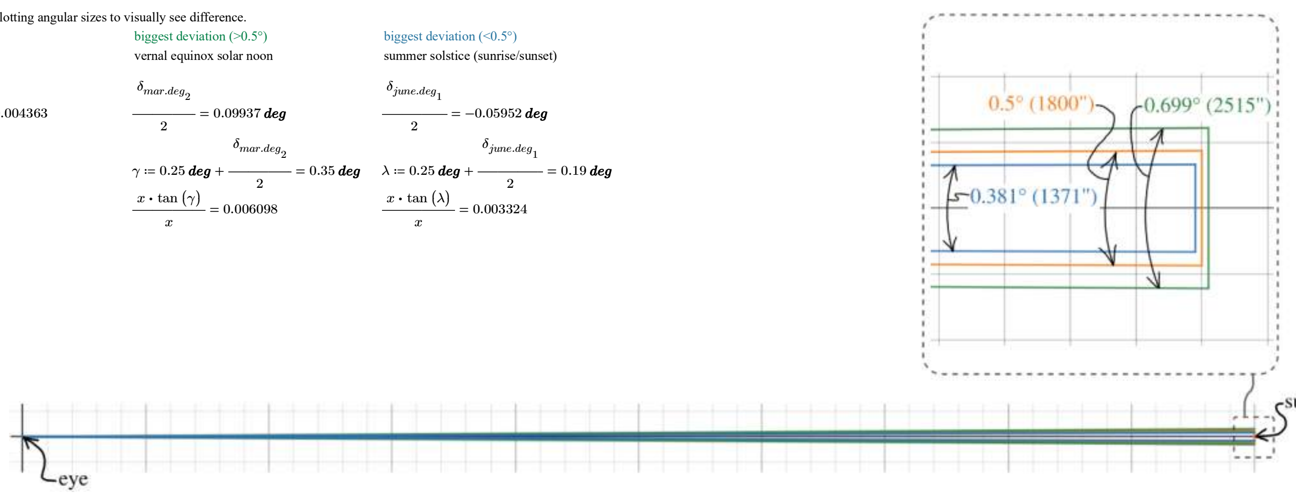
biggest deviation (<0.5°)

summer solstice (sunrise/sunset)

$$\frac{\delta_{june.deg_1}}{2} = -0.05952\ \boldsymbol{deg}$$

$$\lambda := 0.25\ \boldsymbol{deg} + \frac{\delta_{june.deg_1}}{2} = 0.19\ \boldsymbol{deg}$$

$$\frac{x \cdot \tan(\lambda)}{x} = 0.003324$$



Calculations to determine equation for the apparent motion of the sun during the **summer solstice** (June) as observed from Boston.

$x := x_{june.shift} = \begin{bmatrix} 6307.75669 \\ -1236.80766 \\ 3602.92702 \end{bmatrix} mi$ $\quad y := y_{june.shift} = \begin{bmatrix} -2240.24755 \\ -424.48919 \\ 5641.57912 \end{bmatrix} mi$ $\quad z := \begin{bmatrix} 0 \\ z_{sun.june} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3819.26 \\ 0 \end{bmatrix} mi$

$e_1 := \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = \begin{bmatrix} -7545 \\ 1816 \\ 3819 \end{bmatrix} mi$ $\quad e_2 := \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix} = \begin{bmatrix} -2705 \\ 7882 \\ 0 \end{bmatrix} mi$ $\quad n := e_1 \times e_2 = \begin{bmatrix} -30102756 \\ -10330451 \\ -54553631 \end{bmatrix} mi^2$ $\quad \check{n} := \dfrac{n}{\|n\|}$ $\quad u := \dfrac{e_1}{\|e_1\|} = \begin{bmatrix} -0.87231 \\ 0.20994 \\ 0.44159 \end{bmatrix}$ $\quad v := \check{n} \times u = \begin{bmatrix} 0.10911 \\ 0.96394 \\ -0.24274 \end{bmatrix}$ $\quad$ **clear** (h) $\quad$ **clear** (k) $\quad$ **clear** (l)

