

**An Ultrafinitist Approach to the Discrete Unit Circle:  
Geometric Rectification and Morphological Adaptation Lead to  
the Determination of the Exact Area Constant  $k = 25/8$  in Quantized Space**

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**Abstract.** This work introduces a discrete geometric model for the construction of a circle, based on the hypothesis of a quantized space and the strict exclusion of the analytical continuum. The circular domain does not emerge as the limit of regular polygons, but rather from the progressive development of elementary circular sectors of increasing angular width associated with a step-by-step elongation of discrete radius; these sectors extend from the vertical axis of the circle, structuring their evolution into a numerical staircase pattern. A rectification procedure for the profile of this staircase is defined by means of a cutting line passing through the midpoints of the individual steps, thereby isolating right-trapezoid-shaped sections equivalent to the circular sectors of the discrete circle. The ordered accumulation of these elements reduces the measurement of the enclosed area to that of a corresponding equivalent isosceles right triangle. Under specific conditions of structural proportionality—and using a geometric equivalence constant ( $\gamma = \frac{5}{2}$ ) derived from the intrinsic properties of the configuration—it is demonstrated that the construction of this specific discrete model is governed by the exact dimensionless constant  $k = \frac{25}{8}$ , which operates independently of the traditional analytical value of  $\pi$ .

**Keywords:** Ultrafinitism, quantized space, discrete circular domain, geometric rectification, area and boundary measurement, dimensionless geometric constant.

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# 1 Introduction

## 1.1 Theoretical framework and objectives

This work falls within the scope of discrete geometry and the study of geometric models that do not rely on the continuity axiom of the real line. The research determines the constructive properties of the discrete unit circle without resorting to the classical tools of infinitesimal calculus, such as limits, infinite series, or integral calculus. This result is achieved by adopting a purely discrete perspective that, moving beyond the concept of the dimensionless point, models circular geometry on the basis of extended primitive elements.

## 1.2 Structural development and constructive geometry

In the proposed model, the circle is not conceived as a static set of classical geometric points equidistant from a reference center, but rather as a structure generated by defined numerical progressions. The modeling is based on a progressive sectoral expansion of the circle, governed by two interdependent variables: the gradual linear increase of the radius and the quadratic progression of the corresponding circular sectors. This approach allows to obtain the final configuration of the circle as the result of a purely quantized evolution, determined exclusively by the laws of growth and geometric adaptation of the primitive elements.

## 1.3 Axiomatics of the quantized number system

The model is grounded in an axiomatic system based on numerical elements possessing a dimensional value expressed as a rational multiple of an area unit of measurement (a “sample square”). These modules adapt their geometric form to conform to the enclosing circular domain. Quantities vary through discrete, step-like transitions—thereby strictly excluding the analytical continuum—and the analysis unfolds entirely within the field of rational numbers,  $\mathbb{Q}$ . Consequently, the proportionality ratio of the discrete circle emerges as an exact fraction, determined uniquely by this constraint of morphological boundary-fitting of numbers.

## 1.4 Structure of the paper

The article presents a formal derivation of the model and its properties. Although the study constitutes an independent theoretical investigation, the rational value  $\frac{25}{8}$  obtained herein finds a remarkable historical parallel a posteriori in the geometric constants documented in ancient Babylonian mathematics.

The text is organized into six sections, structured as follows:

- **Section 1** introduces the theoretical framework, the fundamental postulates, and the research objectives.
- **Section 2** defines the axiomatic system for quantized space, providing a clarifying example and introducing the dimensionless geometric constant  $k$ .
- **Section 3** defines the system of quantized numbers and the structure of the numerical staircase. Furthermore, it describes in detail the method of geometric rectification of the stepped profile, which isolates right-trapezoid-shaped sections equivalent to the circular sectors of the discrete circle.

- **Section 4** presents the model of the isosceles right triangle equivalent to the discrete circle, introducing the geometric equivalence constant  $\gamma$  and the related criterion of linear proportionality.
- **Section 5** provides the determination of the discrete unit circle along with the exact algebraic derivation of  $k = \frac{25}{8}$  and  $\gamma = \frac{5}{2}$ .
- **Section 6** summarizes the main contributions and outlines the conclusions of the work.

## 2 Structural generation of the discrete circle

The objective of this section is to formalize the construction of a discrete and quantized circular domain, whose surface expands through progressively larger circular sectors each time the radius undergoes step-by-step increments. Upon completion of this progression, this constructive geometry determines the comprehensive configuration of the discrete unit circle.

Taking a Cartesian orthogonal coordinate system as a reference, we assume that the horizontal and vertical axes of the circle coincide, respectively, with the abscissa and ordinate axes and that its center is located at the origin. The modeling is based on a progressive sectorial expansion starting from the positive vertical semi-axis, developing within the plane by means of a clockwise or counterclockwise rotation. In this process, at each sequential increment, the radius increases its linear extension by a constant interval starting from the origin, while the angular development of the sector advances progressively within the plane, describing the area of the domain. When the expansion completes its rotation, realigning with the initial vertical axis, the radius reaches its maximum intended extension, and the circular domain closes in a geometrically exact manner. This evolution is described as a succession of gradually larger discrete domains, characterized by an area proportional to the square of the length progressively reached by the radius, which are equivalent to the corresponding circular sectors of the final circle.

We assume that numerical elements are not dimensionless entities, but rather mathematical objects endowed with a dimensional value expressed as a rational multiple of a unit of area measurement (“sample square”). These modules adapt their geometric shape to conform to the enclosing circular domain. Within this paradigm, the sign of the integer defines the direction of the geometric development: positive numbers are mapped onto the counterclockwise rotation of the domain, whereas negative numbers determine its clockwise development. Based on these premises, the following formal definitions are introduced.

**Definition 2.1** (Scalar Development of the Radius and Orientation of Sectors). The discretized development of the radius occurs exclusively along the abscissa axis and is governed by the integer index  $n \in \mathbb{Z}$ , which acts as both a discrete coordinate and a direction operator. The progression of the radius length  $r_n \in \mathbb{Q}$  is defined as:

$$(2.1) \quad r_n = \frac{n}{N} \cdot R, \quad n \in \mathbb{Z}$$

where  $R \in \mathbb{Q}^+$  represents the maximum extension of the radius and  $N \in \mathbb{Z}^+$  defines the discrete partition factor. While the scalar value of  $r_n$  determines the position along the horizontal axis, the corresponding circular sector undergoes an expansion with respect to the ordinate axis. The sign of  $n$  establishes the direction of the linear development and the consequent orientation of the boundary of the associated sector.

- **Positive orientation** ( $n > 0$ ): The radius takes on positive values along the abscissa semi-axis, and the angular development of the sector with respect to the ordinate axis occurs in a counterclockwise direction.
- **Negative orientation** ( $n < 0$ ): The radius takes on negative values along the abscissa semi-axis, and the angular development of the sector with respect to the ordinate axis occurs in a clockwise direction.
- **Null geometric state** ( $n = 0$ ): Identifies the initial condition ( $r_0 = 0$ ) at which both the radius extension and the sectorial opening with respect to the ordinate axis are null.

**Definition 2.2** (Cumulative Growth of the Domain). The growth of the area enclosed by the discrete domain is modeled by the sequence  $\{a_n\}_{n \in \mathbb{Z}}$ , defined by the quadratic progression:

$$(2.2) \quad a_n = k \cdot \frac{n^2}{N^2} R^2, \quad n \in \mathbb{Z}$$

where  $k \in \mathbb{Q}^+$  represents the rational geometric constant of the model and  $N \in \mathbb{Z}^+$  defines the discrete partition factor. The integer index  $n$  maps the discrete states of the sequential geometric progression. The sequence defines an even function with respect to the index  $n$  ( $a_n = a_{-n}$ ), ensuring that the geometric extension of the surface is identical for both directions of rotation and increases monotonically with the absolute value of  $n$ .

