RESIDUE SEPARATION IN PRIMORIAL SIEVE MATRICES

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Abstract. *A constructive refinement of prime-type classification via nested primorial sieve matrices.* We introduce a primorial-based additive representation of integers (Mo numbers) and prove a *residue separation property*: for any prime q, primes in the arithmetic progression $q \bmod p_s\#$ cannot lie outside the nested Mo-type M_{Tq}. The Mo-types form a strictly descending hierarchy $M_{T2} \supset M_{T3} \supset M_{T5} \supset M_{T11} \supset \cdots$, and we prove each deepest-type class contains infinitely many primes (a constructive refinement of Dirichlet's theorem without analytic density estimates).

We further classify twin prime candidates into coupled types such as $M_{T(5,7)}$ and $M_{T(11,13)}$, showing that the Twin Prime Conjecture is equivalent to the divergence of true twin pairs within $M_{T(5,7)}$. As a rigorous corollary, Zhang's bounded-gap theorem admits a Mo-type formulation: some even gap $N \leq 246$ yields infinitely many prime pairs in a coupled Mo-type progression. The framework reframes classical distribution problems as explicit sieve-matrix counting, leaving open the quantitative density of twin pairs for future work.

1. Introduction

The infinitude of primes in arithmetic progressions is a cornerstone of analytic number theory, first established by Dirichlet via L-functions [1]. While the qualitative result is classical, the *distribution of those primes within an explicit constructive sieve* is rarely examined beyond asymptotic density.

In this note, we define a primorial-based additive system called Mo numbers, generated by row matrices $M(k)$ on intervals $(p_{k-1}\# + 1, p_k\# + 1]$. Every integer admits a *Mo-type* M_{Tq} indexed by a small prime q, with a strictly nested hierarchy $M_{T2} \supset M_{T3} \supset M_{T5} \supset M_{T11} \supset \cdots$ (Section 2.4).

The key is a **complement-exclusion principle**: for $s = n(q)$ and the Dirichlet progression $B_q = \{q + kp_s\#\}$, any element failing the Mo coefficient constraints (e.g. $35 \in B_5$ but not in M_{T5}, while $25 \in M_{T7}$ is composite) lies outside the global Mo set M. Since all primes belong to M (Theorem 4.1), the complement $B_q \setminus M_{Tq}$ contains no primes. Thus Dirichlet primes are forced into a unique deepest-type class.

We then extend this partition to twin prime candidates $M_{T(q,q+2)}$, showing the Twin Prime Conjecture is equivalent to divergence of true pairs in $M_{T(5,7)}$, and give a rigorous Mo-type formulation of Zhang's bounded-gap theorem.

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2. Definitions and the Mo Matrix

2.1. **Primorial Notation.** Let p_k denote the k-th prime number, with $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc. Define the *primorial*

$$p_k\# = \prod_{t=1}^{k} p_t$$

with the convention $p_0\# = 1$.

For an integer $n \geq 2$, let $f^p_{min}(n)$ denote the smallest prime factor of n. If $f^p_{min}(n) = n$, then n is prime.

2.2. **The Matrix $M(k)$.** For $k \geq 1$, the Mo matrix $M(k)$ is constructed on $(p_{k-1}\#+1,\ p_k\#+1]$ using a row-wise sieve. The row vector $F(k)$ consists of integers in the matrix from $M(0)$ to $M(k-1)$ whose smallest prime factor is at least p_k. Matrix $M(k)$ rows are translates $F(k) + r \cdot p_{k-1}\#$ for $1 \leq r \leq p_k - 1$.