$O := [h \;\; k \;\; l]$ $\quad A := [x_1 \;\; y_1 \;\; z_1]$ $\quad B := [x_2 \;\; y_2 \;\; z_2]$ $\quad C := [x_3 \;\; y_3 \;\; z_3]$

$|O - A|^2 = |O - B|^2 = |O - C|^2$ --> $(O - A) \cdot (O - A) = (O - B) \cdot (O - B) = (O - C) \cdot (O - C)$ --> $O^2 - 2 \cdot O \cdot A + A^2 = O^2 - 2 \cdot O \cdot B + B^2 = O^2 - 2 \cdot O \cdot C + C^2$

(1) $\quad -2 \cdot O \cdot A + A^2 = -2 \cdot O \cdot B + B^2$ --> $2 \cdot O \cdot B - 2 \cdot O \cdot A = B^2 - A^2$ --> $O \cdot (B - A) = \dfrac{B^2 - A^2}{2}$

$a_1 := (x_2 - x_1)$ $\quad a_2 := (y_2 - y_1)$ $\quad a_3 := (z_2 - z_1)$ $\quad a_4 := \dfrac{x_2{}^2 + y_2{}^2 + z_2{}^2 - (x_1{}^2 + y_1{}^2 + z_1{}^2)}{2}$ --> $a_1 \cdot h + a_2 \cdot k + a_3 \cdot l = a_4$ --> $h = \dfrac{a_4 - a_2 \cdot k - a_3 \cdot l}{a_1}$

(2) $\quad -2 \cdot O \cdot A + A^2 = -2 \cdot O \cdot C + C^2$ --> $2 \cdot O \cdot C - 2 \cdot O \cdot A = C^2 - A^2$ --> $O \cdot (C - A) = \dfrac{C^2 - A^2}{2}$

$b_1 := (x_3 - x_1)$ $\quad b_2 := (y_3 - y_1)$ $\quad b_3 := (z_3 - z_1)$ $\quad b_4 := \dfrac{x_3{}^2 + y_3{}^2 + z_3{}^2 - (x_1{}^2 + y_1{}^2 + z_1{}^2)}{2}$ --> $b_1 \cdot h + b_2 \cdot k + b_3 \cdot l = b_4$ --> $k = \dfrac{b_4 - b_1 \cdot h - b_3 \cdot l}{b_2}$

(3) $\quad n \cdot (O - A) = 0$

$c_4 := n_1 \cdot x_1 + (n_2 \cdot y_1)$ --> $n_1 \cdot h + n_2 \cdot k + n_3 \cdot l = c_4$ --> $l = \dfrac{c_4 - n_1 \cdot h - n_2 \cdot k}{n_3}$

Guess Values

$h := 1\, mi$ $\quad k := 1\, mi$ $\quad l := 1\, mi$

Constraints

$h = \dfrac{a_4 - a_2 \cdot k - a_3 \cdot l}{a_1}$ $\quad k = \dfrac{b_4 - b_1 \cdot h - b_3 \cdot l}{b_2}$ $\quad l = \dfrac{c_4 - n_1 \cdot h - n_2 \cdot k}{n_3}$

Solver

$solve := \textbf{find}\,(h, k, l)$

Center of circle $\quad \begin{bmatrix} h \\ k \\ l \end{bmatrix} := solve = \begin{bmatrix} 2794.86665247769 \\ 959.249152951129 \\ 1332.54971750383 \end{bmatrix} mi$

$a := \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2} = 8649.1\, mi$ $\quad b := \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = 8648.94\, mi$ $\quad c := \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2} = 8333.02\, mi$

$\Delta := \sqrt{\dfrac{a+b+c}{2} \cdot \left(\dfrac{a+b+c}{2} - a\right) \cdot \left(\dfrac{a+b+c}{2} - b\right) \cdot \left(\dfrac{a+b+c}{2} - c\right)} = 31579237.01\, mi^2$ $\quad r_{cir} := \dfrac{a \cdot b \cdot c}{4 \cdot \Delta} = 4934.86\, mi$