**Definition 2.3** (Development of Incremental Sectors and Sequential Evolution of the Domain). The development of individual incremental sectors is expressed by the sequence of first-order finite differences  $\Delta a_n = a_n - a_{n-1}$ , calculated along the sequential geometric progression defined by the integer domain. For  $n \in \mathbb{Z}$ , the generic incremental term takes the unique linear form:

$$(2.3) \quad \Delta a_n = k \cdot \frac{2n-1}{N^2} R^2$$

This formulation governs the entire evolutionary cycle of the circle as a single sequential progression developing from the negative limit value to the positive limit value:

- **Phase in the negative domain** ( $n < 0$ ): The expansion begins from the maximum negative values with a clockwise orientation. Due to the finite difference calculation, the variation  $\Delta a_n$  takes on a negative sign, algebraically mapping the progressive reduction and contraction of the clockwise-developing area toward the origin.
- **Transition at the null state** ( $n = 0$ ): The index uniquely identifies the formal watershed of the progression, at which the variation takes on the limit value  $-k \frac{1}{N^2} R^2$ . This state exhausts the inverse sequence and serves as a geometric node for the immediate polarity inversion of the boundary, without establishing an intermediate static structural state.
- **Phase in the positive domain** ( $n > 0$ ): Once the transition is passed, the expansion proceeds smoothly toward the positive limits, and the variation  $\Delta a_n$  takes on positive values. The boundary adopts a counterclockwise orientation, generating the progressive sequential increment of the geometric sections.

For analytical completeness, we introduce the measurements of the oriented linear development of the domain boundary and the geometric area-boundary relationship.

**Definition 2.4** (Discrete Scalar Boundary Expansion). The cumulative extension of the discretized boundary  $c_n$  and its sequential partial increment  $\Delta c_n$  are expressed in scalar form by the sequences:

$$(2.4) \quad c_n = 2k \cdot \frac{n}{N}R, \quad n \in \mathbb{Z}$$

$$(2.5) \quad \Delta c_n = c_n - c_{n-1} = 2k \cdot \frac{1}{N}R, \quad n \in \mathbb{Z}$$

Within this formulation, the sign of the index  $n$  defines the algebraic polarity of the linear development of the boundary, determining its alignment with the direction of rotation of the domain boundary.

**Definition 2.5** (Geometric Area-Boundary Ratio). The relationship between the cumulative area  $a_n$  and the development of the scalar boundary  $c_n$  is expressed by the sequence  $\{\rho_n\}_{n \in \mathbb{Z} \setminus \{0\}}$ . Preserving the consistency of the algebraic sign as an indicator of the direction of the domain's development, the ratio reduces linearly as a function of the oriented scalar radius  $r_n$ :

$$(2.6) \quad \rho_n = \frac{a_n}{c_n} = \frac{k \cdot \frac{n^2}{N^2}R^2}{2k \cdot \frac{n}{N}R} = \frac{1}{2} \cdot \frac{n}{N}R = \frac{1}{2}r_n, \quad n \in \mathbb{Z} \setminus \{0\}$$

The geometric ratio thus reflects the sequential development of the model, taking on a negative value during clockwise rotation and a positive value during the counterclockwise rotation phase.

Fig. 1 illustrates the growth of a circle of radius  $R \in \mathbb{Q}^+$  in the case of a counterclockwise rotation with a total number of steps  $N = 5$ . The development is depicted by showing that, at each step of the sequence, the radius is incremented by a fraction equal to  $\frac{1}{5}R$ , generating a partial domain equivalent to the sum of one or more circular sectors of the final circle. The process is structured as follows.

- i) **Step  $n = 1$ :** The initial domain is characterized by a radius  $r_1 = \frac{1}{5}R$  and a cumulative area  $a_1 = \frac{1}{25}kR^2$ , corresponding to the first elementary circular sector  $\Delta a_1$ .
- ii) **Step  $n = 2$ :** The radial increment determines a radius  $r_2 = \frac{2}{5}R$  and a total area  $a_2 = \frac{4}{25}kR^2$ . This configuration is equivalent to the sum of the first two sectors, where  $\Delta a_1 = \frac{1}{25}kR^2$  and the second element provides a partial contribution equal to  $\Delta a_2 = \frac{3}{25}kR^2$ .
- iii) **Step  $n = 3$ :** In the third interval, a radius  $r_3 = \frac{3}{5}R$  and a total area  $a_3 = \frac{9}{25}kR^2$  are established; geometrically, this corresponds to the composition of the preceding sectors plus the third partial contribution  $\Delta a_3 = \frac{5}{25}kR^2$ .
- iv) **Step  $n = 4$ :** The fourth evolutionary phase features a radius  $r_4 = \frac{4}{5}R$  and an area  $a_4 = \frac{16}{25}kR^2$ , consisting of the union of the first three incremental sectors and the fourth element of the area  $\Delta a_4 = \frac{7}{25}kR^2$ .
- v) **Step  $n = 5$ :** The boundary closure condition establishes the maximum radius  $r_5 = R$  and the final total area  $a_5 = kR^2$ . The final area increment is equal to  $\Delta a_5 = \frac{9}{25}kR^2$ .

The entire domain is geometrically comprehensive and equivalent to the total sum of the five analyzed sectors.

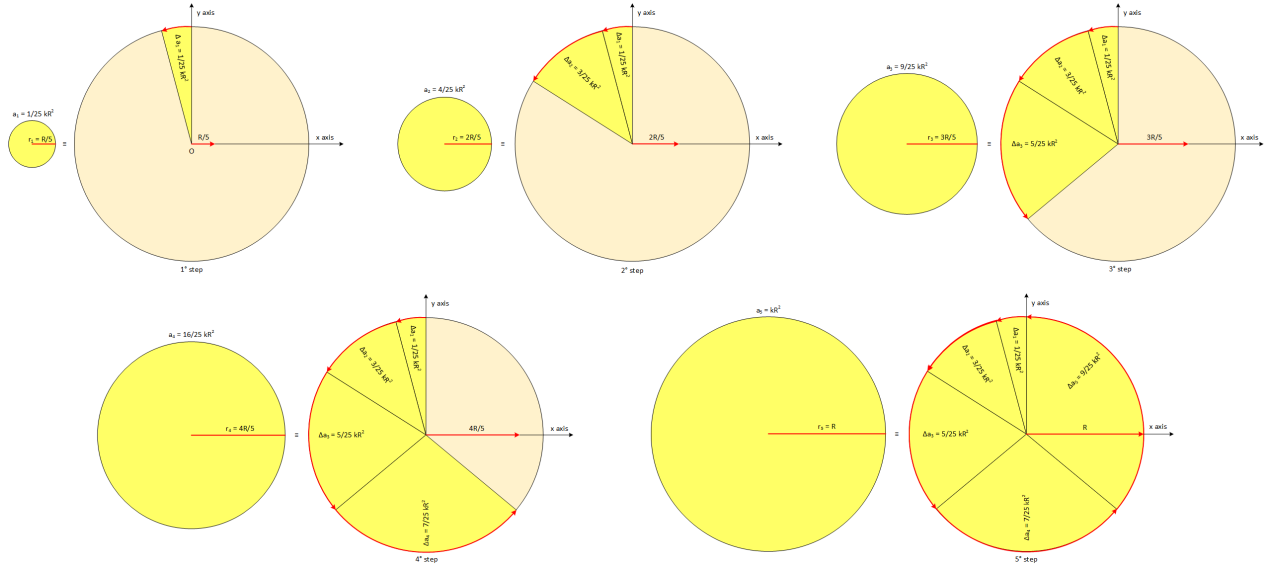


Figure 1: The growth of the circle

### 3 The numerical staircase framework and the corresponding geometric rectification

This section introduces a model based on the properties of discrete space to determine the geometric constant  $k$ , moving beyond the assumptions of continuous differential calculus. The formalization rests on two fundamental postulates:

- i) Numbers are not zero-dimensional point-like entities, but possess a defined geometric extent, calibrated against a *sample square* adopted as the fundamental unit of area;
- ii) Numerical elements are not distributed along a continuous real line oriented from  $-\infty$  to  $+\infty$ ; instead, they are structured in an ordered and discrete manner along a defined two-dimensional *numerical staircase*.