2.3. **Mo numbers.** Let $n(q) = s$ such that integer $q \in M(s)$. An integer m is a *Mo number* if it belongs to some $M(k)$. Equivalently,

$$m = \sum_{t=n(q)}^{k} r_t\, p_{t-1}\# + q, \qquad 0 \leq r_t \leq p_t - 1. \tag{1}$$

2.4. **Mo-types and their nesting.** For prime q, The *Mo-type* M_{Tq} is

$$M_{Tq} = \left\{ m = \sum_{t=n(q)}^{k} r_t\, p_{t-1}\# + q \;\middle|\; k \geq n(q) \right\}. \tag{2}$$

Nesting Tree. The Mo-types satisfy the strict inclusions (viewing each as the set of all its members):

$$\begin{aligned} M_{T2} \supset M_{T3} \supset M_{T5} &\supset M_{T11} \supset M_{T41} \supset \cdots \\ &\supset M_{T7} \supset M_{T13} \supset M_{T43} \supset \cdots \end{aligned}$$

This follows because each base prime admits a lower-level Mo representation: $3 = 1 \cdot p_0\# + 2$, $5 = 1 \cdot p_1\# + 3$, $7 = 2 \cdot p_1\# + 3$, $11 = 1 \cdot p_2\# + 5$, etc. Substituting extends the summation lower bound, proving inclusion.

3. The Lift Lemma

Lemma 3.1 (Lift Lemma). *If $y \in M(k)$ and $f^p_{min}(y) \geq p_{k+1}$, then $r \cdot p_k\# + y \in M(k+1)$ for any $1 \leq r \leq p_{k+1} - 1$.*

Proof. This is precisely the row construction of $M(k+1)$: its rows are $F(k+1) + r \cdot p_k\#$, and $y \in F(k+1)$ by the hypothesis $f^p_{min}(y) \geq p_{k+1}$; see Figure 1. $\square$

4. Proof of $P \subseteq M$

Theorem 4.1. *Every prime $p \geq 2$ belongs to M.*

Proof. Base cases $2, 3, 5, 7$ are verified directly.

Let $p \in (p_n\# + 1, p_{n+1}\# + 1]$. Write $p = r \cdot p_n\# + t$ with $1 \leq r \leq p_{n+1} - 1$, $1 \leq t \leq p_n\#$.

Case 1: $t = 1$. Then $p = (r-1)p_n\# + (p_n\# + 1)$. The boundary term $p_n\# + 1 \in M(n)$; by Lemma 3.1, $p \in M(n+1)$.

Case 2: $t > 1$. Let $q_0 = f^p_{min}(t)$. If $q_0 \le p_n$ then $q_0 \mid p$, contradicting primality. Thus $q_0 \ge p_{n+1}$. We descend: t belongs to some earlier layer $(p_{k-1}\#, p_k\#]$; iterating $t_j = t_{j-1} \bmod p_{i_j-1}\#$ yields a strictly decreasing sequence terminating at $t_d \in \{3,5,7\} \subset M$. Each t_j lifts to t_{j-1} by Lemma 3.1, terminating at $t_d \in M$. Since $q_0 \ge p_{n+1}$, every t_j in the descent has all prime factors $\ge p_{n+1}$, hence $f^p_{\min}(t_j) \ge p_{i_j+1}$ at each lift step, satisfying Lemma 3.1's hypothesis. Hence $t \in M$ and $p = r \cdot p_n\# + t \in M(n+1)$. ☐

5. Type Uniqueness and Complement Exclusion

Lemma 5.1 (Deepest-Type Residue)**.** *If $m \in M_{Tq}$ with $s = n(q)$, then $m \equiv q \pmod{p_s\#}$, and no other $M_{Tq'}$ ($q' \neq q$) shares this residue.*

Proof. Immediate from the constant term in the primorial expansion defining M_{Tq}. ☐

Lemma 5.2 (Complement Exclusion)**.** *Let $s = n(q)$ be the layer index of q (so modulus is $p_s\#$), and $B_q = \{q + k \cdot p_s\# \mid k \ge 0\}$. Then $(B_q \setminus M_{Tq}) \cap M = \emptyset$.*

Proof. If $x \in B_q \setminus M_{Tq}$ and $x \in M$, then $x \in M_{Tq'}$ for some $q' \neq q$. But $x \in B_q \Rightarrow x \equiv q \pmod{p_s\#}$, while $x \in M_{Tq'} \Rightarrow x \equiv q' \pmod{p_s\#}$, contradiction by Lemma 5.1. ☐

Example 1. For $q = 5$, $B_5 = \{5, 11, 17, 23, 29, 35, 41, \dots\}$. Composite $35 = 5 \times 7$ lies in B_5 but not in M_{T5}, and indeed $35 \notin M$.