$u' := r_{cir} \cdot u = \begin{bmatrix} -4304.73318307496 \\ 1036.02472949782 \\ 2179.1716894451 \end{bmatrix} mi$ $\quad v' := r_{cir} \cdot v = \begin{bmatrix} 538.440559588582 \\ 4756.88980099254 \\ -1197.89208807799 \end{bmatrix} mi$

$x_{circle} = \overrightarrow{h + u'_1 \cdot \cos(t) + v'_1 \cdot \sin(t)}$ $\quad y_{circle} = \overrightarrow{k + u'_2 \cdot \cos(t) + v'_2 \cdot \sin(t)}$ $\quad z_{circle} = \overrightarrow{l + u'_3 \cdot \cos(t) + v'_3 \cdot \sin(t)}$

Calculations to determine equation for the apparent motion of the sun during the **autumnal equinox** (September) as observed from Boston.

$x := x_{sept.shift} = \begin{bmatrix} 5294.99299 \\ -2768.391 \\ 1259.20598 \end{bmatrix} mi$ $\quad y := y_{sept.shift} = \begin{bmatrix} -4754.66525 \\ -950.14941 \\ 7004.15539 \end{bmatrix} mi$ $\quad z := \begin{bmatrix} 0 \\ z_{sun.sept} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3227.93 \\ 0 \end{bmatrix} mi$

$e_1 := \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = \begin{bmatrix} -8063 \\ 3805 \\ 3228 \end{bmatrix} mi$ $\quad e_2 := \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix} = \begin{bmatrix} -4036 \\ 11759 \\ 0 \end{bmatrix} mi$ $\quad n := e_1 \times e_2 = \begin{bmatrix} -37956608 \\ -13027224 \\ -79461670 \end{bmatrix} mi^2$ $\quad \check{n} := \dfrac{n}{\|n\|}$ $\quad u := \dfrac{e_1}{\|e_1\|} = \begin{bmatrix} -0.85037 \\ 0.40123 \\ 0.34042 \end{bmatrix}$ $\quad v := \check{n} \times u = \begin{bmatrix} 0.30833 \\ 0.90421 \\ -0.29552 \end{bmatrix}$ $\quad$ **clear** (h) $\quad$ **clear** (k) $\quad$ **clear** (l)

$O := [h \;\; k \;\; l]$ $\quad A := [x_1 \;\; y_1 \;\; z_1]$ $\quad B := [x_2 \;\; y_2 \;\; z_2]$ $\quad C := [x_3 \;\; y_3 \;\; z_3]$

$|O - A|^2 = |O - B|^2 = |O - C|^2$ --> $(O - A) \cdot (O - A) = (O - B) \cdot (O - B) = (O - C) \cdot (O - C)$ --> $O^2 - 2 \cdot O \cdot A + A^2 = O^2 - 2 \cdot O \cdot B + B^2 = O^2 - 2 \cdot O \cdot C + C^2$

(1) $\quad -2 \cdot O \cdot A + A^2 = -2 \cdot O \cdot B + B^2$ --> $2 \cdot O \cdot B - 2 \cdot O \cdot A = B^2 - A^2$ --> $O \cdot (B - A) = \dfrac{B^2 - A^2}{2}$

$a_1 := (x_2 - x_1)$ $\quad a_2 := (y_2 - y_1)$ $\quad a_3 := (z_2 - z_1)$ $\quad a_4 := \dfrac{x_2{}^2 + y_2{}^2 + z_2{}^2 - (x_1{}^2 + y_1{}^2 + z_1{}^2)}{2}$ --> $a_1 \cdot h + a_2 \cdot k + a_3 \cdot l = a_4$ --> $h = \dfrac{a_4 - a_2 \cdot k - a_3 \cdot l}{a_1}$

(2) $\quad -2 \cdot O \cdot A + A^2 = -2 \cdot O \cdot C + C^2$ --> $2 \cdot O \cdot C - 2 \cdot O \cdot A = C^2 - A^2$ --> $O \cdot (C - A) = \dfrac{C^2 - A^2}{2}$