Under these assumptions, the magnitude associated with a specific numerical element is defined not by the cardinality of the infinitesimal points intercepted along a counting direction starting from the origin but rather by the net area occupied by the element within the stepped structure.

To determine the geometric constant  $k$ , it is necessary to relate the morphology of the stepped configuration to the symmetry of the reference geometric domain. Therefore, a method is proposed to model and decompose the stepped structure into subdomains—or areal sections—that are geometrically equivalent to the circular sectors described by the sequence (2.3). This areal quantization allows the calculation of the discrete circle's total area to be reduced to the discrete summation of these equivalent sections.

Based on these premises, the following formal definitions are introduced.

**Definition 3.1** (Sample square). The *sample square*, denoted by  $Q_o$ , is defined as the elementary surface unit derived from the reference *unit segment*  $u$  chosen as the fundamental basis for spatial quantization.

$$(3.1) \quad Q_o = u \times u = 1 u^2$$

**Definition 3.2** (Dimensional value). The *dimensional value* (denoted by  $D.V.$ ) of a quantized quantity is defined by the scaling operator that associates the numerical scalar component  $p \in \mathbb{Q}$  with the unit of area  $Q_o$ :

$$(3.2) \quad D.V. = p \cdot Q_o, \quad p \in \mathbb{Q}$$

**Definition 3.3** (Quantized numbers). A *system of quantized numbers* is defined as a discrete algebraic-geometric structure whose elements exhibit the following topological and dimensional properties:

- i) **Separability**: There exists a strict linear ordering such that every element possesses an immediate successor and predecessor within the discrete domain;
- ii) **Spatial contiguity**: The geometric tessellation is space-filling within the discrete architecture, precluding topological voids or metric gaps between adjacent elements;
- iii) **Internal disjunction**: The elements constitute a proper partition of the space; the intersection of their internal domains is empty, and sharing occurs exclusively along the geometric boundaries modeled on the linear dimensions  $u$  of the reference square  $Q_o$ ;
- iv) **Dimensional invariance**: Each element is intrinsically mapped to a specific and unique dimensional value expressed by the relation (3.2);
- v) **Transitional discontinuity (Quantization)**: Transitions between adjacent numerical states are governed by step functions, analytically excluding the existence of continuous or infinitesimal intermediate values.

**Definition 3.4** (Numerical staircase). A *numerical staircase* is defined as the geometric representation—within an orthonormal Cartesian coordinate system—of an ordered distribution of quantized numbers. This structure takes the form of a step function characterized by constant unit rise and run, calibrated to the unit segment  $u$ .

Fig. 2 illustrates the numerical staircase as a discrete structure in quadrants I and III of the Cartesian plane. The adjacent columns of squares represent integer values. The two parts of the structure are oppositely oriented due to central symmetry with respect to the origin, thereby extending the assignment of negative values to the third quadrant.

The analytical transition between the discrete structure of the stepped arrangement and the continuous model is illustrated through the sequence of panels in Fig. 2. The first panel shows the staircase at the beginning of the transformation. The second panel highlights how the geometry of the discrete cells can be reshaped to generate an area sequence equivalent to the growth dynamics of the circular domain while preserving the invariance of the overall dimensional value.

Consider the partial sum of the first quantized integers  $n$ . Assuming the initial term is zero for  $n = 0$ , the summation corresponds to the well-known Gauss identity:

$$(3.3) \quad \sum_{i=1}^n i = \frac{n^2}{2} + \frac{n}{2}$$

If an infinitesimal rescaling process is applied—reducing the partition step size  $h \rightarrow 0$ —the interval of discrete variation continuously shrinks. In the limit, the step function converges asymptotically to the ideal line defined by equation  $y = x$ , which corresponds to the bisector of the first

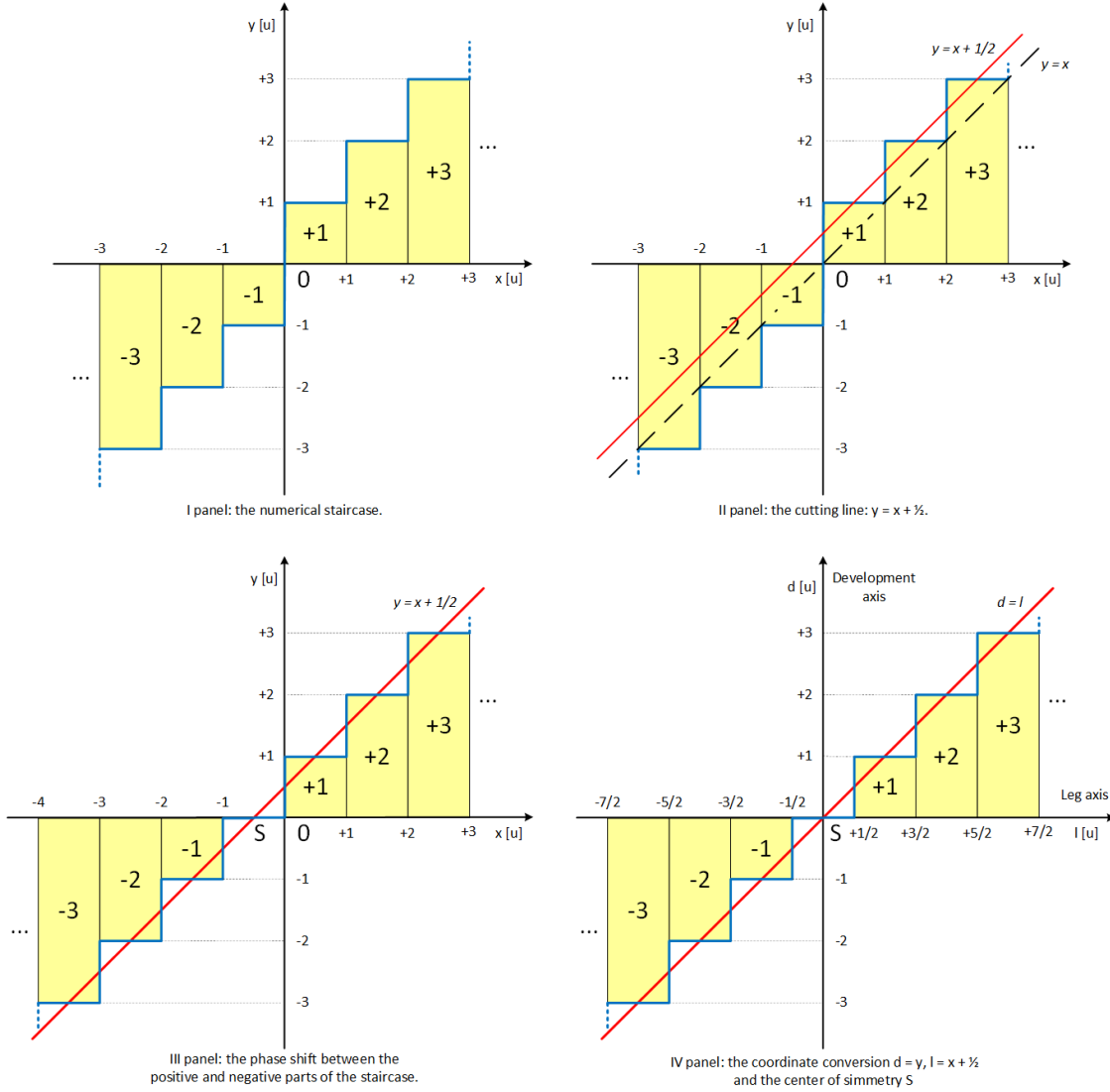


Figure 2: The numerical staircase

and third quadrants (indicated by a dashed black line in the second panel). Assuming a continuous domain, applying the classical Riemann integration operator over the interval  $[0, n]$  yields the following result.

$$(3.4) \quad \int_0^n t \, dt = \frac{n^2}{2}$$

Imposing the condition that the integral over  $[0, n]$  vanishes when  $n = 0$  reveals a structural discrepancy between the discrete formulation (3.3) and the continuous model. This asymmetry lies in the linear term  $\frac{n}{2}$ , which is intrinsic to the summation operator but absent from the corresponding continuous integration.