For $q = 7$, matrix $M(3)$ explicitly contains $25 = 3 \times 6 + 7$. Here $25 \in M_{T7}$ yet $25 = 5^2$ is composite. Thus M_{Tq} admits composites (25), while its complement admits others (35); only primes are forced into the deepest type.

6. Main Result: Infinitude of Each Deepest Type

Theorem 6.1. *For any prime q, the deepest-type class M_{Tq} contains infinitely many primes.*

Proof. By Dirichlet's theorem, B_q contains infinitely many primes. By Theorem 4.1, these primes lie in M. By Lemma 5.1, they are congruent to $q \pmod{p_s\#}$, hence belong to M_{Tq}. Equivalently, Lemma 5.2 states the complement contains no primes. Thus all primes in B_q lie in M_{Tq}. ☐

Corollary 6.2. *Each level of the nesting chain $M_{T2} \supset M_{T3} \supset M_{T5} \supset M_{T11} \supset \cdots$ contains infinitely many primes. In particular, $M_{T5} \supset M_{T11}, M_{T17}, M_{T23}, M_{T29}$ and each subtype is prime-infinite.*

7. Visualisation: The Twin Prime Tree

The nested sieve generates twin prime pairs $(m, m+2)$ counted by $|M_2(k)| = (p_k - 1)\prod_{t=2}^{k-1}(p_t - 2)$. Empirically $|P_2(k)|$ grows as $1, 3, 10, 55, 398, \dots$ while density decays to 0. This reframes the Twin Prime Conjecture as the unboundedness of $|P_2(k)|$.

8. Twin Prime Reconstruction in Mo-Types

8.1. **Type-classified twin candidates.** For a prime $q \ge 5$ with $q+2$ also prime, define the *twin Mo-type*

$$M_{T(q,q+2)} = \{m \,:\, m \in M_{Tq} \text{ and } m+2 \in M_{T(q+2)}\}$$

The first few layers are

$$\begin{aligned}
M_{T(5,7)} &= \{(5,7), (11,13), (17,19), (29,31), (41,43), \dots\},\\
M_{T(11,13)} &= \{(11,13), (41,43), (71,73), (101,103), (131,133), \dots\},\\
M_{T(17,19)} &= \{(17,19), (47,49), (77,79), (107,109), (137,139), \dots\},
\end{aligned}$$

each nested inside $M_{T(5,7)}$ exactly as in the single-prime hierarchy.

8.2. **Candidate counting.** Let $M_2(k)$ be the set of twin candidates generated by the Mo-matrix at level k:

$$M_2(k) = \big\{(m, m+2) \in M_{T(5,7)} \;:\; m \in (p_{k-1}\#, p_k\#]\big\}$$

A direct sieve count gives

$$|M_2(k)| = (p_k - 1) \prod_{t=2}^{k-1} (p_t - 2)$$

Empirically $|P_2(k)|$ (the subset of $M_2(k)$ consisting of true twin primes) grows as $1, 3, 10, 55, 398, \ldots$ for $k = 2, 3, 4, 5, 6$; the density $|P_2(k)|/|M_2(k)| \to 0$ but the absolute count diverges under the classical conjecture.

8.3. **Reformulation of the Twin Prime Conjecture.**

Theorem 8.1 (Mo-type equivalence)**.** *The Twin Prime Conjecture holds* if and only if

$$\lim_{k\to\infty} |P_2(k)| = \infty,$$

i.e. the deepest twin-type $M_{T(5,7)}$ contains infinitely many true twin prime pairs.