$b_1 := (x_3 - x_1)$ $\quad b_2 := (y_3 - y_1)$ $\quad b_3 := (z_3 - z_1)$ $\quad b_4 := \dfrac{x_3{}^2 + y_3{}^2 + z_3{}^2 - (x_1{}^2 + y_1{}^2 + z_1{}^2)}{2}$ --> $b_1 \cdot h + b_2 \cdot k + b_3 \cdot l = b_4$ --> $k = \dfrac{b_4 - b_1 \cdot h - b_3 \cdot l}{b_2}$

(3) $\quad n \cdot (O - A) = 0$

$c_4 := n_1 \cdot x_1 + (n_2 \cdot y_1)$ --> $n_1 \cdot h + n_2 \cdot k + n_3 \cdot l = c_4$ --> $l = \dfrac{c_4 - n_1 \cdot h - n_2 \cdot k}{n_3}$

Guess Values

$h := 1\ mi \qquad k := 1\ mi \qquad l := 1\ mi$

Constraints

$h = \dfrac{a_4 - a_2 \cdot k - a_3 \cdot l}{a_1} \qquad k = \dfrac{b_4 - b_1 \cdot h - b_3 \cdot l}{b_2} \qquad l = \dfrac{c_4 - n_1 \cdot h - n_2 \cdot k}{n_3}$

Solver

$solve := \mathbf{find}(h, k, l)$

Center of circle $\begin{bmatrix} h \\ k \\ l \end{bmatrix} := solve = \begin{bmatrix} 2532.31214974388 \\ 869.123937802192 \\ 397.671409236288 \end{bmatrix} mi$

$a := \sqrt{\left(x_2 - x_3\right)^2 + \left(y_2 - y_3\right)^2 + \left(z_2 - z_3\right)^2} = 9482.19\ mi \qquad b := \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2} = 9482.19\ mi \qquad c := \sqrt{\left(x_3 - x_1\right)^2 + \left(y_3 - y_1\right)^2 + \left(z_3 - z_1\right)^2} = 12432.11\ mi$

$\Delta := \sqrt{\dfrac{a+b+c}{2} \cdot \left(\dfrac{a+b+c}{2} - a\right) \cdot \left(\dfrac{a+b+c}{2} - b\right) \cdot \left(\dfrac{a+b+c}{2} - c\right)} = 44510026.13\ mi^2 \qquad r_{cir} := \dfrac{a \cdot b \cdot c}{4 \cdot \Delta} = 6278.34\ mi$

$u' := r_{cir} \cdot u = \begin{bmatrix} -5338.91879077608 \\ 2519.04177609926 \\ 2137.2710365324 \end{bmatrix} mi \qquad v' := r_{cir} \cdot v = \begin{bmatrix} 1935.7948621751 \\ 5676.95649774048 \\ -1855.37488992769 \end{bmatrix} mi$

$x_{circle} = \overrightarrow{h + u'_1 \cdot \cos(t) + v'_1 \cdot \sin(t)} \qquad y_{circle} = \overrightarrow{k + u'_2 \cdot \cos(t) + v'_2 \cdot \sin(t)} \qquad z_{circle} = \overrightarrow{l + u'_3 \cdot \cos(t) + v'_3 \cdot \sin(t)}$

Calculations to determine equation for the apparent motion of the sun during the **winter solstice** (December) as observed from Boston.

$x := x_{dec.shift} = \begin{bmatrix} 2761.34415 \\ -4299.97433 \\ -1973.13789 \end{bmatrix} mi \qquad y := y_{dec.shift} = \begin{bmatrix} -6761.31462 \\ -1475.80962 \\ 7031.61668 \end{bmatrix} mi \qquad z := \begin{bmatrix} 0 \\ z_{sun.dec} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2043.13632 \\ 0 \end{bmatrix} mi$