To rectify this discrepancy and determine a continuous function whose integral precisely reflects the results of the discrete summation, a linear shift of  $\frac{1}{2}$  is applied along the horizontal axis. This

defines the rectification line described by the equation  $y = x + \frac{1}{2}$ , called the *cutting line* (represented as a solid red line). However, the simple analytic continuation of this function in the third quadrant fails to preserve the dynamic symmetry between the positive and negative branches of the staircase function.

To restore the invariance of the system's central symmetry, an element of controlled discontinuity is introduced: a null step of unit width and zero height interposed between the two half-planes. This configuration dictates that the positive portion of the stepped profile extends from the origin  $O(0,0)$ , while the negative portion branches from the point  $(-1,0)$ . The midpoint of this transition interval—denoted by  $S$  in the third panel of Fig. 2—constitutes the exact center of symmetry of the model. Within this geometric framework, the area subtended by the line  $y = x + \frac{1}{2}$  across the discrete definition intervals corresponds exactly to the sum of the corresponding quantized values.

Finally, as illustrated in the fourth panel, a rigid roto-translation of the reference system is performed to align the new Cartesian origin with the center of symmetry  $S(-\frac{1}{2},0)$ . By defining the new coordinate variables  $d = y$  and  $l = x + \frac{1}{2}$ , the equation of the cutting line takes the canonical linear form:

$$(3.5) \quad d = l$$

The numerical staircase is thus centered with respect to the origin of the new reference system defined by the axes  $l$  and  $d$ , which are identified, respectively, as the *leg axis* and the *axis of discrete circle's development*.

Fig. 3 illustrates the geometric decomposition process of the discrete domains performed by the cutting line  $d = l$ . This line intersects the boundary of the step function exactly at the midpoint of each element's riser, identifying the geometric points at half-integer coordinates.

Note how the condition requiring the line to pass through the midpoint of the steps constitutes a rigid and essential geometric constraint: any asymmetric configuration would result in a lack of congruence between the original domain and the transposed one. Such an asymmetry would entail a loss in the measure of the original domain, violating the principle of invariance of the total dimensional value during the rectification process.

The operation, formally defined as the *geometric rectification of the stepped profile*, transforms the original discrete domains into contiguous sub-regions with a trapezoidal geometry, termed *area sections*. These sections exhibit progressively increasing widths and are bounded at the top by an oblique side with a constant slope of  $45^\circ$ . Under the action of the discrete accumulation operator, the ordered sequence of area sections generates a succession of stable isosceles right triangles within the reference half-plane.

It is essential to reiterate that the numerical values associated with the structure in Fig. 3 do not belong to a purely abstract domain but rather identify a system of quantized numbers. Consequently, their dimensional value is intrinsically linked to the metric of the reference square  $Q_o$ . The linear unit  $u$  used to parameterize the coordinate axes corresponds exactly to the side length of that square. To ensure formal rigor and preserve the dimensional consistency of the geometric relationships, the physical units  $u$  and  $u^2$  will be explicitly included in the subsequent definitions, although normalized to unit values.

**Definition 3.5** (Scalar expansion of the legs). The expansion of the legs associated with isosceles right triangles is governed by the integer index  $m \in \mathbb{Z}$ , which acts as both a discrete coordinate and a direction operator. As a result of the process of rectifying the stepped structure along the cut line

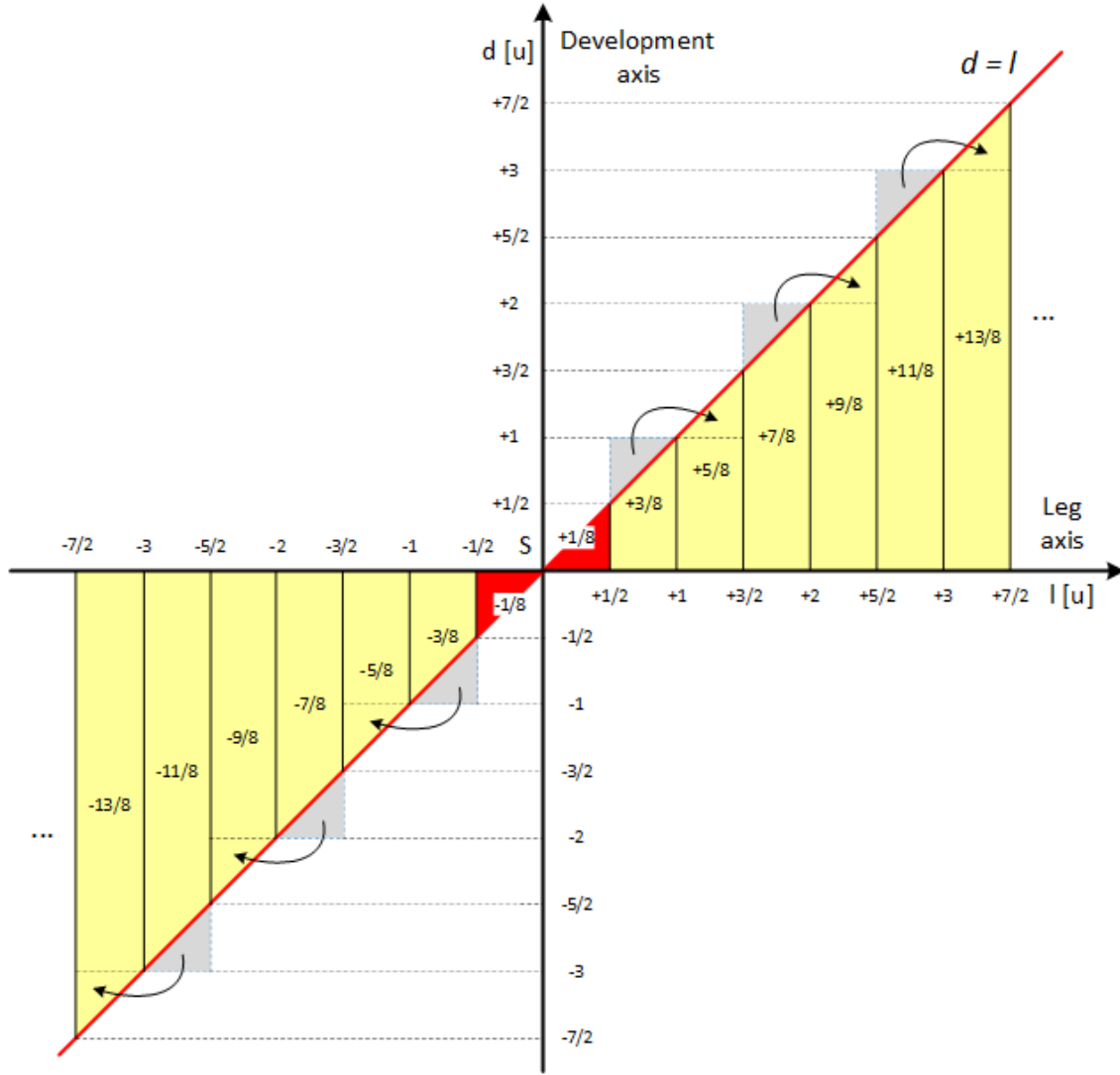


Figure 3: The rectified staircase

$d = l$ , the extent of the legs propagates identically along the coordinate axes, generating the scalar sequence:

$$(3.6) \quad l_m = d_m = \frac{m}{2} \cdot u, \quad m \in \mathbb{Z}$$

The sign of  $m$  determines the direction of the linear expansion along both axes, projecting the geometric progression into the first quadrant for  $m > 0$  and into the third quadrant for  $m < 0$ , with the limiting case  $m = 0$  identifying the null state at the origin of the system.