Proof. Necessity is immediate. Sufficiency: $|P_2(k)| \to \infty$ means infinitely many distinct twin prime pairs arise from the Mo-sieve, which is exactly the Twin Prime Conjecture. □

Thus our framework reframes the conjecture as a *divergence problem for a constructive sieve sequence*, analogous to how Euler's divergence of $\sum 1/p$ proves the infinitude of primes.

8.4. **Quantitative prediction (Hardy–Littlewood).** The first Hardy–Littlewood conjecture [3] gives the quantitative asymptotics

$$\pi_2(x) \sim 2C_2 \int_2^x \frac{dt}{(\log t)^2}, \qquad C_2 = \prod_{p\geq 3}\Big(1 - \frac{1}{(p-1)^2}\Big) \approx 0.66016$$

Translated into Mo-types, this predicts

$$|P_2(k)| \;\sim\; \frac{p_k\#}{(\log p_k\#)^2} \cdot 2C_2 \quad \text{as } k \to \infty$$

so that the Twin Prime Conjecture is equivalent to the non-vanishing of this asymptotic.

8.5. **A provable weak form (Zhang's theorem in Mo-types).** While the full conjecture remains open, Zhang's 2013 bounded-gap theorem [4] yields a rigorous statement within our framework:

Theorem 8.2 (Weak twin prime theorem, Mo-type version)**.** *There exists an even integer $N \leq 7 \times 10^7$ (subsequently improved to $N \leq 246$ [5]) such that for infinitely many primes $q \equiv 5 \pmod 6$, the coupled Mo-type progression $(q \bmod p_s\#, q + N \bmod p_s\#)$ contains infinitely many simultaneous prime pairs.*

Proof sketch. Zhang proved that $\liminf_{p\to\infty}(p' - p) < 7 \times 10^7$. In the Mo-hierarchy this means that for some even N in that range, the arithmetic progression pair $(q \bmod p_s\#,\; q+N \bmod p_s\#)$ contains infinitely many simultaneous prime pairs, each of which lies in a coupled Mo-type by the residue uniqueness of Lemma 5.1. The Polymath project subsequently refined this bound to $N \leq 246$ [5]. □

8.6. **Discussion.** The single-prime case (Theorem 6.1) succeeded because Dirichlet's theorem supplies infinitely many primes in *one* arithmetic progression, and the complement-exclusion argument forces them all into a single Mo-type. For twin primes we require *two* coupled linear forms m and $m+2$ to be simultaneously prime; Dirichlet's theorem alone cannot guarantee this. The Mo-type partition therefore reduces the Twin Prime Conjecture to the divergence of $|P_2(k)|$, which is exactly the quantitative challenge addressed (heuristically) by Hardy–Littlewood and (unconditionally only for bounded gaps) by Zhang.

Future work within this framework includes:

- obtaining lower bounds on $|P_2(k)|$ via sieve estimates inside $M_2(k)$;
- extending the type-classification to prime k-tuples $(q, q+a_2, \ldots, q+a_k)$;
- connecting the Mo-matrix rows to the singular series $\mathfrak{S}(\mathcal{H})$ of the Hardy–Littlewood k-tuple conjecture.

9. Conclusion

We introduced a primorial sieve matrix that partitions integers into nested Mo-types M_{Tq}. Our main constructive result is a residue-separation theorem: combined with Dirichlet's theorem and the fact $P \subseteq M$, every arithmetic progression $q \bmod p_s\#$ contributes infinitely many primes to its deepest Mo-type (Theorem 6.1). The proof is purely combinatorial and avoids analytic estimates.

Second, we showed the framework cleanly reframes the Twin Prime Conjecture as the divergence of $|P_2(k)|$ within the coupled type $M_{T(5,7)}$, and naturally hosts Zhang's bounded-gap theorem ($N \leq 246$) as a coupled Mo-progression result.

While quantitative density of twin pairs remains open (Hardy–Littlewood predicts $|P_2(k)| \sim 2C_2 p_k\#/(\log p_k\#)^2$), the Mo-matrix offers a transparent, algorithmic substrate for future sieve refinements and k-tuple generalisations.