$e_1 := \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = \begin{bmatrix} -7061 \\ 5286 \\ 2043 \end{bmatrix} mi \qquad e_2 := \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix} = \begin{bmatrix} -4734 \\ 13793 \\ 0 \end{bmatrix} mi \qquad n := e_1 \times e_2 = \begin{bmatrix} -28180839 \\ -9673192 \\ -72372152 \end{bmatrix} mi^2 \qquad \check{n} := \dfrac{n}{\|n\|} \qquad u := \dfrac{e_1}{\|e_1\|} = \begin{bmatrix} -0.77992 \\ 0.58378 \\ 0.22566 \end{bmatrix} \qquad v := \check{n} \times u = \begin{bmatrix} 0.51193 \\ 0.80245 \\ -0.30660 \end{bmatrix}$ **clear** (h) **clear** (k)

$O := [h \quad k \quad l] \qquad A := [x_1 \quad y_1 \quad z_1] \qquad B := [x_2 \quad y_2 \quad z_2] \qquad C := [x_3 \quad y_3 \quad z_3]$

$|O - A|^2 = |O - B|^2 = |O - C|^2$ --> $(O - A) \cdot (O - A) = (O - B) \cdot (O - B) = (O - C) \cdot (O - C)$ --> $O^2 - 2 \cdot O \cdot A + A^2 = O^2 - 2 \cdot O \cdot B + B^2 = O^2 - 2 \cdot O \cdot C + C^2$

(1) $-2 \cdot O \cdot A + A^2 = -2 \cdot O \cdot B + B^2$ --> $2 \cdot O \cdot B - 2 \cdot O \cdot A = B^2 - A^2$ --> $O \cdot (B - A) = \dfrac{B^2 - A^2}{2}$

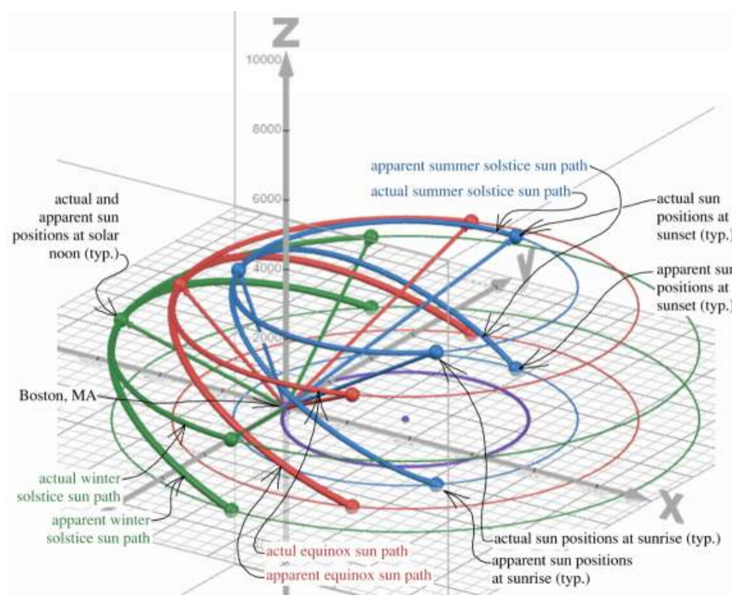
$a_1 := \left(x_2 - x_1\right) \quad a_2 := \left(y_2 - y_1\right) \quad a_3 := \left(z_2 - z_1\right) \quad a_4 := \dfrac{x_2^{\ 2} + y_2^{\ 2} + z_2^{\ 2} - \left(x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2}\right)}{2}$ --> $a_1 \cdot h + a_2 \cdot k + a_3 \cdot l = a_4$ --> $h = \dfrac{a_4 - a_2 \cdot k - a_3 \cdot l}{a_1}$

(2) $-2 \cdot O \cdot A + A^2 = -2 \cdot O \cdot C + C^2$ --> $2 \cdot O \cdot C - 2 \cdot O \cdot A = C^2 - A^2$ --> $O \cdot (C - A) = \dfrac{C^2 - A^2}{2}$

$b_1 := \left(x_3 - x_1\right) \quad b_2 := \left(y_3 - y_1\right) \quad b_3 := \left(z_3 - z_1\right) \quad b_4 := \dfrac{x_3^{\ 2} + y_3^{\ 2} + z_3^{\ 2} - \left(x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2}\right)}{2}$ --> $b_1 \cdot h + b_2 \cdot k + b_3 \cdot l = b_4$ --> $k = \dfrac{b_4 - b_1 \cdot h - b_3 \cdot l}{b_2}$