**Definition 3.6** (Cumulative growth of triangles). The growth of the area of the isosceles right triangles generated by the rectification process is modeled by the symmetric sequence  $\{A_m\}_{m \in \mathbb{Z}}$ . The quadratic progression, determined by the linear extension of the legs, is defined as:

$$(3.7) \quad A_m = \frac{m^2}{8} u^2, \quad m \in \mathbb{Z}$$

The integer index  $m$  represents the discrete states of the stepped structure, generating an area function that is purely positive and symmetric about the origin, calibrated to the metric of the unit segment  $u$ .

**Definition 3.7** (Incremental development of area sections). The development of the individual incremental trapezoidal area sections is expressed by the sequence of first-order finite differences  $\Delta A_m = A_m - A_{m-1}$ , calculated sequentially along the domain of integers. For  $m \in \mathbb{Z}$ , the generic incremental term takes the unique linear form:

$$(3.8) \quad \Delta A_m = \frac{2m-1}{8} u^2$$

This formulation governs the sequential behavior of the system: as a result of the finite difference calculation, the variation  $\Delta A_m$  automatically takes a negative sign in the third quadrant ( $m \leq 0$ ), describing the progressive contraction of the area magnitude as the null state is approached, and a positive sign in the first quadrant ( $m \geq 1$ ), describing the progressive increase of the geometric sections.

The generic  $m$ -th term of the model is analytically described by the configuration of two adjacent and symmetric area sections, whose combination via algebraic summation maps the extent of the quantized domain while remaining strictly within the field of rational numbers  $\mathbb{Q}$ .

As they are directly generated by the linear rectification of a discrete structure, the area sections intrinsically inherit the properties of quantized numerical systems. From a measurement perspective, each section considered in isolation possesses a rational dimensional value; however, the ordered combination of specific adjacent sections allows for mapping the progression of the domain while preserving the system's overall algebraic coherence.

As highlighted by the geometric analysis in Fig. 3, the generic  $m$ -th term of the model can be analytically described by an ordered pair of sections of the following type:

$$(3.9) \quad \left( \frac{-1+4m}{8}, \frac{1+4m}{8} \right), \quad m \in \mathbb{Z}$$

From a geometric perspective, this paired structure reflects the dual nature of each discrete step resulting from the rectification process. While classical numbers are single coordinates on a continuous line, each quantized number  $m$  is represented here by an ordered pair in which the first term,  $\frac{-1+4m}{8}$ , and the second term,  $\frac{1+4m}{8}$ , correspond respectively to the rational area coefficients of the inner and outer boundaries of the  $m$ -th trapezoidal section, calibrated to the unit of area  $u^2$ . The algebraic sum of these boundaries yields exactly the integer value of the index  $m$ , thereby revealing the nature of the quantized number as the area resultant of the elementary domain.

Visually, as illustrated in Fig. 3, these fractions subdivide the continuous space into localized areal domains. The algebraic sum of the pair reduces to  $m$ , restoring the total discrete value of the integer as a pure scalar component, while their arithmetic mean yields the coefficient  $\frac{m}{2}$  which, calibrated to the linear unit  $u$ , precisely defines the scalar extent of the equivalent legs  $l_m = \frac{m}{2} \cdot u$  introduced in (3.6). Consequently, the numbers in this frame of reference are not static points but rather dynamically delimited spatial intervals capable of generating continuous geometric growth through discrete combinations.

This paired structure defines a geometric model that differs from the classical topological representation of numbers along a continuous real line. This formalism lends itself to a curvilinear

mapping in which the discrete coordinates (3.9) can be integrated and projected along discrete geometric trajectories—such as the radial expansion of spiral structures like the Archimedean spiral—consistent with the structural development of the domain.

## 4 The general model of the isosceles right triangle equivalent to the discrete circle

The linear rectification of the stepped profile described in the previous section allows the trapezoidal areal sections to be mapped onto an ordered sequence of stable isosceles right triangles. By stipulating that these areal sections are geometrically equivalent to the discrete circular sectors defined by the relation (2.3), it follows by transitivity that the corresponding isosceles right triangles are equivalent to the discrete circular domains.

Fig. 4 represents the isosceles right triangle equivalent to a generic discrete circle of radius  $R$  subdivided into  $N$  equal partitions. To describe the growth dynamics of this model, an additional geometric axis—termed the *radius axis*—is introduced, defined according to the following constraints:

- i) it runs parallel to the leg axis  $l$ ;
- ii) its origin is fixed at the center of symmetry  $S$ , from which the cutting line  $d = l$  originates;
- iii) the segments defined upon it project into alignment with the upper endpoints of the intervals on the leg axis that delimit the individual areal sections.

The incremental progression (2.1) operates along the radius axis, governed by a constant quantization step size.

$$(4.1) \quad \Delta r = \frac{R}{N}$$

For the model to be consistent, the area increments of the trapezoidal sections must be proportional to the corresponding sectors of the discrete circle. This equivalence is imposed analytically by establishing a one-to-one correspondence between the respective evolutionary states, setting  $m = n$  over the entire domain of integers  $\mathbb{Z}$ .

Equating the general terms of the progressions (3.8) and (2.3) for the same step order yields the area balance equation.

$$(4.2) \quad \frac{2n-1}{8} u^2 = k \frac{2n-1}{N^2} R^2, \quad n \in \mathbb{Z}, \quad N \in \mathbb{Z}^+$$

The cancelation of the common linear factor  $(2n-1)$  from both sides of (4.2) isolates the geometric coefficients as  $\frac{1}{8} u^2 = k \frac{R^2}{N^2}$ . Multiplying by 2 and taking the square root yields the equivalent linear form:

$$(4.3) \quad \Delta l = \frac{1}{2} u = \sqrt{2k} \frac{R}{N}$$

The first term of expression (4.3) represents the constant partial increment of the leg  $\Delta l = l_n - l_{n-1}$  associated with the generation of equivalent triangles, while the second term describes the quantization step  $\Delta r$  weighted by the invariant scale factor  $\sqrt{2k}$ . This factor ensures the geometric proportionality between the triangle's legs and the radius of the discrete circle. This structural relationship is formalized in the following criterion.

*Criterion 1* (Geometric equivalence criterion). Let  $l$  be the leg length of an isosceles right triangle and let  $r$  be the radius of a discrete circular domain. Given the dimensionless geometric constant  $k$  associated with the area of the discrete unit circle, we define the dimensionless geometric equivalence factor  $\gamma$  as:

$$(4.4) \quad \gamma = \sqrt{2k}$$

If the following linear proportionality relationship holds:

$$(4.5) \quad \frac{l}{r} = \gamma$$

then the isosceles right triangle and the discrete circular domain are geometrically equivalent for the purposes of measuring area and discrete perimeter.

By virtue of Criterion 1, the discrete equations governing the expansion of the discrete circle can be fully reformulated in terms of the equivalence constant  $\gamma$  via the following linear system of oriented scalar components:

$$(4.6) \quad \begin{cases} a_n = \frac{1}{2}\gamma^2 r_n^2 = \frac{1}{2}l_n^2 \\ c_n = \gamma^2 r_n = \gamma l_n \end{cases}$$

where  $a_n$  and  $c_n$  denote, respectively, the pure scalar component of the cumulative area and the oriented linear expansion of the boundary at the  $n$ -th step, preserving the sign behavior across the entire domain  $\mathbb{Z}$ .