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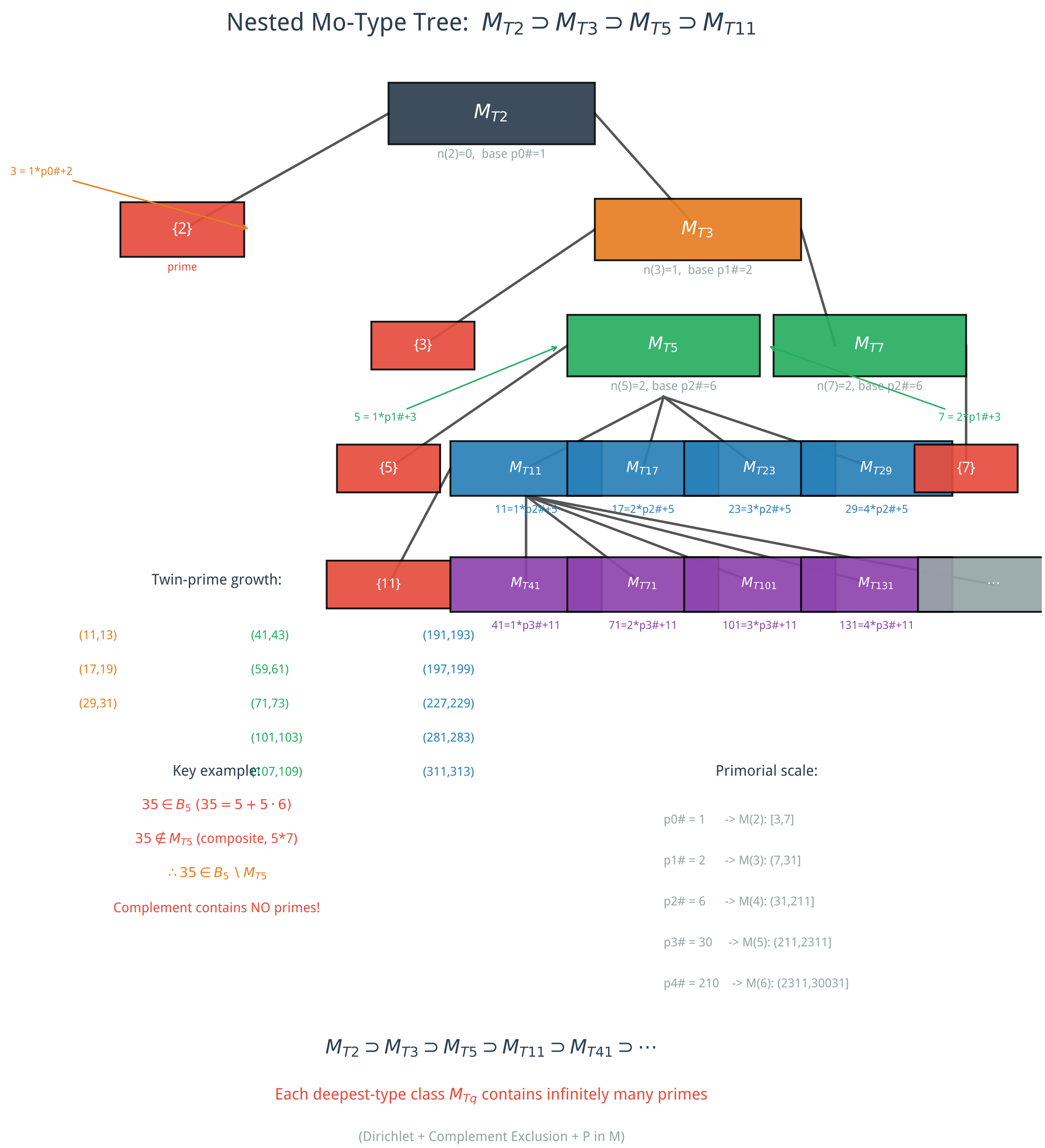

FIGURE 1. The nested Mo-type tree. Arrows show additive decompositions justifying inclusion. Right panel: twin-prime pairs generated at each level.