(3) $n \cdot (O - A) = 0$

$c_4 := n_1 \cdot x_1 + \left(n_2 \cdot y_1\right)$ --> $n_1 \cdot h + n_2 \cdot k + n_3 \cdot l = c_4$ --> $l = \dfrac{c_4 - n_1 \cdot h - n_2 \cdot k}{n_3}$



Guess Values

$h := 1\ mi \qquad k := 1\ mi \qquad l := 1\ mi$

Constraints

$h = \dfrac{a_4 - a_2 \cdot k - a_3 \cdot l}{a_1} \qquad k = \dfrac{b_4 - b_1 \cdot h - b_3 \cdot l}{b_2} \qquad l = \dfrac{c_4 - n_1 \cdot h - n_2 \cdot k}{n_3}$

Solver

$solve := \mathbf{find}(h, k, l)$

Center of circle $\begin{bmatrix} h \\ k \\ l \end{bmatrix} := solve = \begin{bmatrix} 2379.08448022983 \\ 816.504277636962 \\ -863.99669694383 \end{bmatrix} mi$

$a := \sqrt{\left(x_2 - x_3\right)^2 + \left(y_2 - y_3\right)^2 + \left(z_2 - z_3\right)^2} = 9053.45\ mi$

$b := \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2} = 9053.9\ mi$

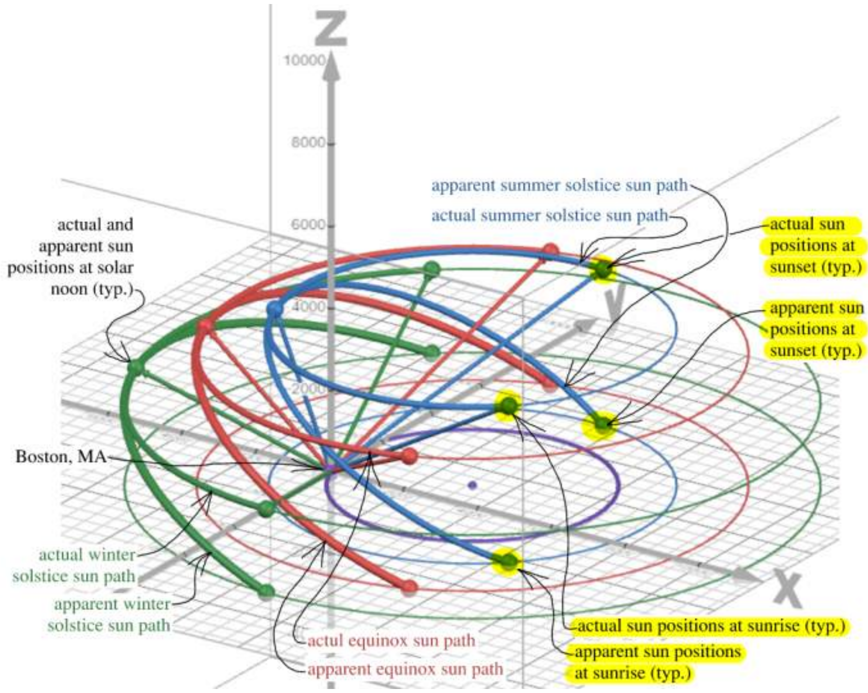
$c := \sqrt{\left(x_3 - x_1\right)^2 + \left(y_3 - y_1\right)^2 + \left(z_3 - z_1\right)^2} = 14582.88\ mi$

$\Delta := \sqrt{\dfrac{a+b+c}{2} \cdot \left(\dfrac{a+b+c}{2} - a\right) \cdot \left(\dfrac{a+b+c}{2} - b\right) \cdot \left(\dfrac{a+b+c}{2} - c\right)} = 39132654.95\ mi^2 \qquad r_{cir} := \dfrac{a \cdot b \cdot c}{4 \cdot \Delta} = 7636.49\ mi$