While the first equation of the system (4.6) expresses the equivalence mapping the area of the discrete circle onto that of an isosceles right triangle with leg length  $l_n$ , the second establishes a geometric decomposition of the boundary. Under ordinary conditions, the numerical value of  $c_n$  corresponds to the area of an ideal rectangle with dimensions defined by the constant  $\gamma$  and the variable leg  $l_n$ . However, when considering the final configuration of the unit circle, the equivalent leg takes on a length of  $l_n = \gamma$ . As a result of this transition, the geometric configuration of the boundary stabilizes into the area of a perfect square with side length  $\gamma$ , yielding an exact numerical quadrature of the boundary equal to  $\gamma^2$ .

The introduction of the constant  $\gamma$  allows the linear expansion of the legs (3.6) to be rescaled into the parameterized form:

$$(4.7) \quad l_n = \gamma \frac{n}{N} R, \quad n \in \mathbb{Z}$$

whose constant incremental rate along the expansion axis takes the form:

$$(4.8) \quad \Delta l = \gamma \frac{R}{N}$$

The adoption of a discrete metric implies the existence of a minimum threshold of spatial divisibility, below which geometric quantities cannot be further subdivided. This minimum threshold is defined as follows.

**Definition 4.1** (Elementary area quantum). The *elementary area quantum* ( $\Delta A_q$ ) is defined as the minimum indivisible geometric surface area generated by the fundamental unit increment of the model's step size. To preserve dimensional consistency, this quantity is expressed by the relation:

$$(4.9) \quad \Delta A_q = \frac{1}{2}(\Delta l)^2 = \frac{1}{2} \left( \gamma \frac{R}{N} \right)^2 = k \frac{R^2}{N^2}$$

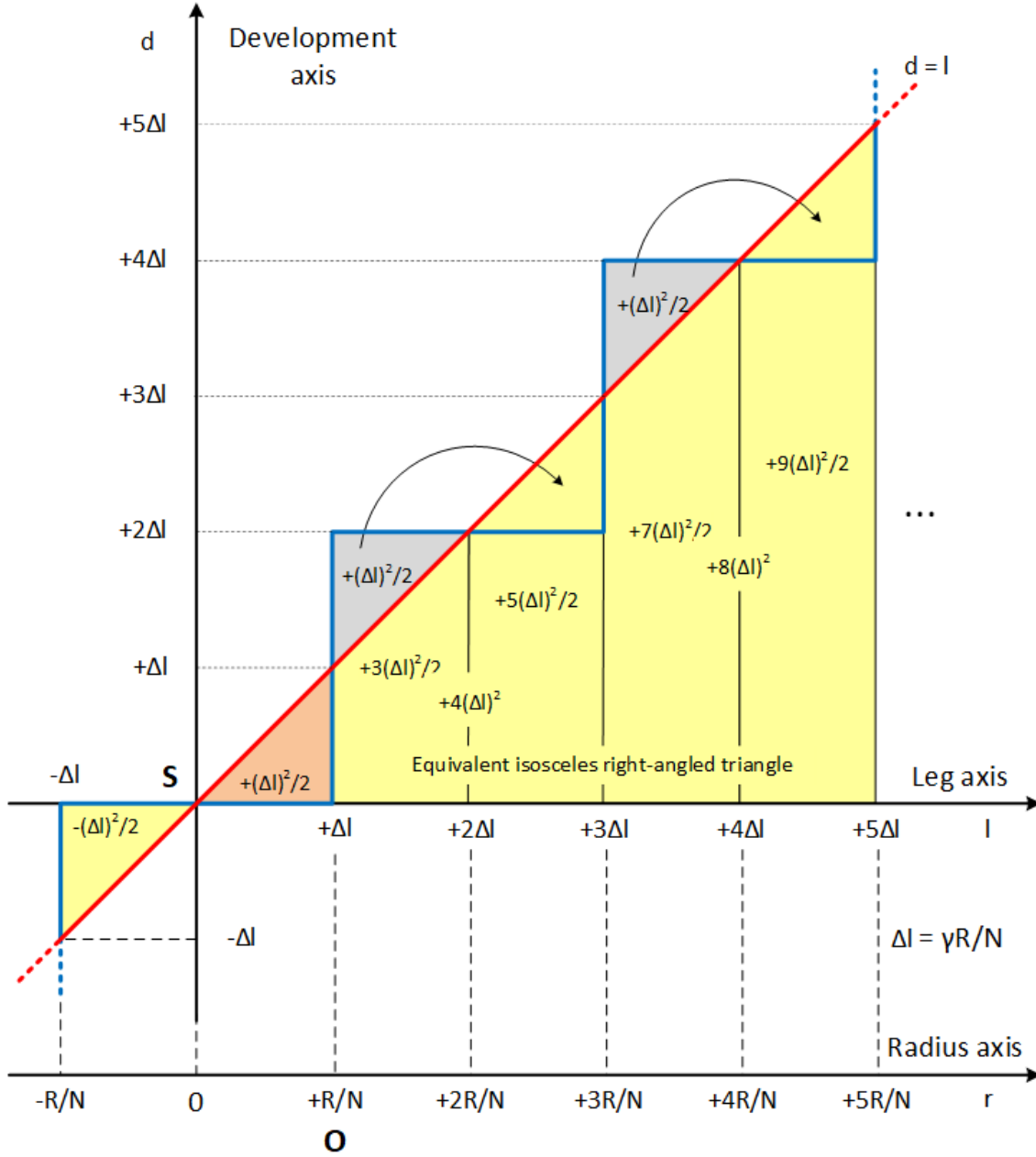


Figure 4: Generalized equivalent isosceles right triangle

where  $k \in \mathbb{Q}^+$  is the dimensionless geometric constant and the magnitude of the interval is strictly determined by the quantization parameter  $N$ .

It should be noted that the value of the elementary area quantum  $\Delta A_q$  coincides exactly with the first elementary circular sector  $\Delta a_1 = k \frac{1}{N^2} R^2$  derived from (2.3) by setting  $n = 1$ . This perfect geometric correspondence demonstrates that the model's fundamental quantum is not a numerical abstraction, but rather identifies the initial structural state from which the domain's entire sectoral expansion originates.

Due to the rectification process, the geometric origin of the discrete circle is shifted along the leg axis relative to the center of symmetry  $S$  by a discrete increment  $\Delta l = \gamma \frac{R}{N}$ . This spatial offset persists through the transition to the continuous limit, retaining a finite discrete magnitude for any quantization parameter  $N$ —a result fully consistent with the axiomatic framework adopted for the quantized space.

Fundamentally, this irreducible discrepancy prevents the discrete domain from collapsing into the standard continuous Euclidean topology, thereby preserving the extent of the fundamental spatial quantum within the system. By maintaining this finite topological gap, the model ensures that all geometric relations remain intrinsically discrete and invariant throughout the entire transformation.

## 5 Determination of the area of the discrete unit circle

As established in the preceding paragraphs, the geometric development of discrete circular sectors and that of trapezoidal area sections prove to be equivalent under specific conditions of linear proportionality between the radius and the legs of the defined model. The growth process entails a coordinated expansion of the associated geometric structures: the addition of a circular sector to the quantized domain corresponds one-to-one to the addition of an area section within the equivalent isosceles right triangle.

The resulting progressions generate geometric figures that, while not congruent in their extended geometric form, exhibit the same discrete surface area. The growth dynamics of these domains is governed by two strict structural constraints.

- i) Conservation of the total area at each evolutionary stage;
- ii) Preservation of the basic generative morphology of the figure (discrete circles and equivalent isosceles right triangles).

However, there is an analytical asymmetry between the two systems: the geometric constant  $k$ , multiplied by the square of the radius, explicitly operates in the progression of the discrete circle, whereas in the progression of the triangle, the configuration depends exclusively on the discrete count of the constituent area sections. It follows that, by identifying the minimum number of structural subdivisions required to close the discrete unit circle, the value of the constant  $k$  can be uniquely determined by equating it with the area of the corresponding equivalent isosceles right triangle.

Let  $N_{\min} \in \mathbb{Z}^+$  be the fundamental quantization parameter defining the minimum number of elementary circular sectors required to complete the geometric closure of the unit circle, starting from the initial null state. Given a discrete unit radius ( $R = 1$ ) and a normalized grid unit ( $u^2 = 1$ ), by equating the cumulative area progression of the equivalent triangle (3.7) with that of the discrete circle (2.2) as a function of the generic step  $n$ , the following relationship—calibrated to the closure parameter—is obtained:

$$(5.1) \quad \frac{n^2}{8} = k \cdot \frac{n^2}{N_{\min}^2}$$

By simplifying the equation and isolating the geometric constant of the system, the relation describing its dependence on the quantization parameter is directly obtained.

$$(5.2) \quad k = \frac{N_{\min}^2}{8}$$

The explicit calculation of the value of  $k$  therefore requires determining the discrete value assumed by  $N_{\min}$ , as formalized in the following theorem.

*Theorem 5.1* (Determination of the area of the discrete unit circle). Within the quantized geometric space based on the unit square metric  $Q_o$ , the area of the discrete unit circle is uniquely determined by the exact rational value:

$$(5.3) \quad k = \frac{25}{8} = 3.125$$

*Proof.* Consider a circular domain with a discrete unit radius ( $R = 1$ ). This domain is structurally constrained within the discrete boundaries of the inscribed square (with an area of 2) and the circumscribed square (with an area of 4). For the domain to preserve the intrinsic properties of a symmetric circular figure without degenerating into the limiting squares, the geometric constant  $k$  must strictly satisfy the open confinement condition:

$$(5.4) \quad 2 < k < 4$$

Since the numerical system is governed by integer partitions quantized along the development axis, the analysis of the discrete values derived from relation (5.2) highlights the following conditions:

- For  $N_{\min} \in \{1, 2, 3\}$ , the values  $k = \frac{1}{8}$ ,  $k = \frac{4}{8}$ , and  $k = \frac{9}{8}$  are obtained, respectively. These solutions are geometrically unacceptable as they fall below the lower limit defined by the inscribed square ( $k < 2$ ).
- For  $N_{\min} = 4$ , the relation yields  $k = \frac{16}{8} = 2$ . This value algebraically corresponds to the area of the inscribed square, resulting in a geometric degeneration of the circular domain.
- For  $N_{\min} = 5$ , the constant takes the value  $k = \frac{25}{8} = 3.125$ . This value satisfies the confinement constraint (5.4), which is perfectly within the admissible range.
- For any value  $N_{\min} \geq 6$ , the condition  $k \geq \frac{36}{8} = 4.5$  holds, systematically violating the upper limit imposed by the area of the circumscribed square ( $k > 4$ ).

Consequently, the parameter  $N_{\min} = 5$  proves to be the only integer solution consistent with the architecture of the discrete space under consideration, thereby demonstrating the validity of the constant  $k = \frac{25}{8}$ .  $\square$

## 5.1 Analysis of area and boundary contributions

By substituting the obtained value of  $k$  into the formula (4.4), the exact magnitude of the geometric equivalence constant of the model is determined.

$$(5.5) \quad \gamma = \sqrt{2k} = \frac{5}{2}$$

The dimensionless geometric constant  $k$ , which scales the total surface area of the domain, can be analytically expressed as the sum of the partial contributions of the first five discrete sectors for  $N = 5$ :

$$(5.6) \quad k = \sum_{n=1}^5 \frac{\Delta a_n}{R^2} = \sum_{n=1}^5 \frac{25}{8} \cdot \frac{2n-1}{25} = \frac{1}{8} + \frac{3}{8} + \frac{5}{8} + \frac{7}{8} + \frac{9}{8} = \frac{25}{8}$$

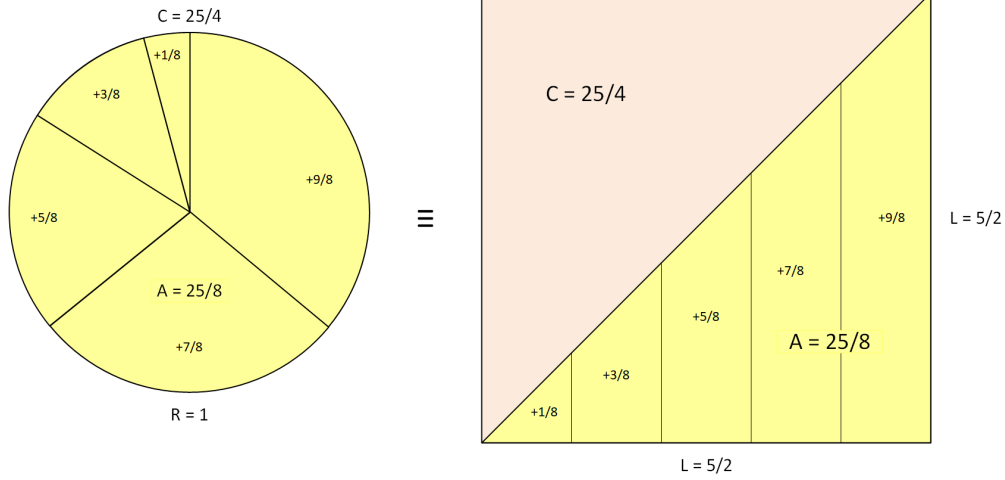


Figure 5: Geometric rectification of the domain: areal equivalence with the isosceles right triangle and boundary equivalence with the calibrated square

Symmetrically, for a unit circle ( $R = 1$ ), the length of the individual circular arcs is locally equal to twice the area of the corresponding circular sectors. The total linear length of the perimeter is determined analytically by summing these discrete contributions, which follow a linear progression of odd numbers—doubled relative to the area component:

$$(5.7) \quad 2k = \sum_{n=1}^5 \frac{\Delta c_n}{R} = 2 \sum_{n=1}^5 \frac{\Delta a_n}{R^2} = \frac{1}{4} + \frac{3}{4} + \frac{5}{4} + \frac{7}{4} + \frac{9}{4} = \frac{25}{4}$$

Fig. 5 illustrates the structural duality of the system through two distinct discrete equivalences governed by the constant  $\gamma$ :

- a) **Area equivalence:** The discrete unit circle is equivalent in total area ( $k = \frac{25}{8}$ ) to the isosceles right triangle whose legs are equal to the equivalence constant  $\gamma = \frac{5}{2}$ .
- b) **Boundary equivalence:** The total linear extent of the domain boundary ( $c = \frac{25}{4}$ ) is equivalent to the area of a square whose side length is equal to the same equivalence constant  $\gamma$ , such that the relation  $\gamma^2 = c$  holds exactly.

This formulation highlights how the single parameter  $\gamma$  simultaneously governs both the regional and boundary distributions of the quantized domain, preventing any dimensional inconsistency during the rectification process.

## 5.2 Analytical verification of the geometric domain closure

To confirm the geometric consistency of the result obtained, an inverse analytical verification is performed: the spatial closure of the domain is verified assuming that the exact value of the dimensionless geometric constant is  $k = \frac{25}{8}$ .