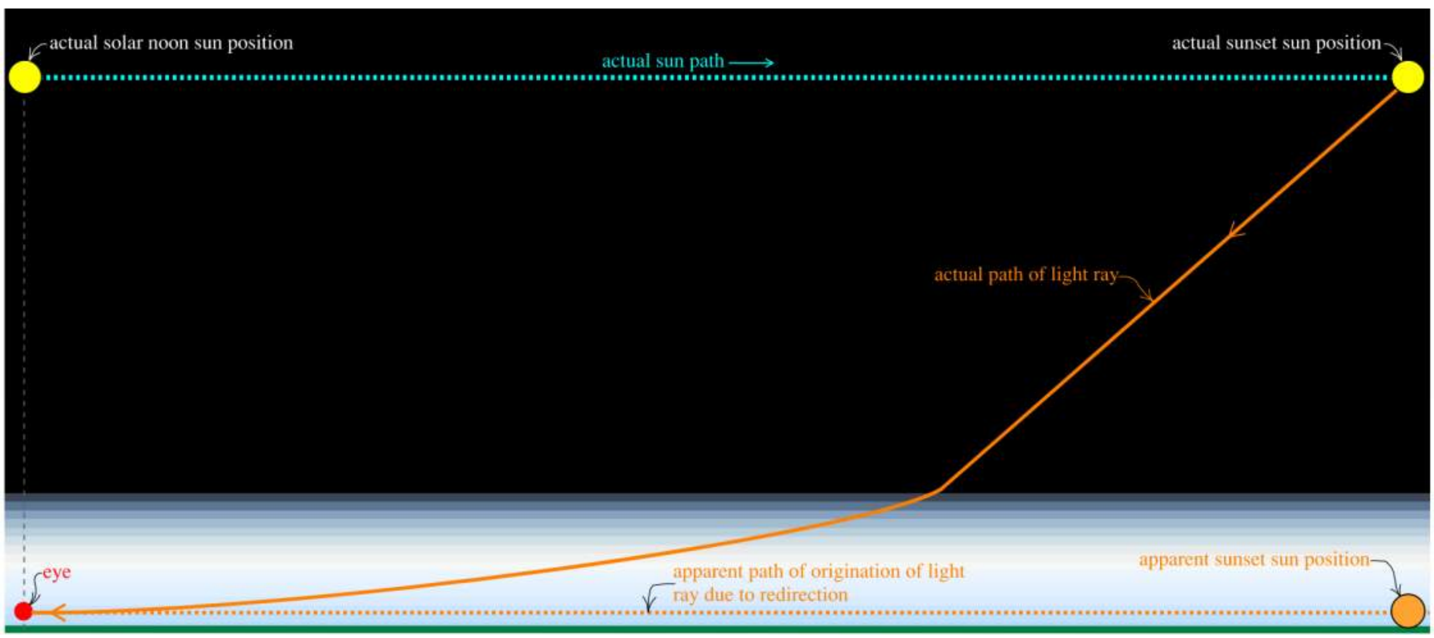
$u' := r_{cir} \cdot u = \begin{bmatrix} -5955.84754344552 \\ 4458.04307656311 \\ 1723.27709482169 \end{bmatrix} mi \qquad v' := r_{cir} \cdot v = \begin{bmatrix} 3909.37673134903 \\ 6127.88602501606 \\ -2341.31125987023 \end{bmatrix} mi$

$x_{circle} = \overrightarrow{h + u'_1 \cdot \cos(t) + v'_1 \cdot \sin(t)} \qquad y_{circle} = \overrightarrow{k + u'_2 \cdot \cos(t) + v'_2 \cdot \sin(t)} \qquad z_{circle} = l + u'_3 \cdot \cos(t) + v'_3 \cdot \sin(t)$

This model (depicted in the diagram on the previous page and shown again below) poses a difficult question to answer. That question is: how is the apparent position of the sun at sunrise or sunset that much lower in our visual perception than the corresponding actual sun positions?



I suppose one possible answer is similar to how mirages are produced. Light rays coming from the sun may be redirected as they travel into the pressure and temperature gradients of the atmosphere.



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