Consider a discrete elementary circular sector with an angular extent equal to one-eighth of the total surface area, having a measure corresponding to the exact fraction  $\frac{25}{64}$ . This subdomain is

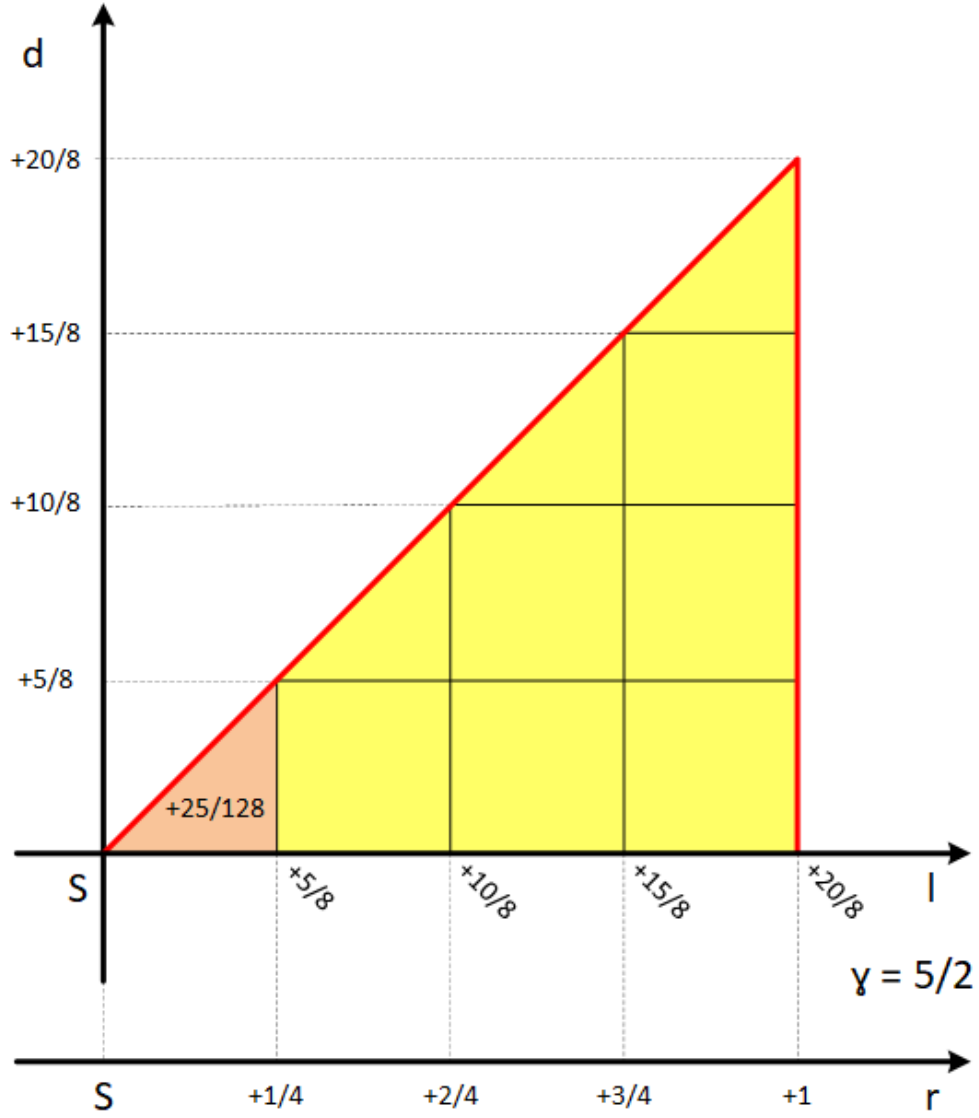


Figure 6: The isosceles right triangle equivalent to the discrete unit circle

geometrically equivalent to a reference square with a side length of  $\frac{5}{8}$ . By subdividing this square along its main diagonal, one isolates an elementary isosceles right triangle with an area of  $\frac{25}{128}$ .

The isosceles right triangle equivalent to the circle can be fully reconstructed by assembling and overlapping sixteen elementary triangular units of magnitude  $\frac{25}{128}$ , arranged according to the geometric scheme illustrated in Fig. 6. Along the leg axis, a specific configuration is identified consisting of four constant partitions of length  $\frac{5}{8}$ , the cumulative growth of which is modeled by the discrete sequence.

$$(5.8) \quad l_n = \left\{ \frac{5}{8}, \frac{10}{8}, \frac{15}{8}, \frac{20}{8} \right\}$$

By applying the geometric equivalence constant  $\gamma = \frac{5}{2}$  derived in (5.5), the linear progression of

the associated segments of the radius for  $N = 4$  is obtained:

$$(5.9) \quad r_n = \frac{l_n}{\gamma} = \left\{ \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1 \right\}$$

Reaching the unit value ( $r_4 = 1$ ) at the end of the fourth partition analytically confirms that the quantized circular domain—with a total area scaled by a factor of  $k = \frac{25}{8}$ —possesses a stable unit radius and that the geometric structure closes without measurement discrepancies when reaching the Cartesian boundary.

## 6 Conclusions and future developments

The value of the discrete geometric constant  $k = \frac{25}{8}$  derived in this work does not invalidate the transcendental constant  $\pi \approx 3.14159 \dots$  within the framework of classical analysis. Both results are mathematically consistent within their respective foundational contexts: the geometric continuum based on the continuity axiom of the real line, on the one hand, and a finite, quantized metric space, on the other. The coexistence of two distinct definitions of area for a unit-radius circular domain does not indicate a logical inconsistency; rather, it reflects the fundamental non-equivalence of the initial axiomatic assumptions.

The fact that the unit circle of the continuum cannot be mapped onto a stable isosceles right triangle highlights the radical topological incompatibility between the standard analytical approach and the rectification model developed here. This discrepancy shifts the debate toward the fundamental premises of classical geometry, particularly the nature of the dimensionless point and topological continuity. Although the proposed methodology represents an unconventional approach, it relies on a rigorous logical-deductive derivation that merits discussion within the scientific community.

To highlight the strengths of the model and outline future research directions, the central conceptual themes emerging from the study are summarized below.

- i) **The sectoral expansion method:** The circular domain is formalized as a dynamic structure that evolves through the coordinated interaction of two discrete progressions: the linear growth of the radius and the quadratic expansion of the elementary sectors. This computational formalism is not limited to the circle alone but lends itself to generalization for modeling morphological growth processes across various fields of discrete geometry and computer graphics.
- ii) **Axiomatics of quantized numbers:** The central hypothesis posits that numerical elements possess a finite extent, determined by rational multiples of the reference base unit. The degeneration of this unit into a standard geometric point would compromise the dimensional coherence of the system. Although departing from the field of real numbers alters the classical completeness criterion of the number line, this mathematical granularity presents intriguing analogies with theoretical models of quantum space-time physics.
- iii) **Rectification of the numerical staircase:** Precisely intersecting the midpoints of the steps with a cutting line allows for the isolation of geometric sections equivalent to right trapezoids with a  $45^\circ$  inclination. The ordered accumulation of these elements maps the discrete surface onto the profile of an equivalent isosceles right triangle. This method links a defined geometric topology to numerical sequences, positing that the algebraic structure precedes and generates the discrete geometric object.

- iv) **Equivalence criterion and proportionality constant:** The existence of an invariant scaling factor  $\gamma = \frac{5}{2}$  governing the linear relationship between the leg of the isosceles right triangle and the radius of the quantized circle has been demonstrated. Reformulating the discrete equations in terms of  $\gamma$  reveals that, within the quantized system, the linear extent of the sector arcs is geometrically equivalent to a sequence of defined rectangles that evolve into a perfect square if the radius is unity. This boundary property opens up further avenues for analytical investigation.
- v) **Analytical determination of the area:** The assignment of the value  $k = \frac{25}{8}$  derives from the open confinement constraint situated between the surfaces of the inscribed and circumscribed squares, associated with the structural closure of the discrete circle at the fundamental step  $N_{\min} = 5$ . The proof is based on the scale and shape invariance properties shared by circular domains and equivalent isosceles right triangles during the coupled growth process in the reference geometric space.

The study also introduces a fundamental proposal on the topology of numerical space, suggesting that the traditional linear Cartesian representation—extending from  $-\infty$  to  $+\infty$ —be replaced by a curvilinear mapping projected along the path of an Archimedean spiral. This approach would enable a native two-dimensional distribution capable of preserving a constant discrete step and ensuring the invariance of the dimensional value across the entire reference plane.

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